Capture zones of the family of functions $\lambda z^m \exp(z)$.

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Abstract

We consider the family of entire transcendental maps given by $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ where $m \geq 2$. All functions $F_{\lambda,m}$ have a superattracting fixed point at $z = 0$, and a critical point at $z = -m$. In the dynamical plane we study the topology of the basin of attraction of $z = 0$. In the parameter plane we focus on the capture behaviour, i.e., λ values such that the critical point belongs to the basin of attraction of $z = 0$. In particular, we find a capture zone for which this basin has a unique connected component, whose boundary is then non-locally connected. However, there are parameter values for which the boundary of the immediate basin of $z = 0$ is a quasicircle.

1 Introduction

Our goal in this paper is to study some dynamical aspects of the families of entire transcendental maps

$$F_{\lambda,m}(z) = \lambda z^m \exp(z), \quad m \geq 2.$$

Observe that $m = 0$ corresponds to the exponential family $E_\lambda(z) = \lambda \exp(z)$, the simplest example of an entire transcendental map with a unique asymptotic value, $z = 0$, in analogy with the well known quadratic family of polynomials $z \to z^2 + c$. The exponential map has been thoroughly studied by many authors (see for example, [Devaney & Krych, 1984], [Devaney & Tangerman, 1986]).

The case $m = 1$ corresponds to $G_\lambda(z) = \lambda z \exp(z)$ which appeared for the first time in [Baker, 1970] as an example of an entire transcendental map whose Julia set is the whole plane (for an appropriate value of λ). Later on this family was studied in [Fagella, 1995] and [Geyer, 2001]. The asymptotic value $z = 0$ of $G_\lambda(z)$ is fixed and its multiplier is $G'_\lambda(0) = \lambda$.

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Hence its dynamical character depends on the parameter λ. Besides this point, the dynamical behavior of G_λ is determined by the orbit of the critical point $z = -1$.

Some functions in the family $F_{\lambda,m} = \lambda z^m \exp(z)$ for $m \geq 2$ have been used in the literature as examples of certain dynamical phenomena (see for example [Bergweiler, 1995], for a Baker domain at a positive distance from any singular orbit). But, to our knowledge, no systematic study has been made before this work.

All functions of the form $F_{\lambda,m}$, with $m \geq 2$, have a superattracting fixed point at $z = 0$ of multiplicity m, which is also an asymptotic value. The only other critical point is $z = -m$. The coexistence of a superattracting fixed point and a free critical point makes this family very much analogous to the family of cubic polinomials $C_a(z) = z^3 - 3a^2 z + 2a^3 + a$ which is described in [Milnor, 1991].

Let f be a transcendental entire function. It is known that the dynamical plane can be decomposed into two invariant sets. The first one is an open set, namely the Fatou set or stable set, denoted by $\mathcal{F}(f)$, and it is formed by points, $z_0 \in \mathbb{C}$ whose iterates $\{f^{\circ n}\}$ form a normal family, in the sense of Montel, in some neihbourhood of z_0. The second one, namely the Julia or chaotic set, denoted by $\mathcal{J}(f)$, is defined as the complement of the Fatou set, that is $\mathcal{J}(f) = \mathbb{C} - \mathcal{F}(f)$. Fatou ([Fatou, 1926]) showed that the Julia set of an entire transcendental function is a completely invariant, closed, nonempty and perfect set. As in the polynomial case, it may also be defined as the closure of the set of repelling periodic points.

We denote by $A(0) = A_{\lambda,m}(0)$ the basin of attraction of $z = 0$, i.e., the set of all z such that $F^{\circ n}_{\lambda,m}(z) \to 0$ as n tends to $+\infty$. We also denote by $A^*(0) = A^*_{\lambda,m}(0)$ the connected component of $A(0)$ containing $z = 0$.

In Sec. 2 we concentrate on the dynamical plane and study the basin of attraction $A(0)$ (see Fig. 1). The skeleton of the main components of $A(0)$ is needed to study later the parameter planes. We summarize the main results with regard to $A(0)$ in the following theorem.

Theorem A.

1. *There exists $\epsilon_0 = \epsilon_0(|\lambda|, m) > 0$, defined as the unique positive solution of $x^{m-1}e^x = 1/|\lambda|$, such that $A^*(0)$ contains the disk $D_{\epsilon_0} = \{z \in \mathbb{C};\ |z| < \epsilon_0\}$.*
2. *There exist $x_0 = x_0(|\lambda|, m) < 0$ and a function $C(x) \geq 0$ such that the open set*
$$H_{|\lambda|,m} = \left\{ z = x + yi \;\middle|\; \begin{array}{ll} x & \in (-\infty, x_0) \\ y & \in (-C(x), C(x)) \end{array} \right\}$$
satisfies $F_{\lambda,m}\left(H_{|\lambda|,m}\right) \subset D_{\epsilon_0}$.
3. *There exist infinitely many strips, denoted by $S^k_{\lambda,m}$, which are preimages of $H_{|\lambda|,m}$. These horizontal strips extend to $+\infty$, and they have asymptotic width equal to π.*

Section 3 is dedicated to parameter planes. For some parameter values, the free critical point $z = -m$ belongs to the basin of attraction of $z = 0$, A(0), in which case we say

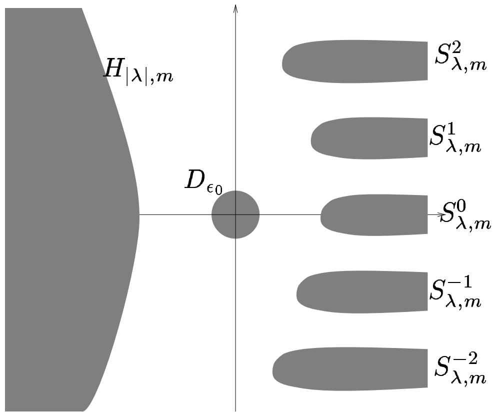


Figure 1: Sketch of the basin of attraction of $z = 0$, satisfying that $D_{\epsilon_0} \subset A^*(0)$, $F_{\lambda,m}(H_{|\lambda|,m}) \subset D_{\epsilon_0}$ and $F_{\lambda,m}\,(S^k_{\lambda,m}) \subset H_{|\lambda|,m}$.

that it has been "captured". The connected components of parameter space for which this phenomenon occurs are called *capture zones*, and they clearly do not exist for members of the family with $m < 2$ such as the exponential. Hence it is natural to study the set of parameters for which the orbit of $z = -m$ is bounded. That is, we define the sets

$$B_m = \{\lambda \in \mathbb{C} \,|\, F^{\circ n}_{\lambda,m}(-m) \nrightarrow \infty\}.$$

In each of these sets, we may also distinguish between two different behaviours. Those parameter values for which $-m \in A(0)$ and those for which this does not occur. Let $\overset{\circ}{B}_m$ denote the interior of B_m. We will study the sets

$$C^n_m = \{\lambda \in \overset{\circ}{B}_m \,|F^n_{\lambda,m}(-m) \in A^*(0) \text{ and } n \text{ is the smallest number with this property}\}$$

Although each B_m contains infinitely many different capture zones, there is one which is dynamically very different from all others. We define the *main capture zone* C^0_m as the set of parameter values λ for which the critical point $z = -m$ belongs to the immediate basin of 0. That is,

$$C^0_m = \{\lambda \in \overset{\circ}{B}_m \mid -m \in A^*(0)\}.$$

As we shall see, this is a quite special component of B_m since its boundary separates the parameter values for which $\partial A^*(0)$ is a Cantor bouquet from those for which it is a Jordan curve (also, this boundary separates the parameter values for which $\mathcal{F}(F_{\lambda,m})$ has one or infinitely many components). The detailed study of this boundary will be the object of a future paper. Our goal in Sec. 3 is to describe the main features of the parameter planes of the functions $F_{\lambda,m}$ and, in particular, the structure of the capture zones (Fig. 2). We summarize some of these facts in the following theorem.

Theorem B.

1. *The critical point $-m$ belongs to $A^*(0)$ if and only if the critical value $F_{\lambda,m}(-m)$ belongs to $A^*(0)$. Hence $C^1_m = \emptyset$.*

2. *The main capture zone C_m^0 is bounded.*

3. *The set C_m^0 contains the disk $\{\lambda \in \mathbb{C};\ |\lambda| < min(\frac{1}{e}, (\frac{e}{m})^m)\}$.*

4. *If $\lambda \in C_m^0$ then $A(0) = A^*(0)$, i.e., the basin of attraction of $z = 0$ has a unique connected component and hence it is totally invariant. Moreover, the boundary of $A^*(0)$ (which equals the Julia set) is a Cantor bouquet and hence it is disconnected and non-locally connected.*

5. *If $\lambda \notin C_m^0$ then $A(0)$ has infinitely many components. Moreover, if $|\lambda| > (\frac{e}{m-1})^{m-1}$, the boundary of $A^*(0)$ is a quasi-circle.*

We also summarize some properties of the most obvious capture zones C_m^2 and C_m^3.

Theorem C.

1. *The set C_m^2 contains an unbounded set to the left or to the right depending on the oddity of m. More precisely, there exists a real constant $D_0(m) > 0$, and a function $\alpha = \alpha(|\lambda|, m) \in (\pi/2, \pi)$, such that*

   - *for m even, the set C_m^2 contains the open set* $\left\{\lambda \in \mathbb{C} \;\middle|\; \begin{array}{ll} |\lambda| & > D_0 \\ |Arg(\lambda)| & > \alpha \end{array}\right\}$
   - *for m odd, the set C_m^2 contains the open set* $\left\{\lambda \in \mathbb{C} \;\middle|\; \begin{array}{ll} |\lambda| & > D_0 \\ |Arg(\lambda)| & < \pi - \alpha \end{array}\right\}$

2. *There exists infinitely many strips in C_m^3. If m is even (resp. odd) then these horizontal strips extend to $+\infty$ (resp. $-\infty$) and they have an asymptotic width equal to $(\frac{e}{m})^m\pi$.*

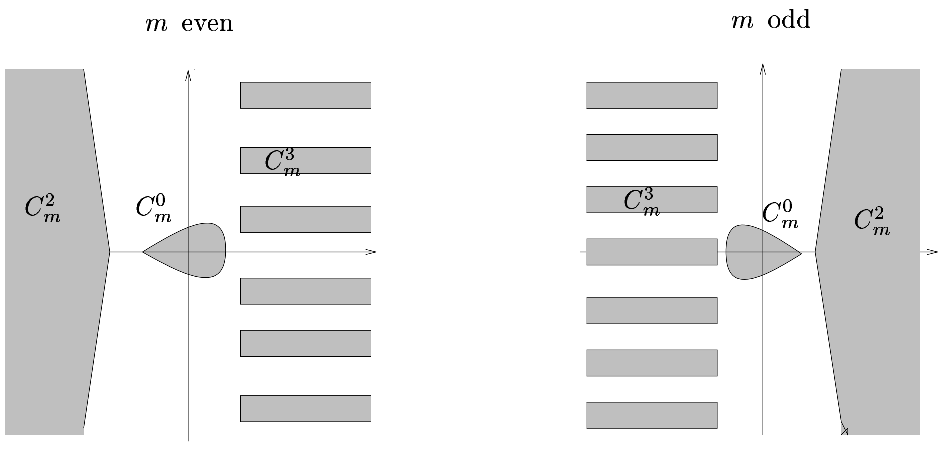


Figure 2: Sketch of capture zones contained in $\overset{\circ}{B}_m$. On the left hand side when m is even and on the right hand side when m is odd.

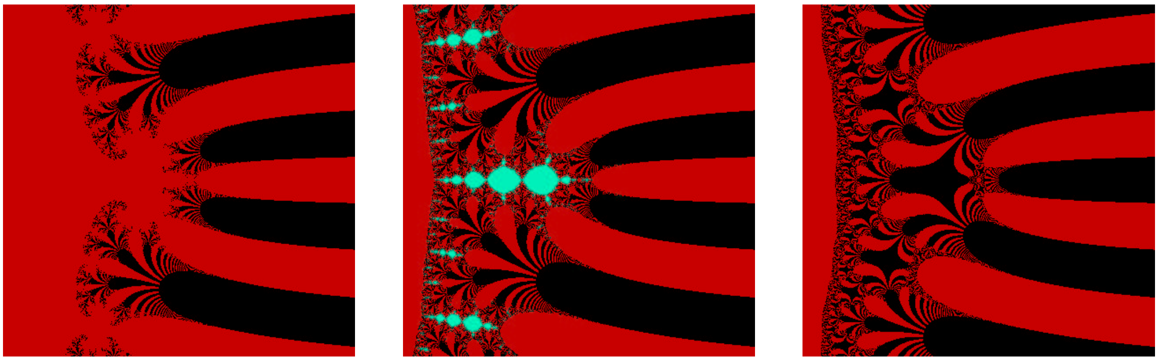
2 The Dynamical Planes

Our goal in this Sec. is to describe the dynamical plane of the families of maps $F_{\lambda,m}$ given by the equation $F_{\lambda,m}(z) = \lambda z^m \exp(z)$, where $m \geq 2$.

The function $F_{\lambda,m}$ is a critically finite entire function, that is, it has a finite number of asymptotic values ($z = 0$), and critical values ($z = 0$ and $z = (-1)^m \lambda (\frac{m}{e})^m$). For this kind of functions there exists a characterization of the Julia set ([Devaney & Tangerman, 1986]), namely as the closure of the set of points whose orbits tend to ∞.

Using the characterization above we can plot an approximation of $\mathcal{J}(F_{\lambda,m})$. Generally, orbits tend to ∞ in specific directions. In our case, if $\lim_{n\to\infty} |F^{\circ n}_{\lambda,m}(z)| = +\infty$, then we have $\lim_{n\to\infty} Re(F^{\circ n}_{\lambda,m}(z)) = +\infty$. Thus, an approximation of the Julia set is given by the set of points whose orbit containts a point with real part greater than, say, 50.

In Figs. 3-4, we display the Julia set of $F_{\lambda,m}$ for differents values of λ and m. The basin of attraction of $z = 0$ is shown in red, while the components of the Fatou set different from A(0) are shown in blue. Points in the Julia set are shown in black.



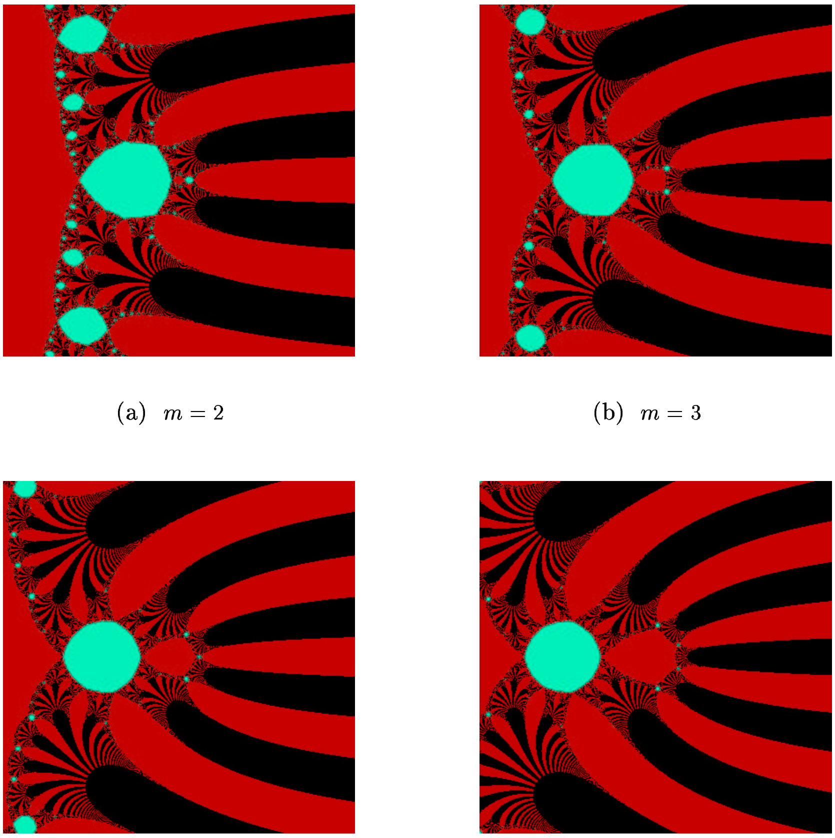
(a) $\lambda = -2.1$ (b) $\lambda = -8$ (c) $\lambda = 6.9$

Figure 3: The Julia set for $F_{\lambda,2}$. Range $(-10, 10) \times (-10, 10)$.

In Fig. 3 we show the dynamical plane of function $F_{\lambda,2} = \lambda z^2 \exp(z)$, for three different values of λ. Apparently, the basin of 0 contains an infinite number of horizontal strips, that extend to $+\infty$ as their real parts tend to $+\infty$. Between these strips we find the well known structures, named Cantor Bouquets which are invariant sets of curves governed by some symbolic dynamics. The existence of this kind of structures in the Julia set are typical for critically finite entire transcendental functions ([Devaney & Tangerman, 1986]).

As we change the parameter λ we observe that the relative position of these bands also changes, but not their width. Also, we can see the existence of an unbounded region that extends to $-\infty$ contained in $A(0)$.

In the next figure (Fig. 4) we show a mosaic of different dynamical planes for some values of m, specifically for $m = 2$, 3, 4, and 5. We choose λ such that all these dynamical planes exhibit the same dynamical behaviour, or more precisely, so that the critical point $z = -m$, is a superattracting fixed point. A simple computation gives $\lambda = (-1)^{m-1} m(\frac{e}{m})^m$. In these dynamical planes we see similar structures as in Fig. 3, even though the values of m are different.



(a) $m = 2$

(b) $m = 3$

(c) $m = 4$

(d) $m = 5$

Figure 4: The dynamical plane of $F_{\lambda,m}$ for different values of m. In every case $\lambda = (-1)^{m-1} m(\frac{e}{m})^m$. Range $(-10, 10) \times (-10, 10)$.

We start with the following general result regarding $A(0)$.

Lemma 2.1. *$A(0)$ has either one or infinitely many connected components. Moreover, connected components different from $A^*(0)$ are unbounded.*

Proof. Using Sullivan's theorem ([Sullivan, 1985]) we have that $A^*(0)$ is the unique fixed connected component of $A(0)$. For all other connected components of $A(0)$ there exists a number $i > 0$ such that $F^i_{\lambda,m}(U) = A^*(0)$, and i is the smallest number with this property. Suppose that there exist a finite number of connected components, and let $U_0 = A^*(0), U_1, U_2, \ldots, U_N$ the connected components of $A(0)$. We may choose the index i in the natural way so that $F^i_{\lambda,m}(U_i) = A^*(0)$. Let $z \in U_N$ such that is not exceptional; then, points in $F^{-1}_{\lambda,m}(z)$ belong to $A(0)$, but not to $U_0 \cup U_1 \cup \ldots \cup U_N$, wich is a contradiction.

Now suppose that U is a connected component of $A(0)$ different from $A^*(0)$, and let $i > 0$ be the smallest number such that $F^i_{\lambda,m}(U) = A^*(0)$. Let $z \in U$, and denote by γ a simple path in $A^*(0)$ that joins $F^i_{\lambda,m}(z)$ and 0. The preimage of γ in U must include a path γ_1 that joins z and ∞, since 0 is an asymptotical value with no other finite preimage than itself. Thus we conclude that U is unbounded. □

2.1 Proof of Theorem A

In this Sec. we describe the basin of attraction of the superattracting fixed point $z = 0$. Since $z = 0$ is a superattracting fixed point, there exists $\epsilon_0 > 0$ such that the open disk $D_{\epsilon_0} = \{z \in \mathbb{C};\ |z| < \epsilon_0\}$ is contained in the immediate basin of attraction of $z = 0$. First, we give an estimate of the size of the immediate basin of attraction, $A^*(0)$ (Proposition 2.3), which will prove the first part of theorem A. Secondly, we find the first preimage of D_{ϵ_0} (Proposition 2.5 and Proposition 2.6), proving the second part of theorem A. Finally, we find the second preimage of D_{ϵ_0} (Proposition 2.8), and prove the third part of theorem A.

Before proving Proposition 2.3 we first look at some properties of the real funtion $h(x) = x^m e^x$ where $m \geq 1$. In Fig. 5 we show the graph of this function. It has a relative extremum at $x = -m$, it is a monotone function on $(-\infty, -m)$ and it satisfies that $|h(x)| \leq |h(-m)|$ for all $x \leq 0$. Also, it is an increasing function in $(0, +\infty)$.

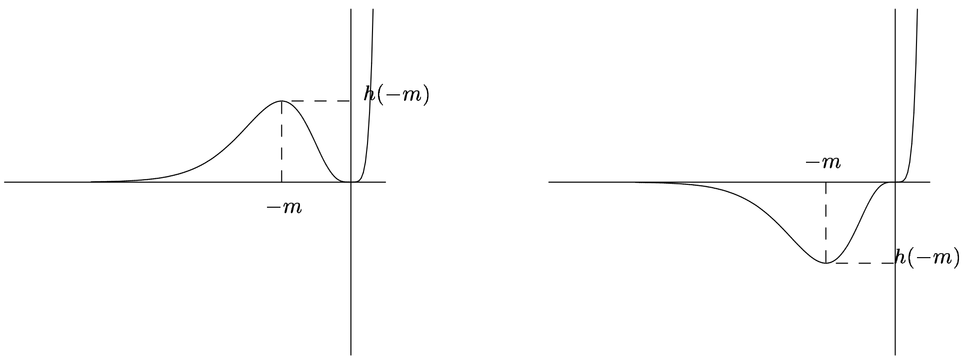


Figure 5: Graph of $h(x) = x^m \exp(x)$. The left hand side corresponds to m even and the right hand side to m odd.

Using these properties it is easy to prove the following auxiliary result.

Lemma 2.2. *Given $r \in (0, |h(-m)|] = (0, (\frac{m}{e})^m]$, the equation $|h(x)| = r$ has a unique solution in $(-\infty, -m]$. Moreover, given $s > 0$, the equation $h(x) = s$ has a unique solution in $(0, +\infty)$.*

We now turn to estimate the function $\epsilon_0(|\lambda|, m)$, and find its dependency on $\lambda \in \mathbb{C} - \{0\}$ and $m \geq 2$.

Proposition 2.3. *If we define ϵ_0 as the unique positive solution of the real equation*

$x^{m-1}e^x = \frac{1}{|\lambda|}$, then $D_{\epsilon_0} = \{z \in \mathbb{C};\ |z| < \epsilon_0\} \subset A^*(0)$. Moreover, if $\lambda \in \mathbb{R}^+$ we have that ϵ_0 lies in $\partial A^*(0)$.

Proof. In order to prove that D_{ϵ_0} is contained in $A^*(0)$, we use Schwartz's lemma. That is, it suffices to prove that if $|z| \leq \epsilon_0$ then $|F_{\lambda,m}(z)| \leq \epsilon_0$.

Suppose $|z| \leq \epsilon_0$, we have

$$|F_{\lambda,m}(z)| = |\lambda z^m e^z| = |\lambda| |z|^m e^{Re(z)} \leq |z| |\lambda| |z|^{m-1} e^{|z|}.$$

Since $|z| < \epsilon_0$ it follows that $|z|^{m-1}e^{|z|} < \epsilon_0^{m-1}e^{\epsilon_0}$, and using that $\epsilon_0^{m-1}e^{\epsilon_0} = 1/|\lambda|$, we conclude

$$|F_{\lambda,m}(z)| \leq |z| |\lambda||z|^{m-1}e^{|z|} \leq |z| \leq \epsilon_0.$$

If $\lambda \in \mathbb{R}^+$, we have that $\lambda\epsilon_0^m \exp(\epsilon_0) = \epsilon_0$, i.e., ϵ_0 is a fixed point. The multiplier of this fixed point is $\epsilon_0 + m > 1$, and hence ϵ_0 lies in the Julia set. By definition we have that $D_{\epsilon_0} \subset A^*(0)$, then ϵ_0 lies in the boundary of $A^*(0)$.

□

In the following auxiliary result we find a lower bound for ϵ_0, which will be used in the next Sec..

Lemma 2.4. *The value of ϵ_0 is always larger or equal than* $\min\{1, (\frac{1}{|\lambda|e})^{\frac{1}{m-1}}\}$

Proof. Suppose $|\lambda| \geq 1/e$. This condition is equivalent to $\frac{1}{|\lambda|e} \leq 1$, hence we must prove that $\epsilon_0 \geq (\frac{1}{|\lambda|e})^{1/(m-1)}$. Using that $x^{m-1}e^x$ is an increasing function on $(0, +\infty)$, this condition is equivalent to

$$\epsilon_0^{m-1}e^{\epsilon_0} \geq \frac{1}{|\lambda|e} \quad e^{(\frac{1}{|\lambda|e})^{\frac{1}{m-1}}}.$$

By definition we have that $\epsilon_0^{m-1}e^{\epsilon_0} = \frac{1}{|\lambda|}$, then

$$\frac{1}{|\lambda|} \geq \frac{1}{|\lambda|e} e^{(\frac{1}{|\lambda|e})^{\frac{1}{m-1}}}$$

or equivalently

$$e \geq e^{(\frac{1}{|\lambda|e})^{\frac{1}{m-1}}}$$

and this follows if $|\lambda| \geq 1/e$.

If $|\lambda| \leq 1/e$, we must prove that $\epsilon_0 \geq 1$. Using the same argument, i.e., that $x^{m-1}e^x$ is an increasing function, it follows that this condition is equivalent to

$$\epsilon_0^{m-1}e^{\epsilon_0} \geq e$$

and this follows if $|\lambda| \leq 1/e$. □

Next we want to find an open set $H_{|\lambda|,m} \subset \mathbb{C}$ such that $F_{\lambda,m}(H_{|\lambda|,m}) \subset D_{\epsilon_0}$ (Fig. 6). To that end, we first obtain a value in $\mathbb{R}^-$, namely $x_0 = x_0(|\lambda|, m)$, such that for all $x \in \mathbb{R}^-$ with $x \leq x_0$ we have that $F_{\lambda,m}(x) \in D_{\epsilon_0}$ (Proposition 2.5). After finding x_0, we will look for an upper bound $C(x) \geq 0$, such that if $z = x + yi$, with $x < x_0$ and $|y| \leq C(x)$, then $F_{\lambda,m}(z) \in D_{\epsilon_0}$ (Proposition 2.6).

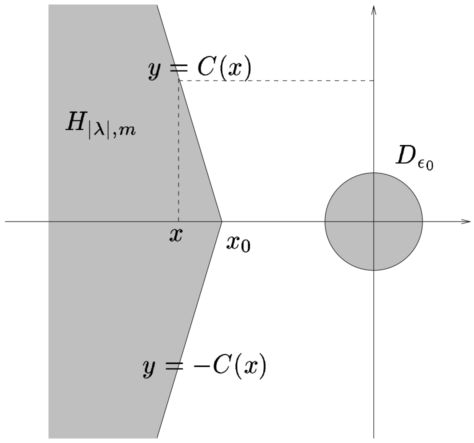


Figure 6: Sketch of $H_{|\lambda|,m}$. satisfying $F_{\lambda,m}\,(H_{|\lambda|,m}) \subset D_{\epsilon_0} \subset A^*(0)$.

Proposition 2.5. *For all $\lambda \in \mathbb{C}$ and $m \geq 2$, there exists $x_0 \in (-\infty, -m]$, such that for all $x \leq x_0$, we have $F_{\lambda,m}(x) \in D_{\epsilon_0}$.*

Proof. We suppose that $z = x + 0i$ and we impose $|F_{\lambda,m}(z)| = \epsilon_0$, that is

$$|F_{\lambda,m}(z)| \;=\; |\lambda||x|^m e^x \;=\; \epsilon_0$$

or equivalently

$$|h(x)| \;=\; |x|^m e^x \;=\; \frac{\epsilon_0}{|\lambda|},$$

where $h(x)$ is the auxiliary function defined above.

If $|h(-m)| \leq \frac{\epsilon_0}{|\lambda|}$, then we take $x_0 = -m$, and for all $x \in (-\infty, x_0)$, we have $|h(x)| \leq |h(-m)| \leq \frac{\epsilon_0}{|\lambda|}$. On the other hand, if $|h(-m)| > \frac{\epsilon_0}{|\lambda|} > 0$, we define x_0 as the unique solution of equation $|h(x)| = \frac{\epsilon_0}{|\lambda|}$ in the interval $(-\infty, -m)$. Since $|h(x)|$ is an increasing function in $(-\infty, -m)$, it follows that for all $x \in (-\infty, x_0)$, then $|h(x)| \leq |h(x_0)| = \frac{\epsilon_0}{|\lambda|}$. ☐

Proposition 2.6. *Let $x_0 = x_0(|\lambda|, m)$ be as in Proposition 2.5. There exists $C(x) \geq 0$ such that the open set (Fig. 6)*

$$H_{|\lambda|,m} \;=\; \left\{ z \;=\; x \;+\; yi \;\middle|\; \begin{array}{ll} x & \in (-\infty, x_0) \\ y & \in (-C(x), C(x)) \end{array} \right\}$$

satisfies $F_{\lambda,m}\,(H_{|\lambda|,m}) \subset D_{\epsilon_0}$.

Proof. We want to calculate an upper bound $C(x) \geq 0$, such that if $z = x + yi$, with $x \in (-\infty, x_0)$ and $|y| \leq C(x)$, then $F_{\lambda,m}(z) \in D_{\epsilon_0}$. If we require that $F_{\lambda,m}(z) \in D_{\epsilon_0}$, we obtain the definition of $C(x)$. Let $z = x \pm yi$, with $x \in (-\infty, x_0)$. Then

$$|F_{\lambda,m}(z)| \;=\; |\lambda||z|^m \exp(Re(z)) \;=\; |\lambda| \left[\sqrt{x^2 + y^2}\right]^m \exp(x) \;=\; |\lambda|(x^2 \,+\, y^2)^{m/2} \exp(x).$$

Using the expression above, and requiring $|F_{\lambda,m}(z)| \leq \epsilon_0$, we obtain

$$|y| \leq +\sqrt{\left[\frac{\epsilon_0}{|\lambda|}\right]^{2/m} \exp(-x\frac{2}{m}) - x^2}.$$

Thus, the right hand side of this inequality gives an analytic expression for the function $C(x)$. □

The next lemma gives a simpler condition to assure that a point $z \in \mathbb{C}$ lies in $H_{|\lambda|,m}$. See Fig. 7.

Lemma 2.7. *The point $z = x + yi$ lies in $H_{|\lambda|,m}$ if there exists $k \geq 1$ and $A \geq 0$, such that $|y| \leq A\,|x|^k$ and $|x|$ is large enough.*

Proof. We will prove that $A|x|^k \leq C(x)$ as $x \to -\infty$. Using the definition of $C(x)$, this is equivalent to showing

$$A|x|^k < \sqrt{\left[\frac{\epsilon_0}{|\lambda|}\right]^{m/2} \exp(-x\frac{m}{2}) - x^2},$$

or

$$\left[A^2|x|^{2k} + x^2\right] \exp(x\frac{m}{2}) < \left[\frac{\epsilon_0}{|\lambda|}\right]^{m/2}.$$

The left hand side of this inequality is a function that tends to zero as x tends to $-\infty$, whereas the right hand side is positive.

□

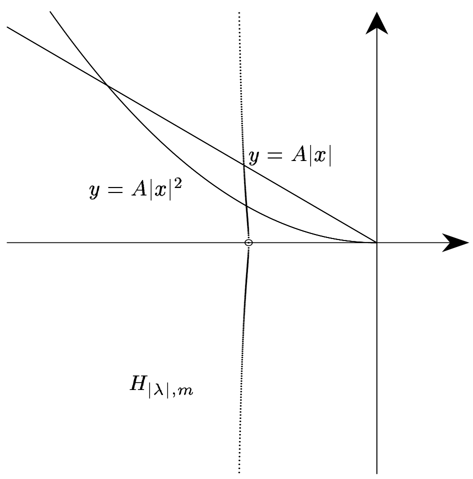


Figure 7: Relation between $H_{|\lambda|,m}$ and $y = A|x|^k$ for $k = 1, 2$

We proceed now to the third iterate, by proving the existence of some strips in dynamical plane, such that the image of these open sets under $F_{\lambda,m}$ is contained in $H_{|\lambda|,m}$ (see Fig. 1).

Before calculating the preimage of the set $H_{|\lambda|,m}$, we first find the preimage of the negative real axis under the function $F_{\lambda,m}$. Hereafter, we denote by $Arg(.) \in (-\pi, \pi]$ the argument. Using the definition of $F_{\lambda,m}$ it is easy to see that

$$Arg(F_{\lambda,m}(z)) = Arg(\lambda) + mArg(z) + Im(z) \qquad (\bmod 2\pi).$$

Finding the preimages of $\mathbb{R}^-$ is equivalent to solving

$$Arg(F_{\lambda,m}(z)) = \pi.$$

We denote $r = |z|$ and $\alpha = Arg(z)$. Then the equation above is equivalent to

$$Arg(\lambda) + m\alpha + rsin(\alpha) = (2k+1)\pi \qquad k \in \mathbb{Z}.$$

Hence, we obtain

$$r = \rho(\alpha) = \frac{(2k+1)\pi - m\alpha - Arg(\lambda)}{sin(\alpha)} \qquad \alpha \in (-\pi, \pi).$$

We denote each of these curves by $\sigma_k = \sigma_k(\lambda, m)$, where the possible values of the argument depend on k. More precisely,

if $m = 2j$ for $j \in \mathbb{Z}$

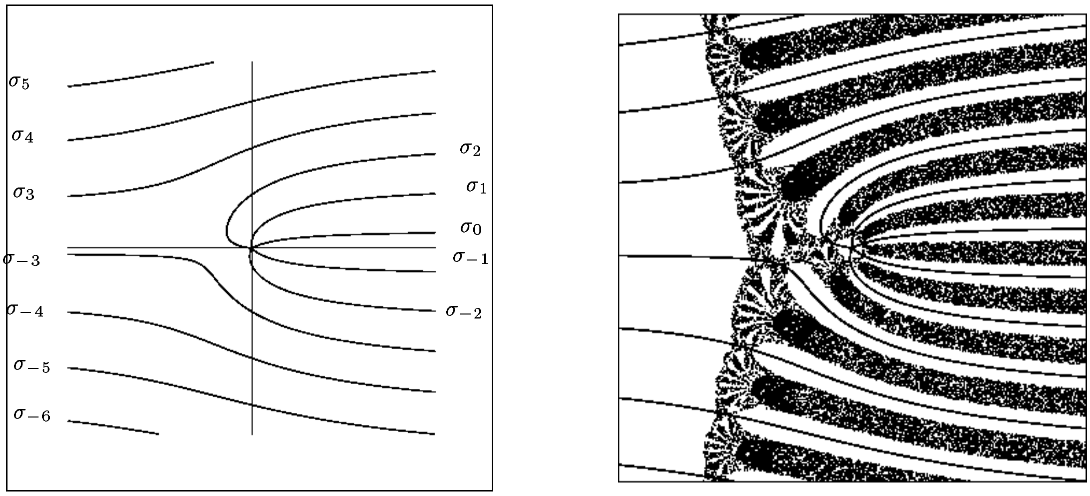
$$\sigma_k = \rho(\alpha)e^{i\alpha} \text{ where } \begin{cases} 0 < \alpha < \pi & \text{if } \quad k \geq j \\ 0 < \alpha < \frac{(2k+1)\pi - Arg(\lambda)}{m} & \text{if } \quad 0 \leq k \leq j-1 \\ \frac{(2k+1)\pi - Arg(\lambda)}{m} < \alpha < 0 & \text{if } \quad -j \leq k \leq 0 \\ -\pi < \alpha < 0 & \text{if } \quad k \leq -(j+1) \end{cases};$$

if $m = 2j+1$ for $j \in \mathbb{Z}$

$$\sigma_k = \rho(\alpha)e^{i\alpha} \text{ where } \begin{cases} 0 < \alpha < \pi & \text{if } \quad k \geq j+1 \\ 0 < \alpha < \frac{(2k+1)\pi - Arg(\lambda)}{m} & \text{if } \quad 0 \leq k \leq j \\ \frac{(2k+1)\pi - Arg(\lambda)}{m} < \alpha < 0 & \text{if } \quad -j \leq k \leq 0 \\ -\pi < \alpha < 0 & \text{if } \quad k \leq -(j+1) \end{cases}$$

In Fig. 8 we show some of these curves for $m = 5$. As their real parts tend to $+\infty$, the σ_k's are asymptotic to the lines $Im(z) = (2k+1)\pi - Arg(\lambda)$. There are m of these curves, that start at the origin and tend to $+\infty$. The others are asymptotic to the lines $Im(z) = (2k+1)\pi - Arg(\lambda)$ when $k < 0$, or $Im(z) = 2k\pi - Arg(\lambda)$ when $k > 0$.

Now we may find preimages of the open set $H_{|\lambda|,m}$. First we find preimages of the interval $(-\infty, x_0)$. There exists one preimage of this interval on each curve σ_k. Moreover, the preimage of a real number tending to $-\infty$, is a complex number on σ_k whose real part tends to $+\infty$. Hence, the preimages of the set $H_{|\lambda|,m}$ contain some strips, namely $S^k_{\lambda,m}$, around σ_k. (See Fig. 8).



(a) Graph of σ_k for $\lambda = 0.45 + 0.35i$ and $m = 5$.

(b) The julia set of $F_{0.45+0.35i,5}$

Figure 8: Strips in the dynamical plane.

Let $z \in \sigma_k$. If we evaluate $F_{\lambda,m}(z)$, we have

$$F_{\lambda,m}(z) = |\lambda||z|^m e^{Re(z)}\, e^{(Arg(\lambda)+mArg(z)+Im(z))i} = -|\lambda||z|^m e^{Re(z)}$$

since each σ_k is a preimage of the negative real axis. The expression above shows that if we keep $Re(z)$ constant, and we increase the index k of the curve σ_k, we obtain values tending to $-\infty$, since $|z|$ increases. Hence, if we denote by $q_0(k)$ the preimage of x_0 on σ_k, its real part decreases as $|k|$ increases (at least after a certain point). This fact explains the apparent arangement of the strips in dynamical plane (Fig. 8).

We now prove that these strips have an asymptotic width equal to π. We fix a value $k \in \mathbb{Z}$, and we recall that σ_k tends asymptotically to the line $Im(z) = (2k+1)\pi - Arg(\lambda)$, as its real part tends to $+\infty$.

Proposition 2.8. *Given any $y \in ((2k+1)\pi - Arg(\lambda) - \frac{\pi}{2}, (2k+1)\pi - Arg(\lambda) + \frac{\pi}{2})$, there exists a real number x_* such that for all $x \geq x_*$, $F_{\lambda,m}(x+yi) \in H_{|\lambda|,m}$.*

Proof. Let $z = x + iy$, where $y \in ((2k+1)\pi - Arg(\lambda) - \frac{\pi}{2}, (2k+1)\pi - Arg(\lambda) + \frac{\pi}{2})$. If we write the parameter λ in polar coordinates, $\lambda = se^{i\beta}$, then

$$\begin{aligned} F_{\lambda,m}(z) = \;& \lambda\, z^m \exp(z) = se^{i\beta}\,(x+yi)^m e^x e^{iy} = s(x+yi)^m\, e^x\, e^{i(y+\beta)} = \\ & \{P(x) + Q(x)i\}\, e^x \{\cos(y+\beta) + \sin(y+\beta)i\} \end{aligned}$$

where

$$\begin{cases} P(x) = & Re(s(x+yi)^m) = sx^m + \mathcal{O}(x^{m-2}) \\ Q(x) = & Im(s(x+yi)^m) = msyx^{m-1} + \mathcal{O}(x^{m-3}). \end{cases}$$

Using the expressions above in $F_{\lambda,m}(z)$, we obtain

$$\begin{aligned} F_{\lambda,m}(z) &= \{P(x)\cos(y+\beta) - Q(x)\sin(y+\beta)\}e^x + \\ &\quad \{P(x)\sin(y+\beta) + Q(x)\cos(y+\beta)\}e^x i \\ &= \{P(x)\cos(y\,+\,\beta)\,+\mathcal{O}(x^{m-1})\}e^x + \\ &\quad \{P(x)\sin(y\,+\,\beta)\,+\mathcal{O}(x^{m-1})\}\,e^x\,i \\ &= \{sx^m\cos(y\,+\,\beta)\,+\,\mathcal{O}(x^{m-1})\}e^x + \\ &\quad \{sx^m\sin(y\,+\,\beta)\,+\,\mathcal{O}(x^{m-1})\}e^x\,i. \end{aligned} \tag{1}$$

We recall that $y \in (\frac{\pi}{2}+2k\pi-\beta\,,\,\frac{3\pi}{2}+2k\pi-\beta)$, or equivalently $y+\beta \in (\frac{\pi}{2}+2k\pi, \frac{3\pi}{2}+2k\pi)$ which implies that $-1 \leq cos(y\,+\,\beta) < 0$

Since $Re(F_{\lambda,m}(z)) = \{sx^m cos(y\,+\,\beta)\,+\,\mathcal{O}(x^{m-1})\}e^x$, it follows that

$$\lim_{x\to+\infty} Re(F_{\lambda,m}(z)) = \lim_{x\to+\infty}\{cos(y\,+\,\beta)x^m s\,+\,\mathcal{O}(x^{m-1})\}e^x = -\infty.$$

Hence, there exists x_* large enough, such that $Re(F_{\lambda,m}(z)) < x_0$ for $x \geq x_*$.

Finally, we use lemma 2.7 to conclude the proof. From equation (1) we have

$$\lim_{x\to+\infty}\frac{Re(F_{\lambda,m}(z))}{Im(F_{\lambda,m}(z))} = \lim_{x\to+\infty}\frac{sx^m\cos(y\,+\beta)\,+\,\mathcal{O}(x^{m-1})}{sx^m\sin(y\,+\beta)\,+\,\mathcal{O}(x^{m-1})} = \frac{\cos(y\,+\beta)}{\sin(y\,+\beta)}.$$

Hence, if $\sin(y\,+\,\beta) \neq 0$, it follows that $Im(F_{\lambda,m}(z)) \approx \frac{\sin(y+\beta)}{\cos(y+\beta)}Re(F_{\lambda,m}(z))$, and we can apply lemma 2.7 (with $A = |\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}|$ and $k=1$). Using this lemma, we conclude that $F_{\lambda,m}(z)$ lies in $H_{|\lambda|,m}$, if x is large enough.

On the other hand, if $\sin(y\,+\,\beta)=0$, then $cos(y\,+\,\beta)=-1$, and we obtain

$$\begin{aligned} F_{\lambda,m}(z) &= \{-P(x)-Q(x)i\}e^x \\ &= \{-sx^m+\mathcal{O}(x^{m-1})\}e^x + \{msyx^{m-1}+\mathcal{O}(x^{m-2})\}ie^x. \end{aligned}$$

Hence

$$\lim_{x\to+\infty}\frac{Re(F_{\lambda,m}(z))}{Im(F_{\lambda,m}(z))} = \lim_{x\to+\infty}\frac{-sx^m+\mathcal{O}(x^{m-1})}{-msyx^{m-1}+\mathcal{O}(x^{m-2})} = +\infty.$$

If x is large enough, there exists $K>0$ such that

$$\left|\frac{Re(F_{\lambda,m}(z))}{Im(F_{\lambda,m}(z))}\right| > K$$

that is, $|Im(F_{\lambda,m}(z))| \leq \frac{1}{K}|Re(F_{\lambda,m}(z))|$. Hence, using lemma 2.7 ($A=\frac{1}{K}$, $k=1$), we also obtain that $F_{\lambda,m}(z)$ lies in $H_{|\lambda|,m}$.

□

3 The Parameter Planes

The orbit of the free critical point $z = -m$, determines in large measure the dynamics of $F_{\lambda,m}$. Indeed, the functions $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ are entire maps with a finite number of critical and asymptotic values. These kind of functions do not have wandering domains nor Baker domains. By the Sullivan classification, we know that if the orbit of $z = -m$ tends to ∞ then the Fatou set must coincide with the basin of 0, i.e., $\mathcal{F}(F_{\lambda,m}) = A(0)$, since no other Fatou components can exist besides those that belong to $A(0)$. The set B_m is defined as before as

$$B_m = \{\lambda \in \mathbb{C} \,|\, F^{\circ n}_{\lambda,m}(-m) \nrightarrow \infty\}.$$

In each of these sets, we may also distinguish between two different behaviours: those parameters values for which $-m \in A(0)$ and those for which this does not occur. Let $\overset{\circ}{B}_m$ denote the interior of B_m.

Definition. Let U be a connected component of $\overset{\circ}{B}_m$. We say that U is a *capture zone* if for all λ in U it is true that $\lim_{n\to+\infty} F^{\circ n}_{\lambda,m}(-m) = 0$, or in other words, $-m \in A(0)$. We then say that the orbit of the critical point is captured by the basin of attraction of the superattracting fixed point $z = 0$.

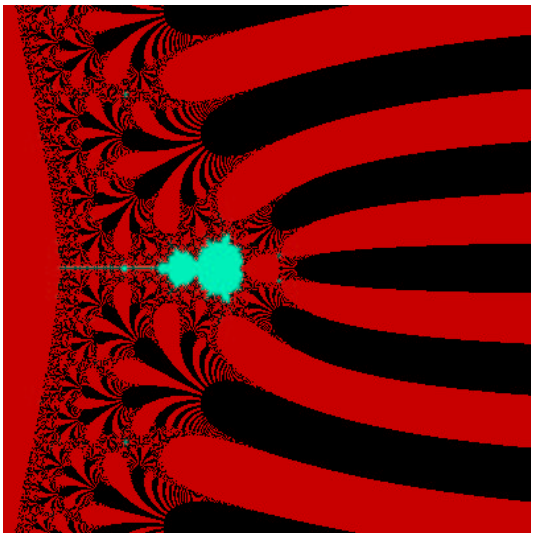
In Figs. 9- 10, we show a numerical approximation of the set B_m for different values of m. The capture zones are shown in red, while other components of B_m are shown in blue. The parameter values for which the orbit of the free critical point tends to ∞ are shown in black. In these sets we can see a countable quantity of horizontal strips. If m is even these strips extend to $+\infty$ as the real part of λ tends to $+\infty$, whereas if m is odd these strips extend to $-\infty$ as the real part of λ tends to $-\infty$. Notsurpringly, the distribution of these capture zones in the parameter plane (Fig. 2) appears to be similar to the distribution of $A(0)$ in the dynamical plane (Fig. 1).

We start with the following simple facts.

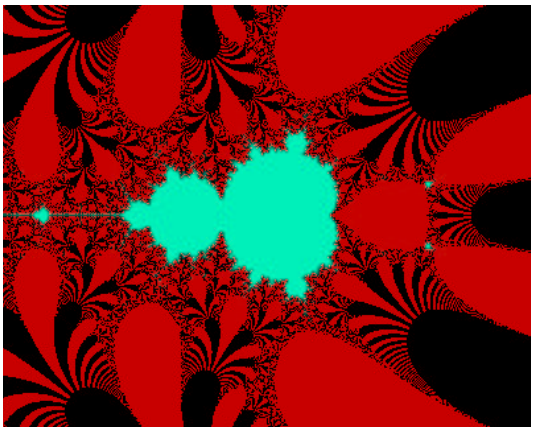
Proposition 3.1. *Let U be a capture zone of B_m and let $\lambda \in U$. Then $\mathcal{F}(F_{\lambda,m}) = A(0)$ and $\mathcal{J}(F_{\lambda,m}) = \partial A(0)$.*

Proof. As in the case of the critical value tending to ∞, since the only free critical point of $F_{\lambda,m}$ lies in the basin of 0, no other components different from those in $A(0)$ can exist in $\mathcal{F}(F_{\lambda,m})$. Let $\mathcal{D}$ be the union of all the components of the Fatou set, then $\mathcal{J}(F_{\lambda,m}) = \partial\mathcal{D}$ ([Carleson & Gamelin, 1993]). In our case, if $\lambda \in U$ and U is a capture zone, then $\mathcal{D} = A(0)$. □

The main objective of this Sec. is to describe the most obvious capture zones contained in B_m, as well as to describe the dynamical plane for parameter values that belong to such components.



(a) Range $(-25, 25) \times (-25, 25)$



(b) Range $(-15, 5) \times (-8, 8)$

Figure 9: Parameter plane for $F_{\lambda,2}$. Color codes are explained in the text.

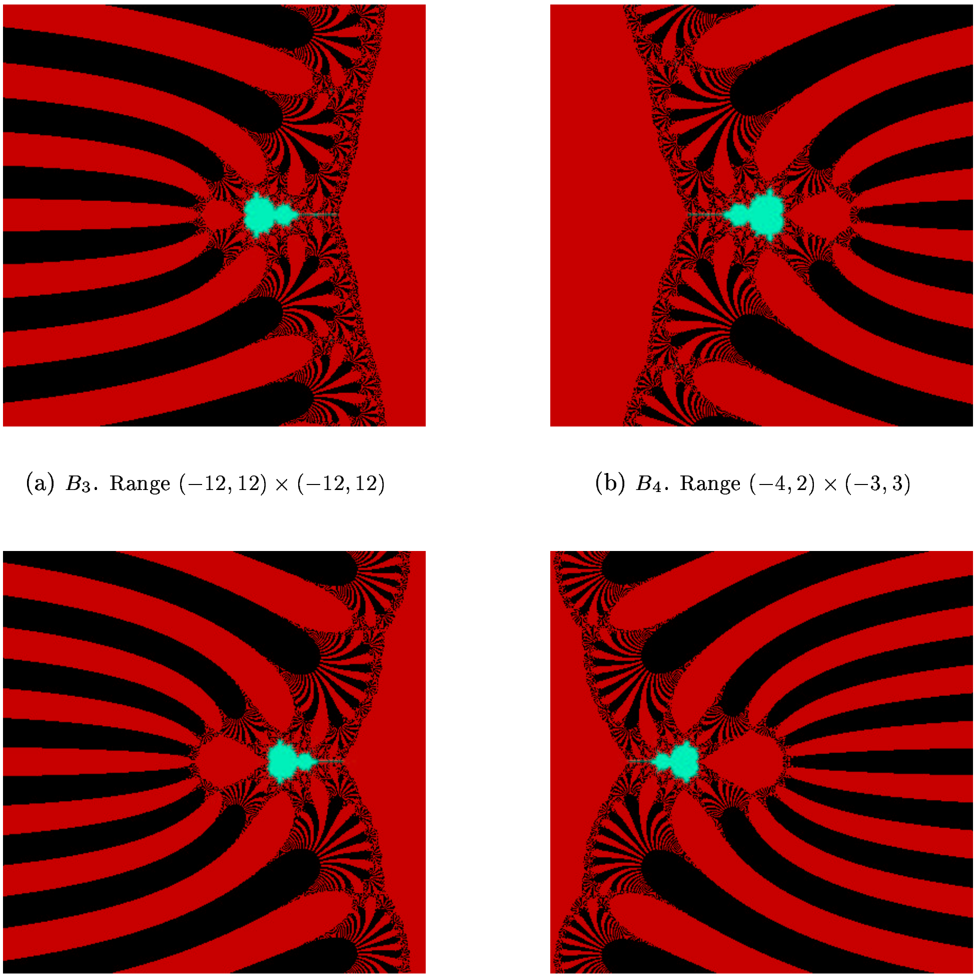
3.1 Proof of Theorem B

In this Sec. we describe the main capture zone C_m^0. We recall that

$$C_m^n = \{\lambda \in \overset{\circ}{B}_m \, | F_{\lambda,m}^n(-m) \in A^*(0) \text{ and } n \text{ is the smallest number with this property}\}$$

We prove each statement of theorem B in a different proposition.

Proposition 3.2. *The critical point $-m$ belongs to $A^*(0)$ if and only if the critical value $F_{\lambda,m}(-m)$ belongs to $A^*(0)$. Hence $C_m^1 = \emptyset$.*



(a) B_3. Range $(-12, 12) \times (-12, 12)$

(b) B_4. Range $(-4, 2) \times (-3, 3)$

(c) B_5. Range $(-0.8, 0.8) \times (-0.8, 0.8)$

(d) B_6. Range $(-0.15, 0.15) \times (-0.15, 0.15)$

Figure 10: Parameter plane for $F_{\lambda,m}$, for differents values of m.

Proof. Supose that $F_{\lambda,m}(-m) \in A^*(0)$. Let γ be a simple path in $A^*(0)$ that joins $F_{\lambda,m}(-m)$ and 0. The set of preimages of γ must include a path γ_1 that joins $-\infty$ with $-m$, and also a path γ_2 that joins $-m$ and 0 (since $-m$ is a critical point and 0 is a fixed point and asymptotic value). Hence $\gamma_1 \cup \gamma_2 \subset A^*(0)$ and so does $-m$. Conversely, if $-m \in A^*(0)$ we have that $F_{\lambda,m}(-m) \in A^*(0)$. □

Proposition 3.3. *The set C_m^0 contains the disk $\{\lambda \in \mathbb{C};\ |\lambda| < \min(\frac{1}{e}, (\frac{e}{m})^m)\}$.*

Proof. We denote D_m the open disk $\{\lambda \in \mathbb{C};\ |\lambda| < \min(\frac{1}{e}, (\frac{e}{m})^m)\}$. Let $\lambda \in D_m$, we will prove that $F_{\lambda,m}(-m)$ lies in D_{ϵ_0} which we know belongs to $A^*(0)$. In order to do so, we use that $\epsilon_0 \geq \min(1, (\frac{1}{|\lambda| e})^{1/(m-1)})$ (lemma 2.4). We choose $\lambda \in D_m$. Then $|\lambda| < \frac{1}{e}$, and hence

$\epsilon_0 \geq 1$. The condition $\lambda \in D_m$ also implies that $|\lambda| < (\frac{e}{m})^m$. Hence

$$|F_{\lambda,m}(-m)| = |\lambda||(-m)^m e^{-m}| = |\lambda|\left(\frac{m}{e}\right)^m < 1 \leq \epsilon_0,$$

and $F_{\lambda,m}(-m)$ lies in $A^*(0)$. □

Proposition 3.4. *The set C_m^0 is bounded. In fact it is contained in the closed disk $\{\lambda \in \mathbb{C};\ |\lambda| \leq (\frac{e}{m-1})^{m-1}\}$.*

Proof. We will prove that $-m \notin A^*(0)$ for all $\lambda \in \mathbb{C}$ such that $|\lambda| > (\frac{e}{m-1})^{m-1}$. Let D the disk centered at 0 of radius $m-1$. If we calculate the image of its boundary, $\{|z| = m-1\}$, we obtain

$$|F_{\lambda,m}(z)| = |\lambda||z|^m e^{Re(z)} \geq |\lambda|(m-1)^m e^{-(m-1)} > m-1$$

where the inequality is obtained using $|\lambda| > (\frac{e}{m-1})^{m-1}$. This shows that $D \subset F_{\lambda,m}(D)$, and hence $A^*(0) \subset D$. Since $-m \notin D$, the proposition follows. □

Proposition 3.5. *If $\lambda \in C_m^0$ then $A(0) = A^*(0)$, i.e., the basin of attraction of $z = 0$ has a unique connected component and hence it is totally invariant. Moreover, the boundary of $A^*(0)$ (which equals the Julia set) is a Cantor bouquet and hence it is disconnected and non-locally connected.*

Proof. Let $\lambda \in C_m^0$. As in proposition 3.2, let γ be a simple path in $A^*(0)$ that joins $F_{\lambda,m}(-m)$ and 0. The preimage of γ must include a path $\tilde{\gamma}$ contained in $A^*(0)$ that joins $-\infty$ with 0 passing through $-m$ ($\tilde{\gamma}$ maps 2-1 to γ). Since $H_{|\lambda|,m}$ intersects $\tilde{\gamma}$ so it follows that $H_{|\lambda|,m} \subset A^*(0)$. All preimages of $\tilde{\gamma}$, are contained in $A^*(0)$ as well, since they all intersect $H_{|\lambda|,m}$. In fact, we have that $A(0) = A^*(0)$ since any preimage of D_{ϵ_0} must contain points of $H_{|\lambda|,m}$. Hence $A(0)$ has a unique connected component. In fact, from [Devaney & Goldberg, 1987],[Baker & Dominguez, 2000], it follows that the Julia set has an uncountable number of connected components and it is not locally connected at any point.

Using [Devaney & Tangerman, 1986] one can show that the Julia set contains a Cantor Bouquet tending to ∞ in the direction of the positive real axis. To see this, it is sufficient to construct a hyperbolic exponential tract on which $F_{\lambda,m}$ has asymptotic direction θ^*. Let B_r an open disk containing $F_{\lambda,m}(-m)$, the preimage of this set is an open set similar to $H_{|\lambda|,m}$. Let D the complement of this set. We have that $F_{\lambda,m}$ maps D onto the exterior of B_r, then D is an exponential tract for $F_{\lambda,m}$. We may choose the negative real axis to define the fundamental domains in D. Since the curves σ_k for $k \in \mathbb{Z}$ are mapped by $F_{\lambda,m}$ onto this axis, it follows that D has asymptotic direction $\theta^* = 0$. Furthermore, since $F_{\lambda,m}(z) = \lambda z^m \exp(z)$, one may check readily that D is a hyperbolic exponential tract. □

Proposition 3.6. *If $\lambda \notin C_m^0$ then $A(0)$ has infinitely many components. Moreover, if $|\lambda| > (\frac{e}{m-1})^{m-1}$, the boundary of $A^*(0)$ is a quasi-circle.*

Proof. Using lemma 2.1 we have that $A(0)$ has either one or infinitely many connected components. If we suppose that $A(0)$ has only one connected component, then $A(0)$ is a completely invariant component of the Fatou set. We have that all the critical values of $F_{\lambda,m}$ are in $A(0)$ (see [Baker, 1984]), and hence we conclude that $-m$ belongs to $A(0)$. However, is imposible if $\lambda \notin C_m^0$.

Let $\lambda \notin C_m^0$ such that $|\lambda| > (\frac{e}{m-1})^{m-1}$. The main idea of this proof is the same as that used by Bergweiler in ([Bergweiler, 1995]). We will show that $F_{\lambda,m}$ is a polynomial-like of degree m in a neighbourhood of 0, which includes the whole immediate basin. From the proof of proposition 3.4, the disc D centered at 0 of radius $m-1$ satisfies $\overline{D} \subset F_{\lambda,m}(D)$, and hence $A^*(0) \subset D$.

Let W be the component of $F_{\lambda,m}^{-1}(D)$ that contains the origin. It is clear that $\overline{W} \subset D$ and $\overline{A^*(0)} \subset W$. Moreover, $F_{\lambda,m}$ is a proper function of degree m from W onto D, (see Fig. 11). In the terminology of polynomial-like mappings, developed by Douady and Hubbard ([Douady & Hubbard, 1985]), the triple $(F_{\lambda,m}; W, D)$ is a polynomial-like mapping of degree m. By the Straightening theorem, there exists a quasiconformal mapping, ϕ, that conjugates $F_{\lambda,m}$ to a polynomial P of degree m, on the set W. That is $(\phi^{-1} \circ F_{\lambda,m} \circ \phi)(z) = P(z)$ for all $z \in W$. Since $z = 0$ is superattracting for $F_{\lambda,m}$ and ϕ is a conjugacy, we have that $z = 0$ is superattracting for P. Hence, after perhaps a holomorphic change of variables, we may assume that $P(z) = z^m$.

Hence, $\partial A^*(0) = \phi(\mathbb{T})$, and the theorem follows. □

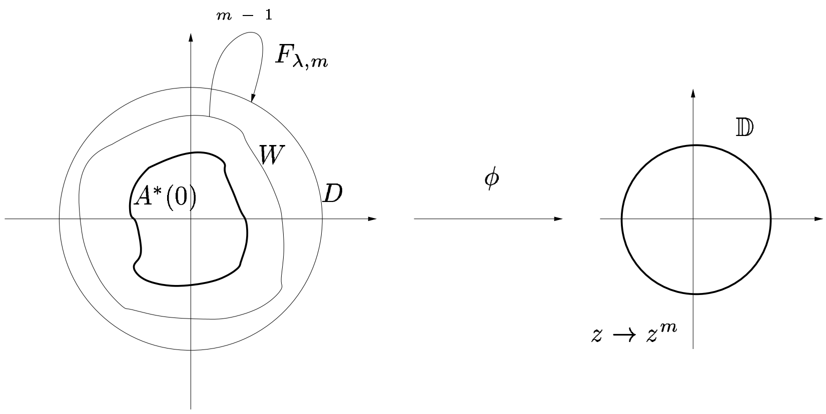


Figure 11: $F_{\lambda,m}$ is a polynomial-like mapping of degree m near the origin.

Remark 3.7. The reason to ask for $|\lambda| > (\frac{e}{m-1})^{m-1}$ as a condition is as follows. We want to find a value $K > 0$ such that if $|z| = K$ then $|F_{\lambda,m}(z)| > K$. This condition is equivalent to

$$|F_{\lambda,m}(z)| \geq |\lambda||z|^m e^{-|z|} = |\lambda|(K)^m e^{-K} > K$$

or equivalently

$$|\lambda| > K^{1-m}e^K.$$

We want to use this argument for the largest possible region of values of λ. Hence, we choose $K > 0$, such that $K^{1-m}e^K$ is minimum. This minimum value is reached exactly at $K = m-1$.

3.2 Proof of Theorem C

In this Sec. we describe the capture zones C_m^2 (Proposition 3.8 and Proposition 3.9) and C_m^3 (Proposition 3.10).

We will construct the open set C_m^2 in two steps. In the first (Proposition 3.8) we obtain an unbounded interval of real numbers I, such that for all $\lambda \in I$, $F_{\lambda,m}^2(-m)$ lies in $A^*(0)$. In the second (Proposition 3.9), we will extend this construction to λ in $\mathbb{C}$. We denote $\lambda = \lambda_1 + \lambda_2 i$, where λ_1 and λ_2, are the real and imaginary parts of λ.

Proposition 3.8. *There exists an unbounded interval, I, such that for all real numbers $\lambda_1 \in I$, we have that $F_{\lambda_1,m}^2(-m) \in D_{\epsilon_0} \subset A^*(0)$.*

Proof. Hereafter we denote $r_m = (\frac{e}{m})^m$. We take $\lambda_1 \in \mathbb{R}$, and we impose that $F_{\lambda_1,m}(-m) \in H_{|\lambda_1|,m}$.

If we calculate $F_{\lambda_1,m}(-m)$ we obtain

$$F_{\lambda_1,m}(-m) \;=\; \lambda_1(-m)^m \exp(-m) \;=\; (-1)^m \frac{\lambda_1}{r_m}.$$

This real value lies in $H_{|\lambda_1|,m}$, if and only if

$$|h(F_{\lambda_1,m}(-m))| \;<\; \frac{\epsilon_0}{|\lambda_1|}.$$

Recall that $h(x) = x^m \exp(x)$, and ϵ_0 only depends on $|\lambda_1|$ and m. This condition is equivalent to

$$\frac{|\lambda_1|^{m+1}}{r_m^m} \exp((-1)^m \frac{\lambda_1}{r_m}) \;<\; \epsilon_0.$$

Using lemma 2.4 we have that $\epsilon_0 \geq \min\{1, (\frac{1}{|\lambda_1|e})^{1/(m-1)}\}$. If we use this explicit lower bound we may impose

$$\frac{|\lambda_1|^{m+1}}{r_m^m} \exp((-1)^m \frac{\lambda_1}{r_m}) \;<\; \min\{1, (\frac{1}{|\lambda_1|e})^{1/(m-1)}\}$$

We define the auxiliary function

$$l(\lambda_1) \;=\; \begin{cases} |\lambda_1|^{m+1+\frac{1}{m-1}} \exp((-1)^m \frac{\lambda_1}{r_m}) \exp(1/(m-1)) & \text{if } |\lambda_1| > 1/e \\ |\lambda_1|^{m+1} \exp((-1)^m \frac{\lambda_1}{r_m}) & \text{if } |\lambda_1| \leq 1/e \end{cases}$$

and the above inequality is transformed into $l(\lambda_1) < r_m^m$.

Using some elementary properties of function $l(\lambda_1)$, one can see that

$$\begin{cases} lim_{\lambda_1 \to -\infty} l(\lambda_1) &= 0 \quad \text{if m is even} \\ lim_{\lambda_1 \to +\infty} l(\lambda_1) &= 0 \quad \text{if m is odd.} \end{cases}$$

Since $l(\lambda_1)$ is continous and positive and it has a finite number of relative maxima and minima, we can find an unbounded interval of real numbers such that $l(\lambda_1) < r_m^m$.

If m is even, we define $I = (-\infty, -D_0)$, where $-D_0 = -D_0(m) \leq 0$, is the smallest of the values such that $l(\lambda_1) = r_m^m$. If m is odd, we choose $I = (D_0, +\infty)$, with $D_0 = D_0(m) \geq 0$, such that D_0 is the largest of the values for which $l(\lambda_1) = r_m^m$. ☐

Proposition 3.9. *Let $D_0(m) > 0$ be as in Proposition 3.8.There exists a function $\alpha = \alpha(|\lambda|, m) \in (\pi/2, \pi)$, such that*

- *for m even, the set C_m^2 contains the open set* $\left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{ll} |\lambda| & > D_0 \\ |Arg(\lambda)| & > \alpha \end{array} \right\}$
- *for m odd, the set C_m^2 contains the open set* $\left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{ll} |\lambda| & > D_0 \\ |Arg(\lambda)| & < \pi - \alpha \end{array} \right\}$

Proof. Given $\lambda_1^* \in I$, we denote by S the circle of radius $|\lambda_1^*|$ and centered at the origin. We will find all complex numbers λ in S, such that $F_{\lambda,m}(-m) \in H_{|\lambda|,m}$.

All complex numbers $\lambda \in S$ have the same $H_{|\lambda|,m}$ set, since this set only depends on $|\lambda|$ and m. We denote it by H_S.

When λ belongs to S, the image of the critical point, $F_{\lambda,m}(-m) = (-1)^m \frac{\lambda}{r_m}$, belongs to another circle, namely $\widetilde{S}$, and its argument verifies

$$Arg(F_{\lambda,m}(-m)) = \begin{cases} Arg(\lambda) & \text{if } m \text{ is even} \\ Arg(\lambda) + \pi & \text{if } m \text{ is odd} \end{cases}$$

This circle is concentric with respect to S and its radius is equal to $\frac{|\lambda_1^*|}{r_m}$ (see Fig. 12), which is larger than the radius of S if $m = 2$, and smaller if $m \geq 3$.

We take λ_1^* on I. Using the construction above of the interval I, we obtain that $F_{\lambda_1^*,m}(-m) \in H_{|\lambda_1^*|,m} = H_S$. This fact assures a non-empty intersection of ∂H_S with $\widetilde{S}$.

Using the analytic definition of H_S (proof of Proposition 2.6), we can calculate $\partial H_S \cap \widetilde{S}$. We find this intersection by solving:

$$\begin{cases} \sqrt{\lambda_1^2 + \lambda_2^2} = \frac{|\lambda_1^*|}{r_m} & \lambda_1 + i\lambda_2 \in \widetilde{S} \\ \lambda_2 = +C(\lambda_1) = \sqrt{\left[\frac{\epsilon_0}{|\lambda_1^*|}\right]^{2/m} \exp(-\lambda_1 \frac{2}{m}) - \lambda_1^2} & \lambda_1 + i\lambda_2 \in \partial H_S \end{cases}$$

It is not difficult to show that this system has two conjugate solutions namely ζ and $\bar{\zeta}$. If we write $\zeta = \widetilde{\lambda_1} + i\widetilde{\lambda_2}$, then

$$\widetilde{\lambda_1} = \ln \frac{\epsilon_0 r_m^m}{|\lambda_1^*|^{m+1}} \qquad \widetilde{\lambda_2} = +C(\widetilde{\lambda_1})$$

Let $\alpha = \alpha(|\lambda_1^*|, m) \in (\pi/2, \pi)$ be the argument of ζ (see Fig. 12).

If m is even then for all complex numbers with modulus equal to $|\lambda_1^*|$, where λ_1^* lies in I, and argument greater than α in absolute value, it is verified that $F_{\lambda,m}(-m) \in H_{|\lambda|,m}$. If m is odd, the same is true for all complex numbers with modulus equal to $|\lambda_1^*|$, where λ_1^* lies in I, and argument, in absolute value, smaller than $\pi - \alpha$. □

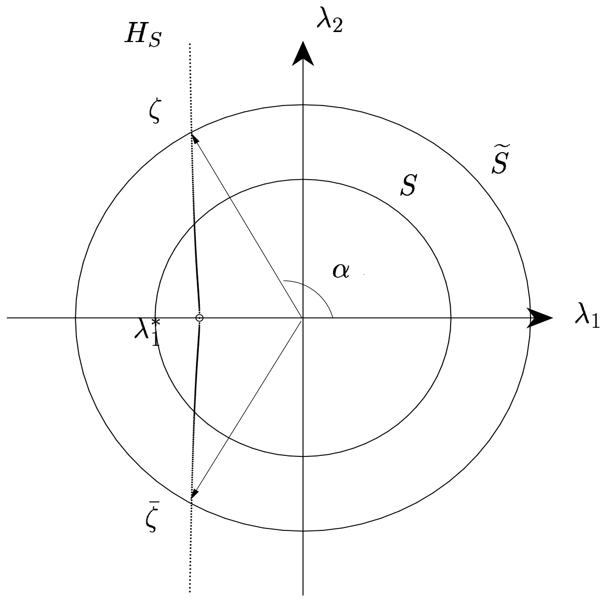


Figure 12: Sketch of construction of the set H_m, $m = 2$.

Parallel to the construction in dynamical plane we will now show the existence of a countable number of horizontal bands, all of which are also capture zones. See Fig. 9- 10. Apparently, when m is even, these strips extend to $+\infty$, while if m is odd they extend to $-\infty$. It also seems that their width decreases as m increases.

In the Sec. above we constructed similar strips around the curves σ_k. These curves were preimages of the negative real axis. In this case, we define

$$\Gamma_k = \{\lambda \in \mathbb{C} \text{ such that } F_{\lambda,m}(-m) \in \sigma_k\}.$$

To find an expression for the curves Γ_k we first write

$$F_{\lambda,m}(-m) = \lambda(-m)^m \exp(-m) = (-1)^m \frac{\lambda}{r_m}$$

where $r_m = (\frac{e}{m})^m$. Recall that $z \in \sigma_k$ when

$$Arg(\lambda) + mArg(z) + Im(z) = (2k+1)\pi.$$

Hence we need that

$$Arg(\lambda) + mArg(F_{\lambda,m}(-m)) + Im(F_{\lambda,m}(-m)) = (2k+1)\pi.$$

If m is even then $Arg(F_{\lambda,m}(-m)) = Arg(\lambda)$, while if m is odd then $Arg(F_{\lambda,m}(-m)) = Arg(\lambda)+\pi$; thus we obtain the condition for $F_{\lambda,m}(-m) \in \sigma_k$

$$\begin{cases} Arg(\lambda) + mArg(\lambda) + \frac{|\lambda|}{r_m}\sin(Arg(\lambda)) = (2k+1)\pi & \text{if m is even} \\ Arg(\lambda) + m(Arg(\lambda)+\pi) + \frac{|\lambda|}{r_m}(-1)\sin(Arg(\lambda)) = (2k+1)\pi & \text{if m is odd} \end{cases}$$

Solving for $|\lambda|$, we obtain a function of $Arg(\lambda)$, which we denote by ϕ. Explicitly, the curve Γ_k can be written as

$$\begin{cases} |\lambda| = \phi(Arg(\lambda)) = r_m \frac{(2k+1)\pi-(m+1)Arg(\lambda)}{\sin(Arg(\lambda))} & -\pi \leq Arg(\lambda) \leq \pi & \text{if m is even} \\ |\lambda| = \phi(Arg(\lambda)) = r_m \frac{(2k+1-m)\pi-(m+1)Arg(\lambda)}{-\sin(Arg(\lambda))} & -\pi \leq Arg(\lambda) \leq \pi & \text{if m is odd} \end{cases}$$

As in the Sec. above, we need to impose $\phi(Arg(\lambda)) \geq 0$. If we denote $\theta = Arg(\lambda)$, we have: if $m = 2j$ for $j \in \mathbb{Z}$

$$\Gamma_k = \phi(\theta)e^{i\theta} \begin{cases} 0 < \theta < \pi & \text{if} \quad k \geq j+1 \\ 0 < \theta < \frac{(2k+1)\pi}{m+1} & \text{if} \quad 0 \leq k \leq j \\ \frac{(2k+1)\pi}{m+1} < \theta < 0 & \text{if} \quad -(j+1) \leq k \leq 0 \\ -\pi < \theta < 0 & \text{if} \quad k \leq -(j+2) \end{cases};$$

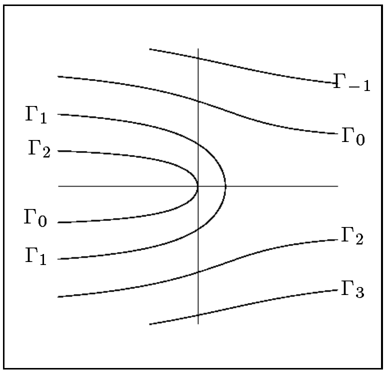
if $m = 2j+1$ for $j \in \mathbb{Z}$

$$\Gamma_k = \phi(\theta)e^{i\theta} \begin{cases} 0 < \theta < \pi & \text{si} \quad k \geq m \\ -\pi < \theta < 0 \cup \frac{2k+1-m}{m+1} < \theta < \pi & \text{if} \quad j+1 \leq k \leq m-1 \\ -\pi < \theta < \pi & \text{if} \quad k = j \\ -\pi < \theta < \frac{2k+1-m}{m+1} \cup 0 < \theta < \pi & \text{if} \quad j-1 \leq k \leq 0 \\ -\pi < \theta < 0 & \text{if} \quad k \leq -1 \end{cases}$$

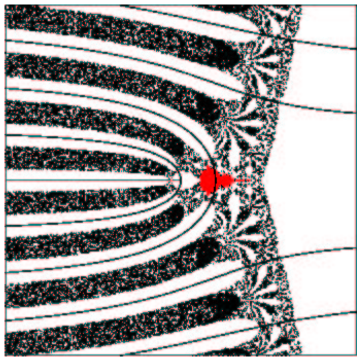
In Fig. 13 we show some of these curves for some values of m. If we suppose that $m = 2j$ is even, then each Γ_k tends asympotically to the line $Im(z) = (2k+1)\pi\, r_m$ as its real part tends to $+\infty$. We can classify these curves in three types. The first one is formed by curves whose real part runs from $-\infty$ to $+\infty$. There are two curves of the second kind, Γ_j and $\Gamma_{-(j+1)}$, with real part in $[-(m+1)r_m, \infty]$. The third group is formed by m curves, starting at the origin and tending to $+\infty$. These m curves have indexes between $j-1$ and $-j$.

If we take m an odd index ($m = 2j+1$), these curves tend asymptotically to the lines $Im(z) = 2(k-m)\pi\, r_m$ as their real part tend to $-\infty$. As above, we can classify these curves in three types. The first one is formed by curves that extend from $-\infty$ to $+\infty$. The second one is formed by the curve Γ_j, has a horseshoe shape, and cuts the real axis at the point $(m+1)r_m$. The third one is formed by $m-1$ curves, starting at the origin and tending to $-\infty$. These m curves have indexes between 0 and $m-1$, except for Γ_j.

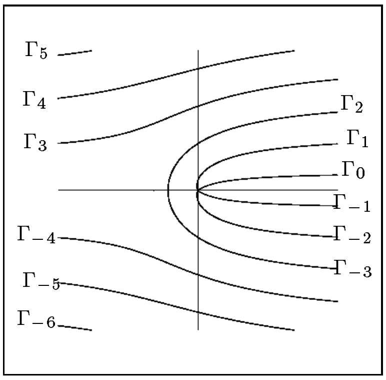
Hence we have obtained some curves Γ_k such that if $\lambda \in \Gamma_k$, then $F_{\lambda,m}(-m) \in \sigma_k$. Hence, choosing $\lambda \in \Gamma_k$ with $Re(F_{\lambda,m}(-m))$ large enough, we obtain that $F_{\lambda,m}(F_{\lambda,m}(-m)) \in H_{|\lambda|,m}$.



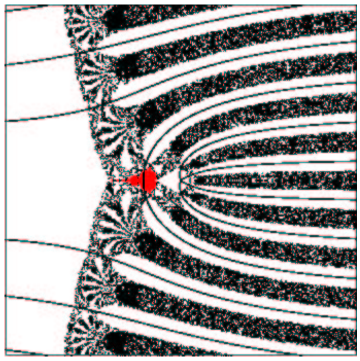
(a) Graph of Γ_k for $m = 3$



(b) The parameter plane of $F_{\lambda,3}$



(c) Graph of Γ_k for $m = 4$



(d) Parameter plane of $F_{\lambda,4}$

Figure 13: Strips in the parameter plane.

Since $Re(F_{\lambda,m}(-m)) = Re(\lambda)(-1)^m r_m$, this corresponds to taking $Re(\lambda)$ or $-Re(\lambda)$ large enough depending on m being even or odd.

By construction, the half curves we just defined belong each to C_m^3. We will now show that a neighbourhood of Γ_k of asymptotic width equal to $r_m\pi$ is also part of C_m^3. We fix a value $k \in \mathbb{Z}$ and we suppose that $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real numbers. We will prove the following result.

Proposition 3.10. *If m is even, for all $\lambda_2 \in (r_m(\frac{\pi}{2} + 2k\pi)\,,\, r_m(\frac{3\pi}{2} + 2k\pi))$ there exists λ_1^* such that, for all $\lambda_1 > \lambda_1^*$ then $F^3_{\lambda,m}(-m) \in A^*(0)$.*

If m is odd, for all $\lambda_2 \in (r_m(-\frac{\pi}{2} + 2k\pi)\,,\, r_m(\frac{\pi}{2} + 2k\pi))$ there exists λ_1^* such that, for all $\lambda_1 < \lambda_1^*$ then $F^3_{\lambda,m}(-m) \in A^*(0)$.

Proof. Assume that m is even (the odd case is completely symetric), and let $\lambda_2 \in (r_m(\frac{\pi}{2} + 2k\pi)\,,\, r_m(\frac{3\pi}{2} + 2k\pi))$. We recall that proposition 2.8 assures that, for all $y \in ((2k+1)\pi - Arg(\lambda) - \frac{\pi}{2}\,,\, (2k+1)\pi - Arg(\lambda) + \frac{\pi}{2})$, there exists $x_* \in \mathbb{R}$ such that, for all $x \geq x_*$, the point

$F_{\lambda,m}(x+yi) \in H_{|\lambda|,m}$.

Using that $F_{\lambda,m}(-m) = \frac{\lambda_1}{r_m} + \frac{\lambda_2}{r_m}i$, it suffices to prove that

$$Im(F_{\lambda,m}(-m)) = \frac{\lambda_2}{r_m} \in (\frac{\pi}{2} + 2k\pi - Arg(\lambda)\,,\, \frac{3\pi}{2} + 2k\pi - Arg(\lambda)).$$

Choosing

$$Re(F_{\lambda,m}(-m)) = \frac{\lambda_1}{r_m} > x^*.$$

The first condition is equivalent to $Arg(\lambda) \in (\alpha_1, \alpha_2)$,

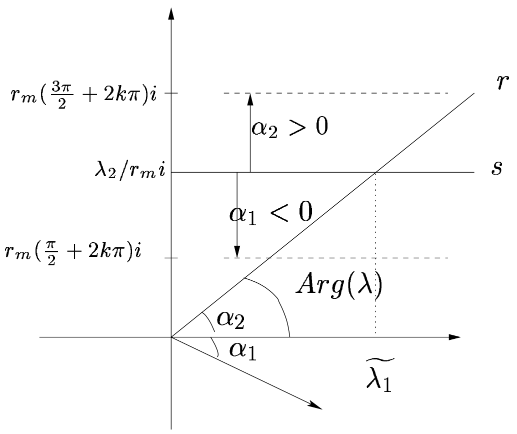


Figure 14: Construction of the value $\widetilde{\lambda_1}$

where $\alpha_1 = \frac{\pi}{2} + 2k\pi - \frac{\lambda_2}{r_m} < 0$ and $\alpha_2 = \frac{3\pi}{2} + 2k\pi - \frac{\lambda_2}{m} > 0$ (Fig. 14).

Suppose that $k > 0$. We denote by r the line through the origin with slope $tan(\alpha_2)$, and let s be the horizontal line through $\frac{\lambda_2}{r_m}i$. We also denote by $\widetilde{\lambda_1}$ the abscissa of the intersection point between the lines r and s (Fig. 14). All values of λ on s, with abscissa greater than $\widetilde{\lambda_1}$ verify that $0 < arg(\lambda) < \alpha_2$. Finally, we define $\lambda_1^* = max\{r_m x_*, r_m\widetilde{\lambda_1}\}$, and for this value both conditions are verified. Therefore, $F^2_{\lambda,m}(-m) \in H_{|\lambda|,m}$, and it follows that $F^3_{\lambda,m}(-m) \in D_{\epsilon_0} \subset A^*(0)$. If $k \leq 0$, we replace the line of slope $tan(\alpha_2)$, by a line with slope $tan(\alpha_1)$.

□

Acknowledgments

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