BORDERLINE WEAK TYPE ESTIMATES FOR SINGULAR INTEGRALS AND SQUARE FUNCTIONS

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ABSTRACT. For any Calderón-Zygmund operator T, any weight w, and $\alpha > 1$, the operator T is bounded as a map from $L^1(M_{L \log \log L (\log \log \log L)^\alpha} w)$ into weak-$L^1(w)$. The interest in questions of this type goes back to the beginnings of the weighted theory, with prior results, due to Coifman-Fefferman, Pérez, and Hytönen-Pérez, on the $L(\log L)^\epsilon$ scale. Also, for square functions Sf, and weights $w \in A_p$, the norm of S from $L^p(w)$ to weak-$L^p(w)$, $2 \le p < \infty$, is bounded by $[w]_{A_p}^{1/2}(1 + \log[w]_{A_\infty})^{1/2}$, which is a sharp estimate.

1. INTRODUCTION AND MAIN RESULTS

We are interested in two different weak-type estimates for singular integrals and square functions, at critical endpoint cases. The first of these is the weak L^1-endpoint for singular integrals. In [7], C. Fefferman and E. M. Stein used the estimate

$$(1.1) \quad \sup_{\lambda > 0} \lambda w(Mf > \lambda) \le C \int_{\mathbb{R}^d} |f(x)| \, Mw(x) \, dx,$$

where w is a general weight on $\mathbb{R}^d$ and M is the Hardy-Littlewood maximal operator, to study vector-valued inequalities for M. Indeed, the connection between weighted inequalities and their vector valued extensions was intensively studied in the next decades (see for instance [1, 22]). Rubio de Francia's extrapolation [20, 21] shows that from the inequality above, one can obtain all the A_p inequalities for the maximal function. See [4] for a recent accounting of that theory, especially [4, Cor 3.10] being relevant to the extrapolation results that follow from our main theorem. A conjecture of B. Muckenhoupt and R. Wheeden [16, p. 134] claimed that the inequality above should be true with the maximal function on the left-hand side replaced by a singular integral operator. This conjecture was disproved by Reguera [18] and Reguera-Thiele [19]. Also see [3].

The focus here is on the positive direction. Inequality (1.1) for singular integral operators is true if Mw is replaced by a larger maximal function. For instance, one can replace Mw by $(Mw^r)^{1/r}$, for exponents $1 < r < \infty$, first proved by Coifman-Fefferman [2],

from which you can derive the result of Benedeck-Calderón-Panzone [1]. Finer variants can be had by considering Orlicz versions of the maximal function. In 1994, C. Pérez [17] already established a version of the inequality with $(Mw^r)^{1/r}$ replaced by $M_{L(\log L)^\epsilon} w$, with $0 < \epsilon < 1$. Closely related to this is the sharp control of a Calderón-Zygmund operator on $L^p(w)$, relative to the A_1 constant of the weight, a theme of Fefferman-Pipher [6]. This matter was also pursued in, for instance, [14, 15]. Recently, T. Hytönen and C. Pérez quantified the estimates of Coifman-Fefferman [2] and Pérez [17].

Theorem A (Hytönen-Pérez, [9]). *For all* $0 < \epsilon < 1$ *and Calderón-Zygmund operator* T,

$$(1.2) \quad \sup_{\lambda>0} \lambda w(T^*f > \lambda) \lesssim \frac{1}{\epsilon} \int_{\mathbb{R}^d} |f(x)| \, M_{L(\log L)^\epsilon} w(x) \, dx.$$

In particular, for every $1 < r < \infty$,

$$(1.3) \quad \sup_{\lambda>0} \lambda w(T^*f > \lambda) \lesssim (1 + \log r') \int_{\mathbb{R}^d} |f(x)| \, M_r w(x) \, dx,$$

where $M_r w := (Mw^r)^{1/r}$, *and if* w *is an* A_1 *weight, there holds*

$$(1.4) \quad \sup_{\lambda>0} \lambda w(T^*f > \lambda) \leq C[w]_{A_1} \log(1 + [w]_{A_\infty}) \int_{\mathbb{R}^d} |f(x)| \, w(x) \, dx.$$

Recall that A_p weights, $1 < p < \infty$ are those non-negative weights w satisfying

$$[w]_{A_p} = \sup_Q \frac{1}{|Q|} \int_Q w \, dx \cdot \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,$$

where the supremum is over cubes in $\mathbb{R}^d$. As p decreases to 1, the condition above strengthens to

$$[w]_{A_1} = \sup_{x \in \mathbb{R}^d} \frac{Mw(x)}{w(x)} < \infty,$$

and as p tends to ∞, the condition weakens to

$$[w]_{A_\infty} = \sup_Q \frac{\int_Q M(w\mathbf{1}_Q)}{w(Q)}.$$

The estimate for M_r, (1.3), is not explicitly mentioned in [9] but it can be derived from (1.2) using the optimization argument in [9, Cor. 1.4]. The result for A_1 weights then follows by an appropriate choice of $r > 1$ based on the sharp reverse Hölder's inequality for the A_∞ constant [8, Thm. 2.3], recalled in (2.5) below.

Here T^* is the maximal truncation of T. Precise definitions of some standard objects, like Calderón-Zygmund operators, are given in section 2. The operator $M_{L(\log L)^\epsilon}$ is the modified Hardy-Littlewood maximal operator with respect to the function $\varphi(t) = t(1 + \log_+ t)^\epsilon$. More precisely,

$$M_{\varphi(L)} w(x) := \sup_{x \in Q} \|w\|_{\varphi(L),Q},$$

and

$$\|w\|_{\varphi(L),Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \varphi\left(\frac{w(x)}{\lambda}\right) dx \leq 1\right\}.$$

Notice that if $r \geq 1$ and $\varphi(t) = t^r$, then $M_{\varphi(L)}w = (Mw^r)^{1/r} = M_r w$. Finally, let us recall some properties of Young functions and Orlicz spaces (see [10] for more details). Throughout the paper, $\varphi : [0,\infty) \to [0,\infty)$ will be a *Young function*, that is, a convex, increasing function such that $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$. From these properties, on can deduce that its inverse φ^{-1} exists on $(0,\infty)$. Moreover, given a Young function, we can define its complementary function ψ by

$$\psi(s) = \sup_{t>0}\{st - \varphi(t)\}.$$

We will assume that $\lim_{t\to\infty}\varphi(t)/t = \infty$ to ensure that ψ is finite valued. Under these conditions, ψ is also a Young function and it is associated with the dual space of $\varphi(L)$. More precisely, one has the following generalized Hölder's inequality:

$$(1.5) \quad \frac{1}{|Q|}\int_Q |f(x)g(x)|dx \lesssim \|f\|_{\varphi(L),Q}\|g\|_{\psi(L),Q}.$$

The main result pushes the prior result to log-log scale as reflected in the tower of height 2 in (1.7).

Theorem 1.6. *Suppose the Young function φ satisfies*

$$(1.7) \quad c_\varphi = \sum_{k=1}^\infty \frac{1}{\psi^{-1}(2^{2^k})} < \infty.$$

Then, for all Calderón-Zygmund operators T, and any weight w on $\mathbb{R}^d$, it holds that

$$\sup_{\lambda>0}\lambda w\{T^*f > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^d}|f(x)|\, M_{\varphi(L)}w(x)\, dx.$$

We will see that this result contains Theorem A. It also contains new estimates such as those presented in Corollary 1.8 below. Notice that in (1.10), we very nearly have a double log in the maximal function, a possibility that was alluded to by Hytönen-Pérez [9]. For simplicity, from now on we will adopt the following notation:

$$\log_1(x) := 1 + \log_+(x) \quad \text{and} \quad \log_k(x) := \log_1\log_{k-1}(x), \text{ for } k > 1.$$

Corollary 1.8. *Under the assumptions of Theorem 1.6, for $1 < \alpha < 2$ it holds that,*

$$(1.9) \quad \sup_{\lambda>0}\lambda w\{T^*f > \lambda\} \lesssim \frac{1}{\alpha-1}\int_{\mathbb{R}^d}|f(x)|\, M_{L(\log_2 L)^\alpha}w(x)\, dx,$$

and in fact, one can also reach

$$(1.10) \quad \sup_{\lambda>0}\lambda w\{T^*f > \lambda\} \lesssim \frac{1}{\alpha-1}\int_{\mathbb{R}^d}|f(x)|\, M_{L\log_2 L(\log_3 L)^\alpha}w(x)\, dx.$$

Most of the prior arguments [8, 9, 14, 17] depend upon extrapolation type arguments, namely establishing a range of L^p inequalities, and then making an appropriate choice of $p \approx 1$ to conclude the argument at L^1. We address these two points in the (short) proof of Theorem 1.6.

- One should work directly with the weak-type norm, avoiding a Calderón-Zygmund decomposition approach involving L^p estimates for some $p > 1$. This is addressed by our decomposition of the sparse operator based upon the function f.
- The quasi-norm nature of the weak-L^1 norm is accounted for by using a restricted weak-type approach, and the specific structure of the operators in question.

Concerning sharpness, it seems very likely that Theorem 1.6 does not have any essential strengthening. But, the main counterexamples have at their heart a counterexample to an L^2 inequality for martingale transforms, as in Reguera's first paper on the subject [18]. Perhaps one could rethink the counterexamples for sparse operators. They must exist, but they seem somewhat involved to construct directly.

We turn to our second result, which concerns square functions. Define the intrinsic square function of M. Wilson [23] G_α, for $0 < \alpha < 1$ by

$$G_\alpha f(x)^2 = \int_{\Gamma(x)} A_\alpha f(y, t)^2 \frac{dy\, dt}{t^{n+1}}$$

where $\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ \ : \ |y| < t\}$ is the cone of aperture one in the upper-half plane, and

$$A_\alpha f(x, t) = \sup_{\gamma \in C_\alpha} |f * \gamma_t(x)|$$

where $\gamma_t(x) = t^{-n}\gamma(xt^{-n})$ and C_α is the collection of functions γ supported in the unit ball with mean zero and such that $|\gamma(x) - \gamma(y)| \leq |x - y|^\alpha$. The sharp weighted strong type norms for the square function have a critical case at $p = 3$, see Lerner [13].

We are concerned with the weak-type bounds, which have a critical case of $p = 2$, at which a $(\log_1[w]_{A_\infty})^{1/2}$ appears in the sharp estimate.

Theorem 1.11. *For $1 \leq p < \infty$, and any weight $w \in A_p$, there holds*

$$\|G_\alpha f\|_{L^{p,\infty}(w)} \leq C_p(w)\|f\|_{L^p(w)}$$

$$\text{where} \qquad C_p(w) = \begin{cases} [w]_{A_p}^{1/p} & 1 \leq p < 2 \\ [w]_{A_p}^{1/2}(\log_1[w]_{A_\infty})^{1/2} & 2 \leq p < \infty \end{cases}$$

In the next section, we recall some basic facts, including the reduction to sparse operators. Theorem 1.6 is then proved, followed by the Theorem 1.11.

2. Background

To say that T is a Calderón-Zygmund operator is to say that for f, g Schwartz functions on $\mathbb{R}^d$, with a positive distance between their supports, there holds

$$\langle Tf, g \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y) f(y) g(x) \, dx \, dy,$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfies the size and smoothness conditions

$$|K(x, y)| \leq \frac{1}{|x - y|^d}, \qquad x \neq y,$$

$$|K(x, y) - K(x', y)| \leq \frac{\omega\left(\frac{|x - x'|}{|x - y|}\right)}{|x - y|^d}, \qquad 2|x - x'| \leq |x - y|,$$

and the same inequality with the roles of the variables reversed also holds. Here the function $\omega : [0, 1] \mapsto [0, 1]$ is a *Dini modulus of continuity*, that is a decreasing function such that

$$\int_0^1 \omega(t) \frac{dt}{t} < \infty.$$

We furthermore assume that T extends to a bounded operator on $L^2(\mathbb{R}^n)$, with norm at most one. As usual, we will define the maximal truncation T^* by

$$T^* f(x) := \sup_{\delta > 0} \left| \int_{|x-y| > \delta} K(x, y) f(y) \, dy \right|.$$

In this setting, a result in [11, Theorem 5.2] provides the pointwise domination of both Tf and $T^* f$ by a sum of, at most, 3^d sparse operators, adapted to (shifted) dyadic grids.

A *sparse operator* is of the form

$$Tf = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \mathbf{1}_Q,$$

where the collection $\mathcal{S}$ consists of dyadic cubes which are sparse, in the sense that for all $Q \in \mathcal{S}$, there holds

$$(2.1) \quad \left| \bigcup_{Q' \in \mathcal{S} \, : \, Q' \subsetneq Q} Q' \right| \leq 8^{-1} |Q|.$$

With this domination result at hand, we can restrict our attention to sparse operators in proving Theorem 1.6.

Concerning square functions, a variant of the argument in [11] shows that the intrinsic square function is dominated by a sum of at most 3^d sparse square functions defined by

$$(2.2) \quad Sf^2 = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q^2 \mathbf{1}_Q.$$

And so, to prove Theorem 1.11, it suffices to prove the same estimate for the square functions Sf.

This elementary lemma on the $\varphi(L)$-norm will be needed.

Lemma 2.3. *Given a cube* $Q \subset \mathbb{R}^d$, *suppose that* $w : E \mapsto [0, \infty)$ *where* $E \subset Q$. *Then,*

$$\langle w \rangle_Q = \frac{1}{|Q|} \int_Q w(x) \, dx \lesssim \frac{\|w\|_{\varphi(L),Q}}{\psi^{-1}(|Q|/|E|)}.$$

Proof. This is just Hölder's inequality (1.5) applied to $w = w\mathbf{1}_E$ and the computation of $\|\mathbf{1}_E\|_{\psi(L),Q}$. $\square$

A final remark is that we will repeatedly use the (very easy) sharp weak-type A_p estimate for the maximal function

$$\lambda^p w\big(Mf > \lambda\big) \lesssim [w]_{A_p} \|f\|_{L^p(w)}^p.$$

Our analysis is entirely elementary, except for an appeal to the sharp reverse-Hölder estimate of Hytönen-Pérez [8, Thm. 2.3].

Theorem 2.4. *There is a dimensional constant* $c > 0$ *so that for* $w \in A_\infty$, *and* $r(w) = 1 + c[w]_{A_\infty}$, *there holds*

$$(2.5) \quad \langle w^{r(w)} \rangle_Q^{1/r(w)} \leq 2 \langle w \rangle_Q, \qquad Q \text{ a cube.}$$

3. Proof of Theorem 1.6 and Corollary 1.8

Let f be a non-negative function. Due to linearity of the weak-type estimate in λ, it suffices to show that

$$w(4 < Tf \leq 8) \lesssim \int_{\mathbb{R}^d} f(x) \, M_{\varphi(L)} w(x) \, dx.$$

Let $\mathcal{E} = \{4 < Tf \leq 8\} \setminus \{Mf > 2^{-2}\}$. In view of the classical inequality (1.1) of Fefferman-Stein, it is enough to check that

$$(3.1) \quad w(\mathcal{E}) \leq \tfrac{1}{4} \int_{\mathcal{E}} Tf(x) \, w(x) \, dx \lesssim \int_{\mathbb{R}^d} f(x) \, M_{\varphi(L)} w(x) \, dx.$$

By getting rid of the set $\{Mf > 2^{-2}\}$, we can eliminate from $\mathcal{S}$ all those cubes Q such that $\langle f \rangle_Q > 2^{-2}$. For $k \geq 2$, define $\mathcal{S}_k$ to be those $Q \in \mathcal{S}$ for which $4^{-k-1} < \langle f \rangle_Q \leq 4^{-k}$, and set

$$T_k f = \sum_{Q \in \mathcal{S}_k} \langle f \rangle_Q \mathbf{1}_Q.$$

The key lemma is the following:

Lemma 3.2. *For each integer* $k \geq 1$, *there is an absolute constant* C *such that,*

$$\int_{\mathcal{E}} T_k f(x) \, w(x) \, dx \leq 2^{-k} w(\mathcal{E}) + \frac{C}{\psi^{-1}(2^{2^k})} \int_{\mathbb{R}^d} f(x) \, M_{\varphi(L)} w(x) \, dx.$$

It is clear that this lemma completes the proof of our main theorem. We just write $Tf = \sum_{k=1}^{\infty} T_k f$, and from (3.1):

$$w(\mathcal{E}) \leq \tfrac{1}{4} \sum_{k=1}^{\infty} \left(2^{-k} w(\mathcal{E}) + \frac{C}{\psi^{-1}(2^{2^k}))} \int_{\mathbb{R}^d} f(x)\, M_{\varphi(L)} w(x)\, dx \right)$$

$$\leq \tfrac{1}{2} w(\mathcal{E}) + C \cdot c_\varphi \int_{\mathbb{R}^d} f(x)\, M_{\varphi(L)} w(x)\, dx,$$

which yields Theorem 1.6. Let us prove the lemma:

Proof. Write $\mathcal{S}_k$ as the union of $\mathcal{S}_{k,\nu}$, for $\nu = 0, 1, \ldots,$, where $\mathcal{S}_{k,0}$ are the maximal elements of $\mathcal{S}_k$, and $\mathcal{S}_{k,\nu+1}$ are the maximal elements of $\mathcal{S}_k \setminus \bigcup_{\ell=0}^{\nu} \mathcal{S}_{k,\ell}$. We are free to assume that $\mathcal{S}_{k,\nu} = \emptyset$ if $\nu > 4^{k+1}$, since for a cube Q in such a family, one would have that $\mathcal{E} \cap Q = \emptyset$. Hence, we need to estimate

(3.3) $$\sum_{\nu=0}^{4^{k+1}} \sum_{Q \in \mathcal{S}_{k,\nu}} \langle f \rangle_Q w(\mathcal{E} \cap Q).$$
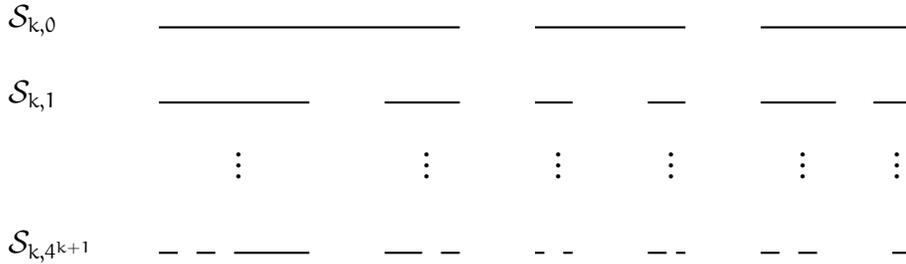

FIGURE 1. Layer decomposition of $\mathcal{S}_k$.

Define

$$E_Q = Q \setminus \bigcup_{Q' \in \mathcal{S}_{k,\nu+1}} Q', \quad \text{for } Q \in \mathcal{S}_{k,\nu}.$$

These sets are disjoint in $Q \in \mathcal{S}_k$ and using that $4^{-k-1} \leq \langle f \rangle_Q \leq 4^{-k}$ together with the sparsity condition (2.1), one can check that

(3.4) $$\int_Q f \leq \frac{8}{3} \int_{E_Q} f.$$

Set $u = 2^k$. It follows again from sparsity that for each $v \geq 0$, and $Q \in \mathcal{S}_{k,v}$,

$$(3.5) \quad |Q_u| \leq 8^{-u}|Q|, \qquad Q_u := \bigcup_{\substack{Q' \in \mathcal{S}_{k,v+u} \\ Q' \subset Q}} Q'.$$

In (3.3), for each $Q \in \mathcal{S}_{k,v}$ we decompose the set $\mathcal{E} \cap Q$ into

$$\mathcal{E} \cap Q = \mathcal{E} \cap \left(Q_u \cup \bigcup_{\ell=0}^{u-1} \bigcup_{\substack{Q' \in \mathcal{S}_{k,v+\ell} \\ Q' \subset Q}} E_{Q'} \right).$$

Using estimates (3.4), (3.5) and Lemma 2.3, for $Q \in \mathcal{S}_{k,v}$,

$$\langle f \rangle_Q w(\mathcal{E} \cap Q_u) \lesssim \int_{E_Q} f(x) \, dx \, \langle w \mathbf{1}_{\mathcal{E} \cap Q_u} \rangle_Q$$

$$\lesssim \frac{1}{\psi^{-1}(2^{2^k})} \int_{E_Q} f(x) \, M_{\varphi(L)} w(x) \, dx.$$

This is the way in which $M_{\varphi(L)} w$ is introduced. We sum this over v and $Q \in \mathcal{S}_{k,v}$ to the main estimate, using the disjointness of the sets E_Q:

$$\sum_{v=0}^{4^{k+1}} \sum_{Q \in \mathcal{S}_{k,v}} \langle f \rangle_Q w(\mathcal{E} \cap Q_u) \lesssim \frac{1}{\psi^{-1}(2^{2^k})} \int_{\mathbb{R}^d} f(x) \, M_{\varphi(L)} w(x) \, dx.$$

Now, the remaining sum is

$$\sum_{v=0}^{4^{k+1}} \sum_{Q \in \mathcal{S}_{k,v}} \sum_{\ell=0}^{u-1} \sum_{\substack{Q' \in \mathcal{S}_{k,v+\ell} \\ Q' \subset Q}} \langle f \rangle_Q w(\mathcal{E} \cap E_{Q'}) \leq 4^{-k} u \sum_{v=0}^{4^{k+1}} \sum_{Q \in \mathcal{S}_{k,v}} w(\mathcal{E} \cap E_Q) \leq 2^{-k} w(\mathcal{E}).$$

This only depends on the fact that $\langle f \rangle_Q \leq 4^{-k}$ and the disjointness of the sets $E_{Q'}$. The proof of the lemma is now complete.

$\square$

Finally, let us check that from here we can deduce both Theorem A and Corollary 1.8. We have three functions φ that we want to study:

1: $\varphi(t) = t(\log_1 t)^\epsilon$, with $0 < \epsilon < 1$,
2: $\varphi(t) = t(\log_2 t)^\alpha$, with $1 < \alpha < 2$,
3: $\varphi(t) = t \log_2 t (\log_3 t)^\alpha$, with $1 < \alpha < 2$.

In these cases, $\varphi(t) = tL(t)$, where L is the corresponding logarithmic part, and it holds that $L(t) \lesssim \psi^{-1}(t)$. Since ψ^{-1} is increasing, it suffices to check that

$$\psi(L(t)) = \sup_{0 < s < t} \{s(L(t) - L(s))\} \lesssim t.$$

This follows by the mean-value theorem and the fact that $rL'(r) \leq C$, with $C > 1$ being a universal constant. Therefore, we can work with c_φ with ψ^{-1} replaced by L.

Case 1. This corresponds to the inequality (1.2). From (1.7) and taking $L(t) = (\log_1 t)^\epsilon$ instead of ψ^{-1},

$$c_\varphi \simeq \sum_{k=1}^\infty \frac{1}{2^{k\epsilon}} \simeq \frac{1}{\epsilon}.$$

Case 2. This corresponds to the inequality (1.9). From (1.7) and taking $L(t) = (\log_2 t)^\alpha$ instead of ψ^{-1},

$$c_\varphi \lesssim \sum_{k=1}^\infty \frac{1}{k^\alpha} \lesssim \frac{1}{\alpha - 1}.$$

Case 3. This corresponds to the inequality (1.10). From (1.7) and taking $L(t) = \log_2 t(\log_3 t)^\alpha$ instead of ψ^{-1},

$$c_\varphi \lesssim \sum_{k=1}^\infty \frac{1}{k(\log k)^\alpha} \lesssim \frac{1}{\alpha - 1}.$$

This proves Theorem A and Corollary 1.8. We also mention that if we apply our theorem to the case $\varphi(t) = t^r$, then $\psi^{-1}(t) \simeq t^{1/r'}$ and the constant that we obtain for $M_{\varphi(L)} = M_r$ is exactly $\log_1 r'$, as in (1.3).

This last estimate (1.3), combined with the reverse Hölder estimate (2.5), proves the estimate (1.4). We remark that an alternate proof of this result can be had by straight forward modification of the argument in the next section. Details are left to the reader.

4. Square Functions: Proof of Theorem 1.11

Recall that it suffices to prove the weak-type bound for the sparse square function Sf defined in (2.2), in which we can assume f is non-negative. The case $1 \le p < 2$ is easy and contained in [12], as so our attention is on the case of $p \ge 2$. The sparse collection of cubes $\mathcal{S}$ is divided according to the approximate size of the average of f. For every integer m, define $\mathcal{S}_m$ by

$$\mathcal{S}_m = \{Q \in \mathcal{S} : 2^{-m-1} < \langle f \rangle_Q \le 2^{-m}\}.$$

The exceptional set for $Q \in \mathcal{S}_m$ is defined relative to $\mathcal{S}_m$ by

$$E_m(Q) = Q \setminus \bigcup_{Q' \subsetneq Q, \, Q' \in \mathcal{S}_m} Q'.$$

By sparsity, we have that

(4.1) $\quad \langle f \mathbf{1}_{E_m(Q)} \rangle_Q \sim \langle f \rangle_Q.$

Also, we set S_m to be the square function associated with $\mathcal{S}_m$

$$(S_m f)^2 := \sum_{Q \in \mathcal{S}_m} \langle f \rangle_Q^2 \mathbf{1}_Q.$$

Thus, trivially,

$$(Sf)^2 := \sum_{Q \in \mathcal{S}} \langle f \rangle_Q^2 1_Q = \sum_{m \in \mathbb{Z}} (S_m f)^2.$$

For the case of moderate m, namely $0 < m \le C \log_1 [w]_{A_\infty}$, we have this estimate.

Lemma 4.2. *For* $p \ge 2$,

$$\|S_m f\|_{L^p(w)} \lesssim [w]_{A_p}^{1/2} \|f\|_{L^p(w)}$$

Here and below we consider the case of $p \ge 2$. The critical case is $p = 2$, but if we were to focus on this case, one would then have to appeal to an A_∞ extrapolation argument for square functions, for which we do not have a clear cut reference. It is easier to simply prove the estimate for all $2 \le p < \infty$.

Proof. Rubio de Francia's extrapolation, as formulated in [5, Thm 3.1], shows that the case of $p = 2$ implies the case of $2 < p < \infty$. The case of $p = 2$ is a standard calculation. We will use the dual weight $\sigma = w^{-1}$, and the standard trick of inserting $1 = [w \cdot \sigma]^{1/2}$ inside a square. By (4.1), we can estimate

$$\sum_{Q \in \mathcal{S}_m} \langle f \rangle_Q^2 w(Q) \lesssim \sum_{Q \in \mathcal{S}_m} \langle f 1_{E_m(Q)} \rangle_Q^2 w(Q)$$

$$\le \sum_{Q \in \mathcal{S}_m} \langle f^2 1_{E(Q)} w \rangle_Q \langle \sigma \rangle_Q w(Q)$$

$$= \int f^2 \Big(\sum_{Q \in \mathcal{S}_m} 1_{E_m(Q)} \langle w \rangle_Q \langle \sigma \rangle_Q \Big) w \le [w]_{A_2} \int f^2 w.$$

$\square$

For large values of m, this estimate is relevant.

Lemma 4.3. *For all integers* $m_0 > 0$,

$$w \Big(\sum_{m=m_0}^{\infty} (S_m f)^2 > 1 \Big) \lesssim [w]_{A_p} \Big(\frac{[w]_{A_\infty}}{2^{m_0}} \Big)^p \|f\|_{L^p(w)}^p.$$

Proof. Write $(S_m f)^2$ as $2^{-2m} b_m$, where

$$b_m \le \sum_{Q \in \mathcal{S}_m} 1_Q$$

and b_m are supported on the set $B_m = \bigcup \{Q \; : \; Q \in \mathcal{S}_m^*\}$. Here $\mathcal{S}_m^*$ are the maximal cubes in $\mathcal{S}_m$. On each cube $Q \in \mathcal{S}_m$, the function b_m is locally exponentially integrable, by sparsity. By the sharp weak-type estimate for the maximal function, $w(B_m) \lesssim 2^{pm} [w]_{A_p} \|f\|_{L^p(w)}^p$.

We then estimate

$$w\left(\sum_{m=m_0}^{\infty}(S_m f)^2 > 1\right) = w\left(\sum_{m=m_0}^{\infty} 2^{-2m} b_m > \sum_{m=m_0}^{\infty} 2^{m_0-m-1}\right)$$

$$\leq \sum_{m=m_0}^{\infty} w(b_m > 2^{m_0+m-1}).$$

We have a very good Lebesgue measure estimate for the *Lebesgue measure* of the sets above. By sparsity, $|\{b_m > 2^{m_0+m-1}\}| \lesssim \exp(-C2^{m_0+m})|B_m|$. Indeed, this estimate is uniform over the cubes $Q \in \mathcal{S}_m^*$: Setting $\beta(Q) := \{x \in Q : b_m(x) > 2^{m_0+m-1}\}$, we have

$$|\beta(Q)| \lesssim \exp(-C2^{m_0+m})|Q|$$

This is converted to w-measure, using the A_∞ property of A_p weights, together with the sharp reverse-Hölder estimate. With $r(w)$ as in (2.5), there holds

$$\langle w\mathbf{1}_{\beta(Q)}\rangle_Q \leq \langle \mathbf{1}_{\beta(Q)}\rangle_Q^{1/r(w)'}\langle w^{r(w)}\rangle_Q^{r(w)}$$

$$\lesssim \left[\frac{|\beta(Q)|}{|Q|}\right]^{(C[w]_{A_\infty})^{-1}} w(Q) \lesssim w(Q)\exp\left(-c\frac{2^{m_0+m}}{[w]_{A_\infty}}\right).$$

Summing over the disjoint cubes in $\mathcal{S}_m^*$, we get

$$w\left(\sum_{m=m_0}^{\infty}(S_m f)^2 > 1\right) \lesssim [w]_{A_p}\|f\|_{L^p(w)}^p \sum_{m=m_0}^{\infty} 2^{mp}\exp\left(-c\frac{2^{m_0+m}}{[w]_{A_\infty}}\right).$$

The last sum is approximated by an integral to finish the proof.

$$\sum_{m=m_0}^{\infty} 2^{mp}\exp\left(-c\frac{2^{m_0+m}}{[w]_{A_\infty}}\right) \leq \int_{m_0}^{\infty} 2^{xp}\exp\left(-c\frac{2^{m_0+x}}{[w]_{A_\infty}}\right)dx$$

$$\approx \int_{2^{m_0}}^{\infty} y^p \exp\left(-c\frac{2^{m_0}}{[w]_{A_\infty}}y\right)\frac{dy}{y}$$

$$= \left(\frac{[w]_{A_\infty}}{2^{m_0}}\right)^p \int_{\frac{2^{2m_0}}{[w]_{A_\infty}}}^{\infty} y^p e^{-y}\frac{dx}{y} \lesssim \left(\frac{[w]_{A_\infty}}{2^{m_0}}\right)^p.$$

$\square$

The Lemmas are finished, and we can turn to the Theorem. Now, it suffices to estimate $w(Sf > \lambda)$, but it suffices to assume that $\lambda = 2$, and $\|f\|_{L^p(w)} = 1$. After division of the sparse collection $\mathcal{S}$ into the subcollections $\mathcal{S}_m$, for $m \in \mathbb{Z}$, estimate for $m_0 \approx \log_1[w]_{A_\infty}$,

$$w((Sf)^2 > 2) \leq w(Mf > 1) + w\left(\sum_{m=1}^{m_0-1}(S_m f)^2 > 1\right) + w\left(\sum_{m=m_0}^{\infty}(S_m f)^2 > 1\right).$$

The first term is controlled by the sharp weak-type estimate for the maximal function, which yields an estimate smaller than what we claim for Sf. The second term is estimated by Chebysheff, Minkowski's inequality as $p \geq 2$, and the norm estimate from Lemma 4.2.

$$
\begin{aligned}
w\Big(\sum_{m=0}^{m_0-1} (S_m f)^2 > 1 \Big) &\leq \Big\| \sum_{m=0}^{m_0-1} (S_m f)^2 \Big\|_{L^{p/2}(w)}^{p/2} \\
&\leq \Big(\sum_{m=0}^{m_0-1} \|(S_m f)^2\|_{L^{p/2}(w)} \Big)^{p/2} \\
&= \Big(\sum_{m=0}^{m_0-1} \|S_m f\|_{L^p(w)}^2 \Big)^{p/2} \leq \Big(m_0 [w]_{A_p} \Big)^{p/2}.
\end{aligned}
$$

For the third term we can just use the estimate from Lemma 4.3:

$$
w\Big(\sum_{m=m_0}^{\infty} (S_m f)^2 > 1 \Big) \leq [w]_{A_p} \Big(\frac{[w]_{A_\infty}}{2^{m_0}} \Big)^p.
$$

Combining these estimates we get

$$
\|Sf\|_{L^{p,\infty}(w)} \lesssim m_0^{\frac{1}{2}} [w]_{A_p}^{\frac{1}{2}} + [w]_{A_p}^{\frac{1}{p}} [w]_{A_\infty} 2^{-m_0} \approx \Big[[w]_{A_p} \log_1 [w]_{A_\infty} \Big]^{\frac{1}{2}}
$$

since $m_0 \approx \log_1 [w]_{A_\infty}$.

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