Reaching weak-type (1,1) estimates via Extrapolation Theory

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2014 Santaló Summer School - September 2014

A_p weights

It is well-known (Muckenhoupt, 1972) that the Hardy-Littlewood maximal operator M satisfies, for $1 \leq p < \infty$:

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where, for $1 < p < \infty$, $w \in A_p$ if

$$\|w\|_{A_p} = \sup_Q \frac{w(Q)}{|Q|}\left(\frac{w^{-p'/p}(Q)}{|Q|}\right)^{p/p'} < \infty,$$

and $w \in A_1$ if

$$Mw(x) \leq Cw(x) \qquad \text{a. e. } x \in \mathbb{R}^n,$$

with $\|w\|_{A_1}$ being the least constant $C > 0$ in the previous expression.

Factorization

One of the most important properties of A_p weights is that they can be characterized in terms of A_1 weights in the following way:

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Therefore, we can think of A_p weights as those of the form:

$$(Mf)^{\delta(1-p)} u,$$

with $f \in L^1_{loc}(\mathbb{R}^n)$, $0 < \delta < 1$ and $u \in A_1$.

Rubio de Francia

In this setting, we have the Rubio de Francia extrapolation theorem:

Theorem (Rubio de Francia's Extrapolation Theorem)

Given a sublinear operator T such that for some $1 \leq p_0 < \infty$ we have

$$\|Tf\|_{L^{p_0,\infty}(w)} \leq C(w,p_0)\|f\|_{L^{p_0}(w)} \qquad \textit{for every } w \in A_{p_0},$$

then, for every $1 < p < \infty$,

$$\|Tf\|_{L^{p,\infty}(w)} \leq C(w,p_0,p)\|f\|_{L^{p}(w)} \qquad \textit{for every } w \in A_{p}.$$

Restricted weak-type

It is also known (Kerman and Torchinsky, 1982) that the Hardy-Littlewood maximal operator M satisfies, for $1 \leq p < \infty$:

$$\|M\chi_E\|_{L^{p,\infty}(w)} \leq Cw(E)^{1/p} \Longleftrightarrow w \in A_p^{\mathcal{R}}$$

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It holds that $A_p \subseteq A_p^{\mathcal{R}} \subseteq A_{p+\varepsilon}$ for every $\varepsilon > 0$ and $A_1^{\mathcal{R}} = A_1$.

A new class of weights

It can be proved that given a locally integrable function f and an A_1 weight u, then

$$(Mf)^{1-p}u \in A_p^{\mathcal{R}},$$

with

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However, we do not know if all $A_p^{\mathcal{R}}$ weights are of this form! Therefore, the class we want to define is exactly

$$\widehat{A}_p = \left\{w = (Mf)^{1-p}u, \quad \text{where } f \in L^1_{loc}, u \in A_1\right\} \subseteq A_p^{\mathcal{R}},$$

with

$$\|w\|_{\widehat{A}_p} = \inf \|u\|_{A_1}^{1/p}.$$

The extrapolation result

Theorem (Carro-Grafakos-Soria)

Given a sublinear operator T such that for some $1 \le p_0 < \infty$ we have

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \le C(w,p_0)w(E)^{1/p_0} \qquad \text{for every } w \in \widehat{A}_{p_0},$$

then, for every $1 \le p < \infty$,

$$\|T\chi_E\|_{L^{p,\infty}(w)} \le C(u,p_0,p)w(E)^{1/p} \qquad \text{for every } w \in \widehat{A}_{p}.$$

From restricted weak-type to weak-type

Looking at this result, we have that if an operator T is of restricted weak-type (p_0, p_0) for every weight in $\widehat{A}_{p_0}$, then it is of restricted weak-type $(1, 1)$ for every weight in A_1.

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For instance, it can be checked that if

$$Tf(x) = K * f(x),$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then T is $(\varepsilon, \delta)-$atomic,

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For instance, it can be checked that if

$$Tf(x) = K * f(x),$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \le p < \infty$, then T is $(\varepsilon, \delta)-$atomic, and if $\{T_n\}_n$ is a sequence of $(\varepsilon, \delta)-$atomic operators, then both

$$T^* f(x) = \sup_n |T_n f(x)|, \quad \text{and} \quad Tf(x) = \left(\sum_n |T_n f(x)|^q \right)^{1/q},$$

are $(\varepsilon, \delta)-$atomic approximable, for every $q \ge 1$.

Applications

Some examples of operators to which one can apply this new extrapolation are:

(i) If $u(x,t) = P_t * f(x)$ is the Poisson integral of f, the Lusin area integral is defined by

$$S_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2},$$

where $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$.

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(iii) The intrinsic square function G_α (introduced by M. Wilson), Haar shift operators, averages of operators satisfying the hypothesis...

Bochner-Riesz

Normally, if one has the proof of the strong-type (p, p) for A_p weights for some operator and it does not rely on the $1 + \varepsilon$ property of the weights, it can be adapted to prove the corresponding restricted weak-type for the $\widehat{A}_p$ class...

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$$\widehat{(T_\lambda f)(\xi)} = \widehat{f}(\xi)(1 - |\xi|^2)_+^\lambda.$$

The **Bochner-Riesz operator at the critical index** $\lambda = (n-1)/2$!!!

We are almost there!

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$$\|T_\lambda(\chi_E)\|_{L^{2,\infty}((M\chi_E)^{1-2}u)} \leq C(u)u(E)^{1/2}.$$

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• The good thing of this approach is that one deals with Banach spaces (whereas in proving a weak-type $(1,1)$ estimate directly forces you to put up with the $L^{1,\infty}$ quasinorm). However, the main drawback is that the $\widehat{A}_p$ class does not have the $1+\varepsilon$ property which has turned out to be really useful when proving A_p estimates.

Gràcies per la vostra atenció!

Gracias por vuestra atención!

Thank you for your attention!