On the accuracy of Restricted Three-Body Models for the Trojan motion

Frederic Gabern,* Àngel Jorba* and Philippe Robutel†

Abstract

In this note we compare the frequencies of the motion of the Trojan asteroids in the Restricted Three-Body Problem (RTBP), the Elliptic Restricted Three-Body Problem (ERTBP) and the Outer Solar System (OSS) model. The RTBP and ERTBP are well-known academic models for the motion of these asteroids, and the OSS is the standard model used for realistic simulations.

Our results are based on a systematic frequency analysis of the motion of these asteroids. The main conclusion is that both the RTBP and ERTBP are not very accurate models for the long-term dynamics, although the level of accuracy strongly depends on the selected asteroid.

Keywords: Symplectic integrators, frequency analysis, Trojan asteroids.

1 Introduction

The Restricted Three-Body Problem models the motion of a particle under the gravitational attraction of two point masses following a (Keplerian) solution of the two-body problem (a general reference is [17]). The goal of this note is to discuss the degree of accuracy of such a model to study the real motion of an asteroid moving near the Lagrangian points of the Sun-Jupiter system.

To this end, we have considered two restricted three-body problems, namely: i) the Circular RTBP, in which Sun and Jupiter describe a circular orbit around their centre of mass, and ii) the Elliptic RTBP, in which Sun and Jupiter move on an elliptic orbit.

The realistic model used to test the results from the restricted models is the so-called Outer Solar System (OSS). This can be shortly described as a restricted five-body problem, in which the motion of the five masses (Sun, Jupiter, Saturn, Uranus and Neptune) and the asteroid is obtained from a numerical integration of the (Newtonian) equations of motion, starting from the present initial conditions of these bodies.

*Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain. e-mails: `gabern@mat.ub.es, angel@maia.ub.es`

†Astronomie et Systèmes Dynamiques, IMCCE-Observatoire de Paris, 77 Av. Denfert-Rochereau, 75014 Paris, France. e-mail: `robutel@imcce.fr`

We are interested in a qualitative comparison of the motion of the asteroids in these three models. More concretely, we want to test whether the three models predict either chaotic or regular behaviour and, in the regular case, we want to compare the frequencies of the motion. The result is that the Restricted Three-Body models are not very accurate models for the long-term motion of the Trojan asteroids, although the accuracy depends on the concrete asteroid.

The paper is divided in several sections. Section 2 is devoted to the specific numerical methods used in the integration of the models, Section 3 contains the frequency analysis for the OSS model, Section 4 discusses the frequency analysis for the Restricted Three-Body Problems and the conclusions can be found in Section 5.

2 Symplectic integrators

In this section, we describe a particular (high order) symplectic integrator. We adapt this integrator to the case of N planets and n particles revolving in quasi-Kepler orbits around the Sun, and we use it for the simulation of the Trojan motion in the Outer Solar System (OSS).

2.1 The general method

The symplectic integrator method of high order that we are going to use is due to J. Laskar and P. Robutel and it can be found in [8]. In this section, for completeness, we give a brief description of it.

Let be the Hamiltonian $H = A + \epsilon B$, where A and B are integrable and ϵ is a small parameter (the meaning of "small" will become clear later on). Let us suppose that we want to do one integration step of time τ. That is, we are interested in computing $e^{\tau L_H}$, where $L_\xi(\cdot)$ is the classical Lie operator, $L_\xi f = \{\xi, f\}$. The well known Baker-Campbell-Hausdorff formula (BCH) ensures that

$$e^{\tau L_H} = e^{\tau L_A} e^{\tau L_{\epsilon B}} + o(\tau).$$

Note that the operator $S_1 = e^{\tau L_A} e^{\tau L_{\epsilon B}}$ gives us the simplest integration method.

It is, then, straightforward to construct an integration scheme of n steps with the following operator:

$$S_n(\tau) = e^{c_1 \tau L_A} e^{d_1 \tau L_{\epsilon B}} \cdots e^{c_n \tau L_A} e^{d_n \tau L_{\epsilon B}},$$

where, of course, $\sum c_i = \sum d_i = 1$.

It is also easy (using the BCH Theorem and the linearity of the Lie operator) to see that there exists a formal operator K such that

$$S_n(\tau) = e^{\tau K}.$$

In general, a symplectic integrator for $H = A + \epsilon B$ of order p is obtained if $K = A + \epsilon B + O(\tau^p)$.

If we restrict to symmetric integrators, that is integrators such that $S_n(\tau)^{-1} = S_n(-\tau)$, we obtain four different symmetric symplectic integration schemes (see [8]):

$$\begin{aligned}
SABA_{2k} &: e^{c_1\tau L_A} e^{d_1\tau L_{\epsilon B}} \cdots e^{d_k\tau L_{\epsilon B}} e^{c_{k+1}\tau L_A} e^{d_k\tau L_{\epsilon B}} \cdots e^{d_1\tau L_{\epsilon B}} e^{c_1\tau L_A},\\
SABA_{2k+1} &: e^{c_1\tau L_A} e^{d_1\tau L_{\epsilon B}} \cdots e^{c_{k+1}\tau L_A} e^{d_{k+1}\tau L_{\epsilon B}} e^{c_{k+1}\tau L_A} \cdots e^{d_1\tau L_{\epsilon B}} e^{c_1\tau L_A},\\
SBAB_{2k-1} &: e^{d_1\tau L_{\epsilon B}} e^{c_2\tau L_A} e^{d_2\tau L_{\epsilon B}} \cdots e^{d_k\tau L_{\epsilon B}} e^{c_{k+1}\tau L_A} e^{d_k\tau L_{\epsilon B}} \cdots e^{c_2\tau L_A} e^{d_1\tau L_{\epsilon B}},\\
SBAB_{2k} &: e^{d_1\tau L_{\epsilon B}} e^{c_2\tau L_A} \cdots e^{c_{k+1}\tau L_A} e^{d_{k+1}\tau L_{\epsilon B}} e^{c_{k+1}\tau L_A} \cdots e^{c_2\tau L_A} e^{d_1\tau L_{\epsilon B}}.
\end{aligned} \tag{1}$$

Now, we take into account that the initial Hamiltonian H is a perturbation of an integrable one (ϵ is "small"). This will help us to obtain higher order symplectic integration schemes. Actually, we will be interested in finding integrators of the type $SABA_n$ and $SBAB_n$ for which the associated Hamiltonian K_{S_n} verifies

$$K_{S_n} = A + \epsilon B + O(\tau^{2n}\epsilon + \tau^2\epsilon^2).$$

Let us remark that in order to construct the operator K_{S_n}, we just have to compute the constants (c_i, d_i) of any integration scheme in (1). The details on how to obtain them can be found in [8].

2.2 Application to the (N planets + n particles) problem

We turn now the sight into the question of the integration of the problem consisting of the mutual gravitational attraction of the Sun, N planets and n massless particles (asteroids, for instance). We will assume that the planets and the particles move in slightly perturbed Keplerian orbits around the Sun.

The Hamiltonians of this problem written in the Jacobi coordinates are essentially the addition of an integrable Keplerian Hamiltonian, let's say A, and a perturbation, ϵB, consisting in the attraction of the pairs planet-planet and planet-particle:

$$\begin{aligned}
H_{PLA} &= \sum_{i=1}^{N} \left(\frac{\eta_i}{\eta_{i-1}} \frac{||\tilde{V}_i||^2}{2m_i} - G\frac{m_i\eta_{i-1}}{||V_i||} \right)\\
&\quad + G\sum_{i=2}^{N} m_i \left(\frac{\eta_{i-1}}{||V_i||} - \frac{m_0}{||R_i||} \right) - G \sum_{0<i<j\leq N} \frac{m_i m_j}{||R_i - R_j||}\\
H_{PAR} &= \sum_{i=1}^{n} \left(\frac{||\tilde{v}_i||^2}{2} - G\frac{\eta_N}{||v_i||} \right) + G\sum_{i=1}^{n} \left(\frac{\eta_N}{||v_i||} - \sum_{j=1}^{N} \frac{m_j}{||r_i - R_j||} \right)
\end{aligned}$$

where $(V_i, \tilde{V}_i)_{i=0,\ldots,N}$ and $(v_i, \tilde{v}_i)_{i=0,\ldots,n}$ are, respectively, the canonical Jacobi coordinates for the planets and the particles, $\eta_i = \sum_{j=0}^{i} m_j$, m_j is the mass of the j-th planet, and R_i and r_k are the heliocentric vector positions, respectively, of the i-th planet and the k-th particle. G is the universal constant of gravitation.

The non-perturbative part of the Hamiltonians is the addition of N Kepler problems corresponding, respectively, to a body of mass m_i attracted by the center of masses of the

i preceding bodies, and n Kepler problems corresponding to a massless body attracted by the center of masses of the system:

$$\begin{aligned} A_{PLA} &= \sum_{i=1}^{N} \left(\frac{\eta_i}{\eta_{i-1}} \frac{||\tilde{V}_i||^2}{2m_i} - G \frac{m_i \eta_{i-1}}{||V_i||} \right), \\ A_{PAR} &= \sum_{i=1}^{n} \left(\frac{||\tilde{v}_i||^2}{2} - G \frac{\eta_N}{||v_i||} \right). \end{aligned}$$

Due to the relative smallness of the masses of the planets with respect to the mass of the Sun, the perturbative part of the Hamiltonians are:

$$\begin{aligned} \epsilon B_{PLA} &= G \sum_{i=2}^{N} m_i \left(\frac{\eta_{i-1}}{||V_i||} - \frac{m_0}{||R_i||} \right) - G \sum_{0<i<j\leq N} \frac{m_i m_j}{||R_i - R_j||}, \\ \epsilon B_{PAR} &= G \sum_{i=1}^{n} \left(\frac{\eta_N}{||v_i||} - \sum_{j=1}^{N} \frac{m_j}{||r_i - R_j||} \right). \end{aligned}$$

2.3 The Outer Solar System model

We focus now on the integration of the five body problem consisting in the Sun and the four main outer planets of the Solar System (Jupiter, Saturn, Uranus and Neptune). This model is known in the literature as the Outer Solar System (OSS, for short). In the frequency analysis that is performed in Section 3, we use the OSS as the basis model for the Trojan motion.

The initial conditions for the planets are taken from the JPL Ephemerides DE405 at the Julian date 2452200.5 (October 10th, 2001). In order to take into account the (small) effect of the inner planets, their masses are added to Sun's one. In Table 1, we show the numerical values of the masses of the bodies that are used in the integration of the OSS. The unit of mass is such that Sun's mass is equal to one. The units of time and length used in the computations are years and A.U., respectively. The chosen integration method for this problem is the symplectic integrator $SABA_4$ with a fixed time step of 1/2 year.

We also study the performance of the integrator $SABA_4$ in this problem by computing the relative variation of the energy H_{PLA} in a long integration. In Figure 1, we have plotted the logarithm of the variation of the system's energy for a time span of 5 Million years for the integrators $SABA_4$. We can see that we have a relative error in the preservation of the energy which is approximately 10^{-11}. In fact, we integrate exactly a Hamiltonian "close" to the one we want to integrate.

3 Frequency analysis

In this section, the refined Fourier analysis (see [6], [7], [14] or [4]) is used in order to study the dynamics of the Trojan swarms. We obtain the basic frequencies of the planets in the

Body		Mass
Sun + inner planets		$0.1000005976999797 \times 10^{+1}$
Jupiter	$(p = 5)$	$0.9547919384243266 \times 10^{-3}$
Saturn	$(p = 6)$	$0.2858859806661309 \times 10^{-3}$
Uranus	$(p = 7)$	$0.4366244043351564 \times 10^{-4}$
Neptune	$(p = 8)$	$0.5151389020466116 \times 10^{-4}$
G		39.476926421373

Table 1: Masses of the bodies used in the Outer Solar System computations. The mass unit is Sun's mass. In the last row, the value of the constant of gravitation is given.

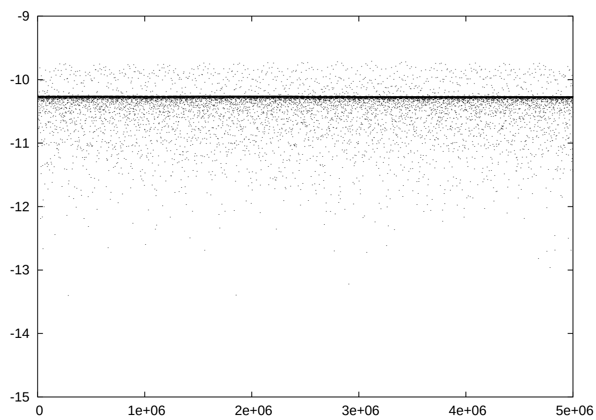

Figure 1: Logarithm of the relative variation of the energy during the integration of the OSS with the $SABA_4$ method. We can see that there is a random variation of the relative error around a fixed energy.

OSS and the proper frequencies of the Trojan asteroids in the same model (obviously, we suppose that the motion of the particles do not affect the motion of the planets).

3.1 Refined Fourier analysis

Let

$$f(t) = \sum_{k\in\mathbb{Z}^m} a_k e^{i<k,\omega>t}, \quad a_k \in \mathbb{C},$$

be a quasi-periodic function for which we know a table of equidistant values in the time span $[-T, T]$. The frequency analysis algorithm given by Laskar will provide the values of the frequencies ω_k and the amplitudes $\tilde{a}_k$ of a function $\tilde{f}(t) = \sum \tilde{a}_k e^{i\omega_k t}$ that approximates $f(t)$ in $[-T, T]$. The iterative scheme goes as follows: First, the frequency ω_1 is found by looking for the maximum amplitude of

$$\Phi(\sigma) =< f(t), e^{i\sigma t} >:= \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\sigma t}\chi(t)dt,$$

where $\chi(t) = 1 + \cos(\pi t/T)$ is a Hanning window filter (see [7] or [4]). Second, the corresponding amplitude itself, $\tilde{a}_1$, is computed by orthogonal projection of $f(t)$ on $e^{i\omega_1 t}$. Then, the process is restarted again with the function $f_1(t) = f(t) - \tilde{a}_1 e^{i\omega_1 t}$ in order to obtain the pairs $(\omega_k, \tilde{a}_k)_{k>1}$. In every step, the set $(e^{i\omega_k t})_k$ should be orthogonalized when projecting the functions f_i. See [7], for details.

Another method for approximating the frequencies is the one given by [4]. Their procedure consists, basically, in equating the discrete Fourier transforms of the sampled initial data and of the quasi-periodic approximation. This alternative method is used in further computations in order to check the results.

3.2 Basic frequencies of the OSS

It is known that the basic frequencies of the planets play an important role when studying the proper elements and the proper frequencies of any group of asteroids in the Solar System (there is an endless list of references on this topic; see for example [11], [5], [12] and references therein).

For the motion of a planet, we basically have to deal with three basic frequencies: one related with the mean motion (the fast one) and two secular ones, related with perihelion and node precessions. They appear as basic frequencies (in most cases with the biggest amplitude) when we apply the frequency analysis to the following three complex functions:

$$\begin{aligned}
\alpha_p(t) &= a_p(t)\,\exp(i\lambda_p(t)), \\
\beta_p(t) &= e_p(t)\,\exp(i\varpi_p(t)), \\
\gamma_p(t) &= \sin\left(\frac{i_p(t)}{2}\right)\exp(i\Omega_p(t)),
\end{aligned} \tag{2}$$

where the symbols denoting the osculating elliptic elements at time t for a planet p are: a_p the semi-major axis, e_p the eccentricity, i_p the inclination, λ_p the mean longitude ($\lambda_p =$

Planet	ν_p (deg)	g_p (sec)	s_p (sec)	Sidereal Orbit Period (yr)
Jupiter ($p=5$)	5.2968048×10^{-1}	4.24512464	0.0	11.862
Saturn ($p=6$)	2.1329183×10^{-1}	28.2468157	−26.3382192	29.458
Uranus ($p=7$)	7.4782874×10^{-2}	3.08791946	−2.99175127	84.019
Neptune ($p=8$)	3.8134035×10^{-2}	0.67409606	−0.70379108	164.766

Table 2: Basic frequencies of the planetary orbits in the Outer Solar System.

$\varpi_p + M_p$, with M_p the mean anomaly), ϖ_p the longitude of the perihelion ($\varpi_p = \Omega_p + \omega_p$, with ω_p the argument of the perihelion) and Ω_p the longitude of the ascending node.

Thus, we integrate the OSS with the symplectic integrator $SABA_4$ (see Section 2) for a time span of 5 Million years with a time step of 1/2 year. We tabulate the elliptic elements of the orbits of the planets in a mesh of approximately $100,000$ points and use them to make the frequency analysis of the functions (2). Finally, we obtain the basic frequencies of the planets: ν_p (proper mean motion), g_p (perihelion frequency) and s_p (node frequency). Their values for Jupiter, Saturn, Uranus and Neptune can be found in Table 2. In order to check this computation, we can compare this basic set of frequencies with others obtained in prior works (such as [6] or [13]) or we can repeat the computations by halving the integration step. We also have tested it by using the method of the running box (see [9]). The agreement, in all the cases, is good enough for our purposes.

3.3 Basic frequencies of the Trojan orbits

The computation of asteroid proper elements and basic frequencies for the Trojan case is a topic of research that has already been considered by many authors. See, for example, the works of Bien and Schubart ([15], [1] and [16]), or the papers by Milani ([9] and [10]).

In this section, we are interested in finding the basic frequencies that arise from the analysis of the orbits of the Trojan asteroids when we integrate them in the OSS with the same conditions as in 3.2. That is, we integrate the OSS together with 420 Trojans with the symplectic integrator $SABA_4$ for a time span of 5 Million years. We tabulate the osculating elliptic elements of the Trojan orbits in a mesh of approximately $100,000$ equidistant points. We evaluate the functions

$$\begin{aligned}
\alpha_k(t) &= a_k(t)\,\exp(i(\lambda_k(t) - \lambda_5(t))),\\
\beta_k(t) &= e_k(t)\,\exp(i\varpi_k(t)),\\
\gamma_k(t) &= \sin\left(\frac{i_k(t)}{2}\right)\exp(i\Omega_k(t)),
\end{aligned} \tag{3}$$

in this mesh of points for every asteroid (where $(a_k, e_k, i_k, \lambda_k, \varpi_k, \Omega_k)$ are the osculating orbital elements at time t of the Trojan asteroid with catalog number k and λ_5 is the mean longitude of Jupiter) and we use the refined Fourier analysis to obtain the firsts 10 frequencies that have maximum amplitude. From them, after removing the basic frequencies of the planets, we obtain the Trojans proper frequencies. The results for the

C.N.	Name	$\nu\,(deg)$	$g\,(sec)$	$s\,(sec)$
588	Achilles	2.434	344.40	-11.02
617	Patroclus	2.350	310.62	-6.76
624	Hektor	2.316	335.74	-12.99
659	Nestor	2.467	355.01	-16.48
884	Priamus	2.424	353.41	-12.24
911	Agamemnon	2.290	317.35	-8.69
1143	Odysseus	2.439	365.38	-10.91
1172	Aneas	2.366	331.54	-7.92
1173	Anchises	2.371	378.19	-23.67
1208	Troilus	2.199	258.54	-0.45
1404	Ajax	2.310	334.43	-15.15
1437	Diomedes	2.213	331.25	-20.05
1583	Antilochus	2.181	282.18	-10.79
1647	Menelaus	2.438	359.34	-9.96
1749	Telamon	2.421	366.33	-13.79
1867	Deiphobus	2.233	286.24	-5.76
1868	Thersites	2.316	345.09	-20.48
1869	Philoctetes	2.382	388.78	-20.44
1870	Glaukos	2.429	363.50	-9.48
1871	Astyanax	2.311	401.29	-25.28

Table 3: Basic frequencies of the Trojan asteroids in the Outer Solar System model. In the first column we display the catalog number of the asteroid and in the second one the asteroid's name. The frequency of libration can be seen in the third column, while in the fourth and fifth columns the two secular frequencies are shown.

20 firsts numbered Trojan asteroids can be seen in Table 3, where the first column is the catalog number (C.N.) of the asteroid and the second contains its name. The third, fourth and fifth columns are the three basic frequencies (ν, g and s, respectively). For a complete list of the 420 Trojans studied, see [3].

ν is the frequency of libration. It appears (usually when the frequencies of the planets are removed) as one of the basic frequencies of the function $\alpha_k(t)$ and its related period is about 150 years. g and s are known as the secular frequencies and they correspond, respectively, to the secular motion of the asteroid's perihelion and node.

In Figure 2, the Trojan swarm is plotted in the frequency space. The projection into the ν-g plane is given in the first plot and into the g-s plane in the second one.

As a first test of this computation, we have repeated the integration with a time step of 0.25 years. The agreement of the frequencies computed in this case and in the prior one is good enough for the precision required. As a second check, we have compared the values in Table 3 with the ones obtained by Milani [9]. The differences are quite small and can come from the fact that we possibly use slightly different values for the parameters and for the initial conditions.

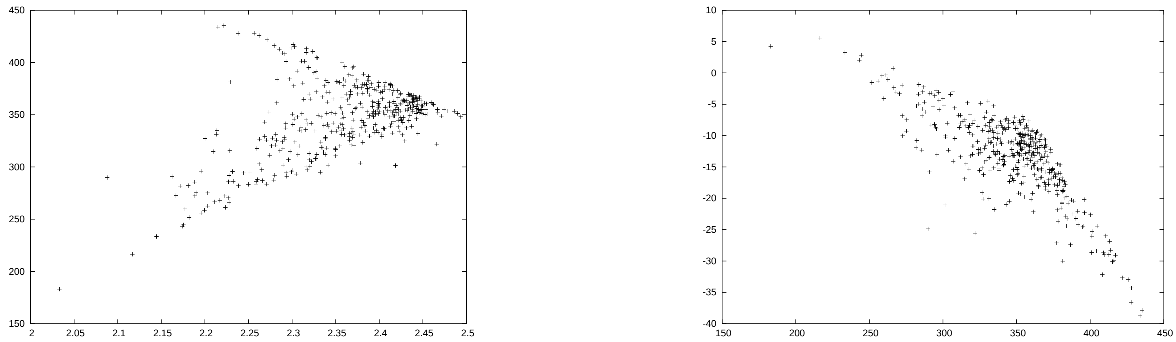

Figure 2: Distribution of the Trojan asteroids in the frequency space. Left: g in sec/yr versus ν in deg/yr. Right: s versus g, both in sec/yr.

4 Frequency analysis in three-body models

In this section, we describe the results of a Fourier analysis of the Trojan orbits in the RTBP and its elliptic version, the ERTBP. The goal of these computations is to study the performance of these restricted three body models to predict the dynamics of the Trojan asteroids.

In order to achieve this, we can compute the three basic frequencies ν, g and s for every Trojan orbit in the two models by making a Fourier analysis of the functions (3). Afterwards, we compare the results of these computations with the ones obtained with the OSS model (that will be considered close to the reality).

4.1 The Restricted Three Body Problem

First, the initial conditions for the 420 Trojan asteroids are taken from the Bowell Catalog [2] at the Julian date 2452200.5 (October 10th, 2001), in the same way as in 3.3. Second, we recompute these initial conditions in the RTBP frame of coordinates. Finally, we integrate the orbit of each Trojan in the circular RTBP for a time span of 2 Myr. and tabulate the functions (3) in a net of approximately $100,000$ equidistant points. Finally, in order to obtain the basic frequencies ν, g and s, a Fourier analysis is performed.

In this case, some asteroids escape from the system before ending a 4 Myrs. trajectory (we follow the integration in a second interval of 2 Myrs.). For some other asteroids, the frequency analysis of the orbits does not give a precise enough result; thus they are skipped over. The asteroids Catalog Numbers (C.N.) for the escaped ones are 3801, 7815, 15440, 15539, 16956, 24449 and 24471, and for the chaotic ones are 1868, 2146, 2895, 4060, 5264, 7641, 12444, 12916, 12929, 18046, 19844, 23694 and 24531. Recall that none of these asteroids escape in the OSS under the same conditions.

In Table 4, the three proper frequencies of 20 asteroids, are shown. For the complete frequency catalog, see [3].

Let us make a side comment: It is usual in the literature to study the dynamics near the triangular points of the Sun-Jupiter RTBP by using some normal form of the

C.N.	Name	$\nu\,(deg)$	$g\,(sec)$	$s\,(sec)$
588	Achilles	2.370	320.95	-2.97
617	Patroclus	2.394	327.13	-4.53
624	Hektor	2.349	375.03	-12.79
659	Nestor	2.369	312.65	-5.72
884	Priamus	2.294	318.32	-10.13
911	Agamemnon	2.378	361.83	-7.02
1143	Odysseus	2.238	294.05	-4.61
1172	Aneas	2.169	246.16	2.39
1173	Anchises	2.264	306.65	-7.64
1208	Troilus	2.308	303.50	1.04
1404	Ajax	2.397	345.87	-7.24
1437	Diomedes	2.252	352.93	-14.89
1583	Antilochus	1.978	152.22	10.90
1647	Menelaus	2.200	289.80	-7.75
1749	Telamon	2.354	326.43	-3.65
1867	Deiphobus	2.202	263.21	3.47
1869	Philoctetes	2.269	326.03	-9.94
1870	Glaukos	2.208	272.46	0.84
1871	Astyanax	2.061	250.42	-15.12
1872	Helenos	2.039	202.41	-4.38

Table 4: Basic frequencies of the Trojan asteroids in the circular Restricted Three Body Problem. In the first column we display the catalog number of the asteroid and in the second one the asteroid's name. The frequency of libration can be seen in the third column, while in the fourth and fifth columns the two secular frequencies are shown.

Hamiltonian. This normal form can be written as

$$H(I,\theta) = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3 + \mathcal{R}(I,\theta),$$

where $I = (I_1, I_2, I_3)$ and $\theta \in \mathbb{T}^3$ are the usual action-angle variables, $\mathcal{R}(I,\theta) = O(||I||^2)$ and $I_1 = I_2 = I_3 = 0$ corresponds to the triangular point L_4 or L_5. Thus, the "frequencies" ω_1, ω_2 and ω_3 can be taken as the frequencies of the fixed point L_4 or L_5 and their concrete numerical values for the Sun-Jupiter RTBP are:

$$\begin{aligned} \omega_1 &= \texttt{-0.080463875837}, \\ \omega_2 &= \texttt{0.99675752552}, \\ \omega_3 &= \texttt{1.0}. \end{aligned}$$

If a Trojan asteroid move quasi-periodically (or almost quasi-periodically) "near" one of the triangular points, it is natural to think that the basic frequencies of the motion of the Trojan should be related, in some way, with a perturbation of these ω_1, ω_2 and ω_3. Let be $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_3$ these perturbations (they normally appear when a Fourier analysis of the functions $f_x = \frac{x^2+p_x^2}{2}$, $f_y = \frac{y^2+p_y^2}{2}$ and $f_z = \frac{z^2+p_z^2}{2}$ is performed). Then, it turns out that this three basic frequencies seem to be related with the three proper frequencies ν, g and s in the following manner:

$$\begin{aligned} \nu &\approx \tilde{\omega}_1 \frac{2\pi}{T_5}\frac{180}{\pi}, \\ g &\approx (1-\tilde{\omega}_2)\frac{2\pi}{T_5}\frac{180}{\pi}3600, \\ s &\approx (1-\tilde{\omega}_3)\frac{2\pi}{T_5}\frac{180}{\pi}3600, \end{aligned} \tag{4}$$

where $T_5 = 11.862$ is the sidereal period (in years) of Jupiter's orbit.

In Table 5, we show the values of $\tilde{\omega}_1$ for the different models. The differences between these values and ω_1 gives us an idea of "how far", in the RTBP, is the asteroid from the librating point. For the other models, we should compare them with the frequency of the corresponding invariant object.

4.2 The Elliptic Restricted Three Body Problem

We integrate the ERTBP with the same conditions as in the former sections. The phase of the time-dependent perturbation is obtained by computing the actual perihelion argument of Jupiter's orbit in the Julian date 2452200.5.

In this case, there are also some asteroids that escape from the Solar System before ending a 4 Myr. integration. The Catalog Numbers of the escaped particles are 1868, 1873, 3801, 4946, 15440, 15539, 24471 and 24531. There are some cases for which the proper frequencies cannot be well determined. The corresponding C.N. are 2146, 2363, 7641, 7815, 11395, 12444, 12916, 12929, 15527, 15977, 16956, 18046, 19844, 24449 and 24587. Thus, in a similar way to the Circular RTBP, the behaviour of some asteroids is qualitatively very different of what happens in the OSS.

Asteroid		$\tilde{\omega}_1$		
C.N.	Name	OSS	RTBP	ERTBP
588	Achilles	0.080195	0.078084	0.078524
617	Patroclus	0.077431	0.078879	0.079341
624	Hektor	0.076299	0.077394	0.077441
659	Nestor	0.081280	0.078065	0.077198
884	Priamus	0.079874	0.075595	0.077358
911	Agamemnon	0.075440	0.078353	0.078122
1143	Odysseus	0.080375	0.073745	0.074371
1172	Aneas	0.077967	0.071468	0.071942
1173	Anchises	0.078138	0.074612	0.075012
1208	Troilus	0.072473	0.076044	0.076437
1404	Ajax	0.076125	0.078972	0.077430
1437	Diomedes	0.072922	0.074216	0.074698
1583	Antilochus	0.071864	0.065181	0.065448
1647	Menelaus	0.080348	0.072482	0.072651
1749	Telamon	0.079785	0.077557	0.077767

Table 5: $\tilde{\omega}_1$ values for the OSS and the three-body models. They are computed from (4).

The concrete values of the proper frequencies of the rest 397 Trojan asteroids can be found in [3].

5 Conclusions

We have computed the proper frequencies of 420 Trojan asteroids in the Outer Solar System. For every asteroid, there are 14 frequencies that seem to play some role: we have 11 proper frequencies (four fast and seven secular ones) coming from the planets and the three asteroid proper ones. The proper frequencies of the asteroid are the frequency of libration (around the corresponding triangular point) and two secular ones (corresponding to node and perihelion precessions).

Moreover, we have performed a similar analysis in the Circular RTBP and in the Elliptic RTBP.

The main qualitative difference between these models is that there are some asteroids that escape in the three-body problems and do not escape in the OSS. It is clear that, for these bodies, the three-body models are not suitable to study its dynamics.

In Table 6, the relative differences between the frequencies ν and g of the three models are shown. We see that the restricted three-body models simulate in an acceptable way the librational motion around the (corresponding) triangular point, since the relative differences between the frequencies corresponding to this motion in the OSS and in the models is less than 10%. For what concerns the secular motion, the analytic models simulate it in a worse way. In fact, we do not show the values corresponding to the secular frequency s, because of the big differences between the restricted three-body models and

Asteroid		Freq. of libration (ν)			Perihelion's freq. (g)		
C.N.	Name	OSS	RTBP	ERTBP	OSS	RTBP	ERTBP
588	Achilles	2.434	0.026	0.021	344.40	0.07	0.07
617	Patroclus	2.350	0.019	0.025	310.62	0.05	0.05
624	Hektor	2.316	0.014	0.015	335.74	0.12	0.12
659	Nestor	2.467	0.040	0.050	355.01	0.12	0.10
884	Priamus	2.424	0.054	0.031	353.41	0.10	0.12
911	Agamemnon	2.290	0.039	0.036	317.35	0.14	0.16
1143	Odysseus	2.439	0.082	0.075	365.38	0.20	0.20
1172	Aneas	2.366	0.083	0.077	331.54	0.26	0.26
1173	Anchises	2.371	0.045	0.040	378.19	0.19	0.19
1208	Troilus	2.199	0.049	0.055	258.54	0.17	0.17
1404	Ajax	2.310	0.037	0.017	334.43	0.03	0.08
1437	Diomedes	2.213	0.018	0.024	331.25	0.07	0.06
1583	Antilochus	2.181	0.093	0.089	282.18	0.46	0.46
1647	Menelaus	2.438	0.098	0.096	359.34	0.19	0.19
1749	Telamon	2.421	0.028	0.025	366.33	0.11	0.11

Table 6: Relative differences between the frequencies of libration and the Perihelion's frequencies of the 15 firsts numbered Trojan asteroids computed in the different models. In the first and second columns we display the catalog number and the name of the asteroid. The frequencies for the OSS model can be seen in the third and sixth column. Finally, in columns four and five, and seven and eight, the relatives differences are shown. More concretely, we compute respectively $\left|\frac{\nu^{OSS}-\nu^{model}}{\nu^{OSS}}\right|$ and $\left|\frac{g^{OSS}-g^{model}}{g^{OSS}}\right|$.

the OSS. This was expected, since the Trojan secular frequencies mainly come from the indirect action of the perturbative planets (Saturn, Uranus and Neptune) to Jupiter's motion that, of course, is not present in the restricted three body models.

From these calculations, it is clear that the results obtained by means of Restricted Three-Body models cannot be used, in general, to derive conclusions about the real motion of a Trojan. On the other hand, there are asteroids for which either the Circular or the Elliptic RTBP give reasonable dynamical predictions. Therefore, we want to stress that the level of accuracy of these models is not uniform but it strongly depends on the asteroid considered.

Acknowledgements

The authors thank J. Laskar and J.M. Mondelo to let them use their frequency analysis programs. F.G. and A.J. have been supported by the Spanish CICYT grant BFM2000–0623, the Catalan CIRIT grant 2001SGR–70 and DURSI.

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