Invariant subspaces for the shift operator

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Hardy spaces

Definition (Hardy spaces)

Let $1 \leq p < \infty$. *We define the space* H^p *by*

$$H^p = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H^p}^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt < \infty \right\}.$$

We also define

$$H^\infty = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Hardy spaces

The following result connects H^p-spaces to L^p-spaces:

Theorem ($H^p \subseteq L^p(\mathbb{T})$)

Let $1 \leq p \leq \infty$. A function f belongs to H^p if, and only if, it is the Poisson integral of some $g \in L^p(\mathbb{T})$ whose Fourier coefficients satisfy

$$\hat{g}(k) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) e^{-ikt} dt = 0 \qquad \forall k < 0.$$

Moreover,

$$g(e^{it}) = \lim_{r \to 1^-} f(re^{it})$$

exists for almost every $0 \leq t < 2\pi$, and $\|g\|_{L^p(\mathbb{T})} = \|f\|_{H^p}$.

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exists for almost every $0 \leq t < 2\pi$, and $\|g\|_{L^p(\mathbb{T})} = \|f\|_{H^p}$.

With this result, for all $1 \leq p \leq \infty$, the Hardy space H^p can be interpreted as a subspace of $L^p(\mathbb{T})$,

$$H^p = \{f \in L^p(\mathbb{T}) : \hat{f}(k) = 0, \ \forall k < 0\}.$$

Hardy spaces

Proposition ($H^2 \leftrightarrow \ell^2$)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$. *Then,*

$$f \in H^2 \iff \{a_n\}_n \in \ell^2.$$

Moreover, we have that $\|f\|_{H^2} = \|\{a_n\}_n\|_2$.

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$$f \in H^2 \Longleftrightarrow \{a_n\}_n \in \ell^2.$$

Moreover, we have that $\|f\|_{H^2} = \|\{a_n\}_n\|_2$.

As a consequence, we have that H^2 is isometrically isomorphic to ℓ^2 (thus, it is a Hilbert space) and

$$H^2 = \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \{a_n\}_n \in \ell^2 \right\}.$$

Hardy spaces

Remark (Density of polynomials in H^2)

The last proposition also implies that the space of polynomials is dense in H^2, since every function $f \in H^2$ can be approximated by the partial sums of its power series $\sum_{n=0}^{\infty} a_n z^n$.

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The last proposition also implies that the space of polynomials is dense in H^2, since every function $f \in H^2$ can be approximated by the partial sums of its power series $\sum_{n=0}^{\infty} a_n z^n$.

Indeed,

$$\left\| f - \sum_{n=0}^{N} a_n z^n \right\|_{H^2} = \left\| \sum_{n=N+1}^{\infty} a_n z^n \right\|_{H^2} = \left(\sum_{n=N+1}^{\infty} |a_n|^2 \right)^{1/2} \xrightarrow{N} 0.$$

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If f is an inner function, then $|f(z)| \leq 1$ for all $z \in \mathbb{D}$.

Indeed, $f \in H^\infty$ and thus, for all $z \in \mathbb{D}$,

$$\begin{aligned} |f(z)| =& |(\mathcal{P}f)(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})||P_z(e^{it})|dt \\ =& \frac{1}{2\pi} \int_0^{2\pi} P_z(e^{it})dt = 1. \end{aligned}$$

Inner functions

Definition (Blaschke product)

Let $\{a_n\}_n \subseteq \mathbb{D} \setminus \{0\}$ *satisfying* $\sum_{n=0}^{\infty} (1 - |a_n|) < \infty$ *and* $m \geq 0$. *We define the Blaschke product associated with* $\{a_n\}_n$ *and* m *by*

$$B(z) := z^m \prod_{n=0}^{\infty} \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - \overline{a_n} z}, \qquad z \in \mathbb{D}.$$

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Proposition

Under these conditions, B *defines a function in* H^∞ *and* $|B| = 1$ *a.e. on* $\mathbb{T}$*.*

Inner functions

Theorem (Characterization of inner functions)

Suppose that λ is a constant with $|\lambda| = 1$, B is a Blaschke product and μ is a finite, positive, Borel measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure. Then

$$G(z) = \lambda B(z) \exp\left\{ -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\}, \qquad z \in \mathbb{D},$$

is an inner function. Moreover, every inner function is of this form.

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EXAMPLE TIME!!

Outer functions

Definition (Outer function)

If φ *is a positive, measurable function on* $\mathbb{T}$ *such that* $\log \varphi \in L^1(\mathbb{T})$*, then*

$$Q(z) = \lambda \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log \varphi(e^{it})dt\right\}, \qquad z \in \mathbb{D},$$

is called an outer function. Here λ *is a constant with* $|\lambda| = 1$*.*

Factorization theorem

Theorem (Factorization)

Let $1 \leq p \leq \infty$ *and assume that* $f \in H^p$ *is not identically zero. Then, there is an outer function* $Q_f \in H^p$ *(whose constant factor is* $\lambda = 1$*) and an inner function* G_f *such that*

$$f = G_f Q_f.$$

Moreover, this decomposition is unique.

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Moreover, this decomposition is unique.

The functions G_f and Q_f are called the inner and outer factors of f, respectively.

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Invariant subspaces and cyclic vectors

Definition (Invariant subspace)

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$$T(F) \subseteq F.$$

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Definition (Cyclic element)

We say that an element $x \in E$ is cyclic for $T \in \mathcal{L}(E)$ if

$$\mathcal{O}_T(x) := \overline{\langle x, Tx, T^2x, \ldots \rangle} = E.$$

Summing up...

- $H^p = \{f \in L^p(\mathbb{T}) : \widehat{f}(k) = 0 \; \forall k < 0\}.$

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- $H^p = \{f \in L^p(\mathbb{T}) : \widehat{f}(k) = 0 \ \forall k < 0\}$.
- H^2 is a Hilbert space.
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- Inner functions: $f \in H^\infty : |f| = 1$ a.e. on $\mathbb{T}$ ($\Rightarrow f(z) \leq 1$ on $\mathbb{D}$).

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- Inner functions are $G(z) = \lambda B(z) S_\mu(z)$.

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- For all $f \in H^p$, $f = G_f Q_f$ in a unique way.
- $x \in E$ is cyclic for T iff $\mathcal{O}_T(x) = \overline{\langle x, Tx, T^2x, ...\rangle} = E$.

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The shift operator

Definition (Shift operator)

Let H be a separable Hilbert space and let $\{\xi_n\}_{n\geq 0} \subseteq H$ be an orthonormal basis. We define the shift operator S on H as the continuous, linear operator satisfying

$$S(\xi_n) = \xi_{n+1}, \qquad n \geq 0.$$

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PROBLEM: We want to study the invariant subspaces for the shift operator on a Hilbert space H.

Identification $H \leftrightarrow H^2$

We know that $\{e^{ikt}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$. By Fischer-Riesz's theorem, this is equivalent to saying that, for all $f \in L^2(\mathbb{T})$,

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikt}, \quad \text{in } L^2(\mathbb{T}).$$

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We have that

$$H^2 = \{f \in L^2(\mathbb{T}) : \hat{f}(k) = 0, \ \forall k < 0\},$$

so every function in H^2 has the form

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Therefore, $\{e^{int}\}_{n\geq 0}$ is an orthonormal basis for the Hilbert space H^2.

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Even more, if we write $z = e^{it} \in \mathbb{T}$, then $e^{int} = z^n$ for all $n \geq 0$ and the shift operator becomes multiplication by z on H^2.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \Longrightarrow (Sf)(z) = \sum_{n=0}^{\infty} a_n z^{n+1} = zf(z).$$

Beurling's theorem

Given a closed subspace $M \subseteq H^2$, M will be *invariant* under S if $zM \subseteq M$. Equivalently, M is invariant if and only if $p(z)M \subseteq M$ for every polynomial p.

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Theorem (Beurling)

A non-zero subspace $M \subseteq H^2$ is invariant under S if and only if there exists an inner function G such that

$$M = GH^2 = \{Gf : f \in H^2\}.$$

Moreover, G is unique up to a constant factor of modulus 1.

The proof

Assume that $M = GH^2$.

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- M is S-invariant: If $f \in H^2$,

$$S(Gf)(z) = zG(z)f(z) = G(z)[zf(z)] \in GH^2.$$

The proof

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$$|f| = \frac{|G_1|}{|G_2|} = 1 \text{ and } |g| = \frac{1}{|f|} = 1 \text{ a.e. on } \mathbb{T},$$

so both f and $g = 1/f$ are inner functions.

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Therefore, $|f| = 1$ on $\mathbb{D}$ and by the maximum principle, $f = \lambda$ with $|\lambda| = 1$, as we wanted to show.

The proof

Conversely, let $M \neq \{0\}$ be an invariant subspace. Consider k the least non-negative integer so that there exists a function $f \in M$ satisfying

$$f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots, \qquad \text{with } a_k \neq 0.$$

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Using the minimality of k, we have that $f \notin zM$, and by hypothesis, $zM \subseteq M$, so zM is a proper subspace of M. Moreover, zM is closed in M, and by the orthogonal projection theorem, we have

$$M = zM \oplus (zM)^{\perp_M},$$

where

$$(zM)^{\perp_M} := \{f \in M : f \perp zg \quad \forall g \in M\},$$

and $(zM)^{\perp_M} \neq \{0\}$.

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That is, writing $z = e^{it}$,

$$\begin{aligned} 0 =& \frac{1}{2\pi} \int_0^{2\pi} G(e^{it}) e^{-int} \overline{G(e^{it})} dt \\ =& \frac{1}{2\pi} \int_0^{2\pi} |G(e^{it})|^2 e^{-int} dt, \qquad \forall n \geq 1. \end{aligned}$$

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Furthermore, if we conjugate the previous equation, since $|G(e^{it})|^2 \in \mathbb{R}$, we obtain

$$0 = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{it})|^2 e^{int} dt, \qquad \forall n \geq 1.$$

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$$|G|^2 = 1 \text{ a.e. on } \mathbb{T},$$

and hence, G is an inner function.

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With this, we prove that

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Therefore, if we prove that $(GH^2)^{\perp_M} = \{0\}$, we are done.

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Take $h \in M$ with $h \perp GH^2$.

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Combining both identities, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\overline{G(e^{it})}e^{-ikt}dt = 0, \qquad k \in \mathbb{Z},$$

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This implies that $h\overline{G} = 0$ a.e. on $\mathbb{T}$.
Since $|G| = 1$ a.e. on $\mathbb{T}$, we get that $h = 0$ as an H^2-function and we complete the proof.

Contents

Bijection

We have that inner functions are the ones of the form

$$G(z) = \lambda B(z) \exp\left\{ -\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \right\}, \qquad z \in \mathbb{D},$$

where λ is a constant with $|\lambda| = 1$, B is a Blaschke product and μ is a finite, positive, Borel measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure.

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where λ is a constant with $|\lambda| = 1$, B is a Blaschke product and μ is a finite, positive, Borel measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure.
Therefore, there is a bijection

$$\{(\{a_n\}_n, m, \mu)\} \longleftrightarrow \{\text{Invariant subspaces for } S\},$$

where $\{a_n\}_n \subseteq \mathbb{D} \setminus \{0\}$ satisfies the Blaschke condition $(\sum_n (1 - |a_n|) < \infty)$ and $m \geq 0$.

Basic invariant subspaces

Given a function $f \in H^2$, we may wonder what invariant subspace is the smallest one containing f. Such subspace is given by

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We have the following proposition:

Proposition

Let $f \in H^2$ and $f = G_f Q_f$ be its factorization as a product of an inner and an outer function. Then,

$$\mathcal{O}_S(f) = G_f H^2.$$

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But GG_h is another inner function, so by the uniqueness of this factorization, $Q_f = Q_h$ and $G_f = GG_h$. In particular, $G_f \in GH^2$, and by invariance and density of the polynomials in H^2, we conclude that

$$G_f H^2 \subseteq GH^2 = \mathcal{O}_S(f),$$

and we complete the proof.

cyclic vectors

This result yields an easy description of cyclic vectors:

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A function $f \in H^2$ is cyclic for S if and only if f is an outer function.

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EXAMPLE TIME!!

Greatest common divisor of a family of inner functions

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- One can check that G_0 is the greatest common divisor of $\mathcal{G}$.

=)

The End!