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Exponentially small splitting of separatrices in a weakly hyperbolic case

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Abstract

We validate the Poincaré–Melnikov method in the singular case of high-frequency periodic perturbations of the Hamiltonian $h_0(x, y) = (1/2)y^2 - x^3 + x^4$ under appropriate conditions, which among other things, imply that we are considering the bifurcation case when the character of the fixed point changes from parabolic in the unperturbed case to hyperbolic in the perturbed one. The splitting is exponentially small.
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1. Introduction

Given a one-degree-of-freedom Hamiltonian system $h_0(x, y)$ with a homoclinic connection γ_0 associated to some fixed point p and a perturbation of it

$$h_0(x, y) + \varepsilon h_1(x, y, t, \varepsilon), \tag{1}$$

the Poincaré–Melnikov method [Me] is a tool to detect transversal intersection of the perturbed invariant manifolds. Moreover, in case of intersection, it provides asymptotics for the area of the lobe generated by the invariant manifolds between two consecutive homoclinic points and for the angle of the invariant manifolds at homoclinic points.

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The standard theory applies to hyperbolic fixed points and regular perturbations, that is when h_1 is of class C^r, $r \geqslant 3$.

The case of singular perturbations $h_1(x, y, t/\varepsilon, \varepsilon)$ is very important because it appears when one reduces two degrees of freedom near integrable systems near a periodic orbit. In this case, if the manifolds split, the area and the angle are exponentially small with respect to ε.

More generally, exponentially small splitting of invariant manifolds of invariant tori appears in near integrable Hamiltonian systems and it is a very important issue in the study of Arnold diffusion [Ar]. It is a difficult problem and satisfactory results have only appeared recently. However, since we deal with perturbations of a one-degree-of-freedom Hamiltonian, our result does not apply in this higher-dimensional setting.

Exponentially small phenomena also appear in one step discretizations of autonomous differential equations [FiS].

However this case was already encountered by Poincaré [Po]. He studied a model which is a special perturbation of the pendulum and he found that the splitting of separatrices is exponentially small in a perturbation parameter. He overcome the difficulties introducing an extra parameter and letting it to be exponentially small with respect to the other one.

Much later Neishtadt [Ne] provided exponentially small upper bounds for the splitting in the singular case with only one parameter. Asymptotic expressions have appeared recently, mainly for particular non-perturbed systems, such as the pendulum, the Duffing equation, etc. [An,DS1,Ge1,HMS,Tr]. In these examples the asymptotics are of the form $c\varepsilon^r \exp(-a/\varepsilon)$. However this is not always the case as it is shown by an example presented in [SMH] (see also the discussion in [GL]).

Exponentially upper bounds for general systems with sharp exponents are found in [Fo1,Fo2,FoS].

The papers [DS2,Ge2] address the problem of obtaining, under certain conditions, the asymptotics from the formal Melnikov function although this is not always the case [Tr].

Poincaré maps associated to (1) are near identity area preserving maps.

Lazutkin [La] gave the asymptotic formula for the splitting for the standard map and introduced new analytic ideas to study the problem. The proof of the formula was later completed by Gelfreich [Ge3].

There exists also a Poincaré–Melnikov theory for the setting of maps. For the regular case see [DR1,Ea]. For a singular case see [DR2]. A more detailed account of these results, both for maps and one and a half degrees of freedom Hamiltonians, can be found in [GL].

The case of a parabolic fixed point is much less studied. In this case the first problem is to ensure the existence of invariant manifolds for the perturbed system. This strongly depends on the higher order terms at the fixed point. For the regular parabolic case see [CFN].

In [BF] we consider the singular parabolic case. We consider non-perturbed Hamiltonians $h_0(x, y) = \frac{1}{2}y^2 + V(x)$ with $V(x) = a_n x^n + \cdots$, $n \geqslant 3$, and perturbations which do not destroy the parabolic character of the fixed point. Under appropriate hypotheses we prove that the formal Melnikov function gives the right exponentially small asymptotics.

In the present paper we consider a bifurcation case, that is, the fixed point is parabolic for the unperturbed system but is hyperbolic for the perturbed one and hence it has small eigenvalues. It is important to mention that the main part of this work is related to finding suitable parameterizations of the invariant manifolds of the fixed point of the perturbed system.

Once we have the parameterizations we can apply some of the results obtained in [BF] which also apply in this case. Due to some technical difficulties, we restrict ourselves to the particular non-perturbed Hamiltonian $h_0(x, y) = \frac{1}{2}y^2 - x^3 + x^4$.

This paper is organized as follows. In Section 2, we introduce notation and the hypotheses. In Section 3, we state the main results. In Section 4, we prove the existence of suitable parameterizations of the invariant manifolds of the perturbed system and finally in Section 5 we give the sketch of the proof of the asymptotic formulas for the area of the lobes and the angle between the invariant manifolds at a homoclinic point which are exponentially small in ε. Actually, under the stated hypotheses on h_1, we get that the formal Melnikov function associated to the problem gives the right asymptotics.

2. Notation and hypotheses

We consider Hamiltonian systems of the form

$$H(x, y, t/\varepsilon, \mu, \varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon), \qquad \varepsilon > 0, \tag{2}$$

where

$$h_0(x, y) = \frac{y^2}{2} + V(x) \qquad \text{and} \qquad V(x) = -x^3 + x^4.$$

The unperturbed system has the homoclinic orbit $\gamma_0 = (\alpha_0, \beta_0)$ given by

$$\alpha_0(t) = \frac{2}{2+t^2}, \qquad \beta_0(t) = -\frac{4t}{(2+t^2)^2}. \tag{3}$$

Note that α_0 has two poles of order 1 at $t = \pm i\sqrt{2}$.

2.1. Hypotheses

H1. The function $h_1(x, y, \theta, \mu, \varepsilon)$ is C^0, 2π-periodic in θ, has zero average:

$$\int_0^{2\pi} h_1(x, y, \theta, \mu, \varepsilon)\, d\theta = 0$$

and is real analytic with respect to (x, y, μ).

H2. The function $h_1(x, y, \theta, \mu, \varepsilon)$ is a polynomial of order 2 and degree κ in the (x, y) variables:

$$h_1(x, y, \theta, \mu, \varepsilon) = \sum_{i+j=2}^{\kappa} b_{ij}(\theta, \mu, \varepsilon)x^i y^j.$$

We introduce the functions B_{ij}, determined by the conditions:

$$\partial_\theta B_{ij} = b_{ij}, \qquad \int_0^{2\pi} B_{ij}(\theta,\mu,\varepsilon)\,d\theta = 0.$$

H3. With the above-introduced notation

$$\int_0^{2\pi} b_{11}(\theta,0,0)B_{20}(\theta,0,0)\,d\theta > 0.$$

Consider the terms $b_{ij}(\theta,0,0)x^i y^j$ of h_1 evaluated on γ_0. We define ℓ to be the greatest order of the poles $\pm i\sqrt{2}$ corresponding to $b_{ij}(\theta,\mu,\varepsilon)\alpha_0^i(u)\beta_0^j(u)$. That is:

$$\begin{aligned}\ell \;=\; &\max\{i+2j : \forall \mu_0,\varepsilon_0>0\ \exists(\theta,\mu,\varepsilon)\in[0,2\pi]\\ &\times(-\mu_0,\mu_0)\times(0,\varepsilon_0) \text{ s.t. } b_{ij}(\theta,\mu,\varepsilon)\neq 0\}.\end{aligned} \tag{4}$$

Also we define $v = p-\ell$ and we ask that:
H4. The constant v is greater or equal than 0.

Remark 2.1. The previous hypotheses imply $p\geqslant 3$. Indeed, by hypothesis H3, $b_{11}\neq 0$. The order of the pole of the term $b_{11}xy$ evaluated at the homoclinic orbit is 3, hence, by definition of ℓ, $\ell\geqslant 1+2=3$.

Remark 2.2. We will study in detail the associated Poincaré map and we will see H3 implies that the origin is a saddle point when $\mu\neq 0$ and $\varepsilon>0$ small.

Remark 2.3. Hypothesis H4 controls the growth of the perturbation term evaluated at the homoclinic orbit for values of time close to the singularities.

3. Main results

3.1. Parameterizations of the stable and unstable manifolds

First we introduce some notation. Given T, $\tau>0$, we define the sets

$$\begin{aligned} D^{\mathrm{s}} &= D^{\mathrm{s}}(T,\tau) = \{(t,s)\in\mathbb{R}\times\mathbb{C} : t+\mathrm{Re}\ s\geqslant T,\ |\,\mathrm{Im}\, s|\leqslant\tau\},\\ D^{\mathrm{u}} &= D^{\mathrm{u}}(T,\tau) = \{(t,s)\in\mathbb{R}\times\mathbb{C} : t+\mathrm{Re}\ s\leqslant -T,\ |\,\mathrm{Im}\, s|\leqslant\tau\}\end{aligned}$$

and for $\rho>0$, $k,l\in\mathbb{R}$, $(k,l\geqslant 0)$ we define the space $\mathcal{X}_k^l = \mathcal{X}_k^l(\rho)$ of functions $h : D^{\mathrm{s}}\to\mathbb{C}$ such that

(a) h is continuous,
(b) for t fixed, $s\mapsto h(t,s)$ is analytic,
(c) $h(t,s+2\pi\varepsilon) = h(t+2\pi\varepsilon,s)$ for all $(t,s)\in D^{\mathrm{s}}$,
(d) $\|h\|_{k,l} := \sup\{(t+\mathrm{Re}\ s)^k e^{\rho l(t+\mathrm{Re}\, s)}|h(t,s)| :\ (t,s)\in D^{\mathrm{s}}\} < \infty$.

We can prove that $\mathcal{X}_k^l$ is a Banach space with the norm $\|\cdot\|_{k,l}$ and that

$$\mathcal{X}_{k_2}^l \subset \mathcal{X}_{k_1}^l \quad \text{if } k_2 > k_1, \quad \text{and } \mathcal{X}_k^{l_2} \subset \mathcal{X}_k^{l_1} \quad \text{if } l_2 > l_1.$$

In an analogous way we define the space $\widetilde{\mathcal{X}}_k^l$ of functions defined on D^{u}. The next result gives the existence and some properties of a special parameterization of the stable and the unstable invariant manifolds.

Theorem 3.1. *Assuming hypotheses* H1–H4, *there exist $T > 0$ big enough and parameterizations $\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t,s)$, $\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t,s)$ of the local stable and unstable invariant manifolds of the origin of* (2), *defined in $D^{\mathrm{s}}(T,\sqrt{2})$, $D^{\mathrm{u}}(T,\sqrt{2})$, respectively, such that* ($*$ *stands for* s *or* u):

(1) $t \mapsto \gamma^*_{\mu,\varepsilon}(t,s)$ *is a solution of the equation associated to* (2) *and* $s \mapsto \gamma^*_{\mu,\varepsilon}(t,s)$ *is real analytic. Moreover the map* $(t,s,\mu,\varepsilon) \mapsto \gamma^*_{\mu,\varepsilon}(t,s)$ *is continuous, C^1 with respect to t and analytic with respect to* (s,μ).

(2) *For all* $(t,s) \in D^*(T,\sqrt{2})$, $\gamma^*_{\mu,\varepsilon}(t \pm 2\pi\varepsilon, s) = \gamma^*_{\mu,\varepsilon}(t, s \pm 2\pi\varepsilon)$, $+$ *for* $* =$ s *and* $-$ *for* $* =$ u.

(3) *For* $\mu = 0$, $\gamma^*_{\mu,\varepsilon}(t,s)$ *coincides with the restriction of the homoclinic solution* $\gamma_0(t+s)$ *to* $D^*(T,\sqrt{2})$, *and for* $\mu \neq 0$ *the following estimate holds*:

$$\gamma^*_{\mu,\varepsilon}(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1} G_{\mu,\varepsilon}(\gamma_0(t+s), t/\varepsilon) + O(\mu\varepsilon^{p+2}), \quad (t,s) \in D^*(T,\sqrt{2}),$$

where $\partial_\theta G_{\mu,\varepsilon}(x,y,\theta) = (\partial_y h_1(x,y,\theta,\mu,\varepsilon), -\partial_x h_1(x,y,\theta,\mu,\varepsilon))$ *and has zero average*.

(4) $\gamma^*_{\mu,\varepsilon}(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1}\sigma^*_{\mu,\varepsilon}(t,s)$ *where* $\sigma^*_{\mu,\varepsilon}(t,s) \in \mathcal{X}_2^0 \times \mathcal{X}_2^0$ *if* $* =$ s *and* $\sigma^*_{\mu,\varepsilon}(t,s) \in \widetilde{\mathcal{X}}_2^0 \times \widetilde{\mathcal{X}}_2^0$ *if* $* =$ u.

The proof of Theorem 3.1 is similar to that of Theorem 3.1 in [BF], but here, for $\mu \neq 0$ the behavior of γ^* is exponential in time and hence we face to a competition between the algebraic ($\mu = 0$) and the exponential ($\mu \neq 0$) characters. Therefore we have to take a different first approximation of $\gamma^*_{\mu,\varepsilon}$ and we have to be much more explicit in some computations.

3.2. Asymptotic formula for the splitting of separatrices

Let $M(s,\mu,\varepsilon)$ be the Melnikov function defined by

$$M(s,\mu,\varepsilon) = \int_{-\infty}^{\infty} \{h_0, h_1\}(\gamma_0(t+s), t/\varepsilon, \mu, \varepsilon)\, dt.$$

We denote by P^{t_0} the Poincaré map from t_0 to $t_0 + 2\pi\varepsilon$ of system (2), by A the area of the lobe generated by the stable and unstable manifold between two consecutive primary homoclinic points and by ϑ the angle between the stable and unstable invariant manifolds at a homoclinic point. We recall that, since the Poincaré map is area preserving, the area A will not depend on the concrete primary homoclinic points we consider.

Theorem 3.2. *Under hypotheses* H1–H4, *for* $\varepsilon \to 0^+$, $\mu \to 0$, *the following formulas hold*:

$$A = \mu\varepsilon^p \int_{s_0}^{\bar{s}_0} M(\upsilon, \mu, \varepsilon)\, d\upsilon + O(\mu^2\varepsilon^{2\nu+2}, \mu^2\varepsilon^{\nu+p+1}, \mu\varepsilon^{p+2})e^{-\sqrt{2}/\varepsilon},$$

$$\sin\vartheta = \mu\varepsilon^p \frac{M'(s_0, \mu, \varepsilon)}{|\dot{\gamma}_0(t_0+s_0)|^2} + O(\mu^2\varepsilon^{2\nu}, \mu^2\varepsilon^{\nu+p-1}, \mu\varepsilon^p)e^{-\sqrt{2}/\varepsilon},$$

where $s_0 < \bar{s}_0$ *are the two zeros of the Melnikov function* (*associated to two consecutive homoclinic points*), *closest to zero*.

We define the function $J(x, y, \theta, \mu, \varepsilon) = \{h_0, h_1\}(x, y, \theta, \mu, \varepsilon)$. This function is 2π-periodic in θ and has zero average with respect to θ. Let $J_k(x, y, \mu, \varepsilon)$ be its Fourier coefficients. It is clear that, for all $k \in \mathbb{Z}\backslash\{0\}$, $J_k(\gamma_0(u), 0, 0)$ has a pole of order at most $\ell + 1$ at $u = \pm i\sqrt{2}$, then, near the singularities $u = \mp i\sqrt{2}$, $J_k(\gamma_0(u), 0, 0)$ has the form

$$J_k(\gamma_0(u), 0, 0) = \frac{1}{(u \pm i\sqrt{2})^{\ell+1}} \left(J^{\pm}_{k,0} + \sum_{m \geqslant 1} J^{\pm}_{k,m}(u \pm i\sqrt{2})^m \right).$$

We introduce the further hypothesis:

H5. The Fourier coefficients $J_{\pm 1}$ evaluated on $(x, y) = \gamma_0(u)$, $\mu = 0$, $\varepsilon = 0$, that is $J_{\pm 1}(\gamma_0(u), 0, 0)$, have singularities of order exactly $\ell + 1$ at the points $u = \pm i\sqrt{2}$.

Remark 3.3. Hypothesis H5 is generic because it is equivalent to assume that some coefficient of the Laurent expansion of $J_{\pm 1}(\gamma_0(u), 0, 0)$ is different from zero.

Under this additional hypothesis we can obtain an explicit asymptotic expression of the Melnikov function which provides the asymptotics for the area and the angle.

Corollary 3.4. *If* H1–H5 *hold, then for* $\varepsilon \to 0^+$, $\mu \to 0$,

$$A \sim \mu\varepsilon^{\nu+1} 8\pi |J^-_{1,0}| \frac{1}{\ell!} e^{-\sqrt{2}/\varepsilon},$$

$$|\sin\vartheta| \sim \mu\varepsilon^{\nu-1} 4\pi |J^-_{1,0}| \frac{1}{\ell!} e^{-\sqrt{2}/\varepsilon} \frac{1}{|\dot{\gamma}_0(t_0+s_0)|^2}.$$

3.3. An example

In order to illustrate the practical application of formulas given in Corollary 3.4 we provide an easy example. Consider the Hamiltonian given by $H(x, y, t/\varepsilon, \mu, \varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$ with

$$h_1(x, y, t/\varepsilon, \mu, \varepsilon) = b_{20}(t/\varepsilon, \mu, \varepsilon)x^2 + b_{11}(t/\varepsilon, \mu, \varepsilon)xy + b_{02}(t/\varepsilon, \mu, \varepsilon)y^2$$

and satisfying hypotheses H1 and H3. An easy computation proves that, near the singularity $u = i\sqrt{2}$,

$$\{h_0, h_1\}(\gamma_0(u), t/\varepsilon, \mu, \varepsilon) = \frac{1}{(u - i\sqrt{2})^5}(-2b_{02}(t/\varepsilon, \mu, \varepsilon) + O(u - i\sqrt{2})).$$

Let b_{02}^k be the k-Fourier coefficient of b_{02} when $\mu = 0$, $\varepsilon = 0$. Hypothesis H5 is equivalent to assume that $b_{02}^1 \neq 0$ which is the constant $-J_{1,0}^-/2$. In this case, $\ell = 4$. Then, if $p \geqslant \ell = 4$, we have the following asymptotic formulas:

$$A \sim \mu\varepsilon^{p-3} 16\pi |b_{02}^1| \frac{1}{\ell!} e^{-\sqrt{2}/\varepsilon},$$

$$|\sin\vartheta| \sim \mu\varepsilon^{p-5} 8\pi |b_{02}^1| \frac{1}{\ell!} e^{-\sqrt{2}/\varepsilon} \frac{1}{|\dot{\gamma}_0(t_0 + s_0)|^2}.$$

4. Proof of Theorem 3.1

In this section we prove the existence of special parameterizations of the stable and unstable manifolds in domains independent of the parameters μ and ε. In fact, we prove the existence of such parameterization for the stable manifold but it is easy to see that, with slight changes, the proof works for the unstable one.

Since the time parameterization of the homoclinic orbit of the unperturbed system near the fixed point (that is, when $t \to \pm\infty$) has an algebraic character, and we know that the parameterization of the stable manifold near a hyperbolic fixed point (which will be the case for the perturbed system) is exponential in time, it seems natural to suspect that the homoclinic orbit of the unperturbed system is not a good approximation of the stable curve of the perturbed one. Actually, for μ small, there is a competition between the algebraic and the exponential character. Therefore we need a better initial approximation for the stable manifold, which will be obtained as a parameterization of the stable manifold of an auxiliary system. First we will have to obtain well adapted coordinates.

4.1. Averaging and Floquet theory

As in [BF] we perform some steps of averaging in order to obtain a suitable change of coordinates to deal with. The Floquet theory is used to reduce the linear part of the system to a system with constant coefficients.

We introduce the following notation:

Definition 4.1. Let U be an open subset of $\mathbb{C}^2$. Given $l \in \mathbb{Z}^+$, we denote by $\mathcal{P}_l$ the set of functions $p : U \times \mathbb{R} \times B(0, \mu_0) \times [0, \varepsilon_0) \to \mathbb{C}$ that are continuous, 2π-periodic in θ, analytic in (x, y, μ), and have order l, i.e. they can be represented in the form,

$$p(x, y, \theta, \mu, \varepsilon) = \sum_{i+j=l}^{\infty} a_{i,j}(\theta, \mu, \varepsilon) x^i y^j,$$

where the coefficients $a_{i,j}(\theta, \mu, \varepsilon)$ are continuous, 2π-periodic in θ and analytic in μ.

To simplify the notation we will not write the dependence on μ, ε of certain functions unless we want to stress it.

In this subsection we will prove the following result:

Proposition 4.2. *There exists a change of variables $\mathcal{C}$, defined in a neighborhood of the origin, which transforms the Hamiltonian equations associated to H into*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \mu^2\varepsilon^{2p+1}(bx - cx^2) - V'(x) \end{pmatrix} + \mu\varepsilon^{p+4}F_3(x, y, t/\varepsilon), \tag{5}$$

where $b = b(\mu, \varepsilon) = b_0(1 + O(\mu, \varepsilon))$, $b_0 = (2/\pi)\int_0^{2\pi} b_{11}(\theta, 0, 0)B_{20}(\theta, 0, 0)\, d\theta > 0$, and $c = c(\mu, \varepsilon)$ do not depend on t/ε and $F_3 \in \mathcal{P}_3$.

Moreover, the change $\mathcal{C}$ is continuous, C^1 and 2π-periodic in t/ε, analytic in (x, y, μ) and is of the form,

$$\mathcal{C}(x, y, t/\varepsilon, \mu, \varepsilon) = (x, y) + \mu\varepsilon^{p+1}G_{\mu,\varepsilon}(x, y, t/\varepsilon) + \mu\varepsilon^{p+2}r_2(x, y, t/\varepsilon), \tag{6}$$

where $G_{\mu,\varepsilon}$ satisfies $\partial_\theta G_{\mu,\varepsilon} = (\partial_y h_1, -\partial_x h_1)$ and has zero average, and $r_2 \in \mathcal{P}_2$.

First we scale the time by $\theta = t/\varepsilon$. We get the Hamiltonian system $\varepsilon H(x, y, \theta, \mu, \varepsilon)$.

In order to move the contribution of the perturbation to terms of higher order in the parameters we will do some steps of averaging. For this we quote Lemma 3.2 in [BF]:

Lemma 4.3. *Let $\varepsilon H = \varepsilon h_0 + \mu\varepsilon^{p+1}h_1$, with $h_0(x, y) = y^2/2 + V(x)$, $V(x) = O(x^n)$ and $h_1(x, y, \theta, \mu, \varepsilon) = O(|(x, y)|^k)$. Assume that V is analytic, h_1 is C^0, analytic with respect to (x, y, μ) and 2π-periodic in θ. Then, there exists a canonical change of variables $(x, y) = \mathcal{C}_0(\bar{x}, \bar{y}, \theta, \mu, \varepsilon)$ which is C^0 in $(\bar{x}, \bar{y}, \theta, \mu, \varepsilon)$, C^1 and 2π-periodic in θ and analytic in $(\bar{x}, \bar{y}, \mu)$ that transforms the Hamiltonian εH into*

$$\varepsilon\mathcal{H}_0 = \varepsilon h_0 + \mu\varepsilon^{p+2n+3}F_{2n-2} + \mu^2\varepsilon^{2p+2}R_{2k-2}$$

in a neighborhood of the origin, where $F_{2n-2} \in \mathcal{P}_{2n-2}$ and has zero average with respect to θ, $R_{2k-2} = \partial_y h_1 \partial_x S_1^1 + \varepsilon r_{2k-2} \in \mathcal{P}_{2k-2}$, with S_1^1 such that $\partial_\theta S_1^1 = -h_1$ and has zero average, and $r_{2k-2} \in \mathcal{P}_{2k-2}$. Moreover $\mathcal{H}_0$ is continuous in $(\bar{x}, \bar{y}, \theta, \mu, \varepsilon)$ and analytic in $(\bar{x}, \bar{y}, \mu)$.

From the proof of Lemma 4.3 we obtain that

$$\mathcal{C}_0(x, y, \theta, \mu, \varepsilon) = (x, y) + \mu\varepsilon^{p+1}G_{\mu,\varepsilon}(x, y, \theta) + \mu\varepsilon^{p+2}r_2(x, y, \theta),$$

where $G_{\mu,\varepsilon}$ satisfies $\partial_\theta G_{\mu,\varepsilon} = (\partial_y h_1, -\partial_x h_1)$ and has zero average.

Applying Lemma 4.3 to εH with $n = 3$ and $k = 2$ we obtain

$$\varepsilon\mathcal{H}_0(x, y, \theta, \mu, \varepsilon) = \varepsilon h_0(x, y) + \mu\varepsilon^{p+9}F_4(x, y, \theta, \mu, \varepsilon) + \mu^2\varepsilon^{2p+2}R_2(x, y, \theta, \mu, \varepsilon),$$

where $F_4 \in \mathcal{P}_4$ and has zero average with respect to θ, $R_2 = \partial_y h_1 \partial_x S^1 + \varepsilon r_2 \in \mathcal{P}_2$ with S^1 such that $\partial_\theta S^1(x, y, \theta) = -h_1(x, y, \theta)$ and has zero average with respect to

θ, and $r_2 \in \mathcal{P}_2$. Computing in detail the expression for R_2 we obtain

$$R_2(x, y, \theta) = -[2b_{11}(\theta)B_{20}(\theta)x^2 + [b_{11}(\theta)B_{11}(\theta) + 4b_{02}(\theta)B_{20}(\theta)]xy \\ +2b_{02}(\theta)B_{11}(\theta)y^2] + \varepsilon r_2(x, y, \theta) + R_3(x, y, \theta) \tag{7}$$

with $R_3 \in \mathcal{P}_3$.

To make the quadratic terms of $\varepsilon\mathcal{H}_0$ independent of θ we apply Floquet's theory.

We introduce $z = (x, y)$ and we let

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A(\theta) = A_{\mu,\varepsilon}(\theta) = \begin{pmatrix} \partial_{yx}R_2 & \partial_{yy}R_2 \\ -\partial_{xx}R_2 & -\partial_{xy}R_2 \end{pmatrix}, \tag{8}$$

where the derivatives of R_2 are evaluated at $z = 0$. Then, the linear part of the equation associated to $\varepsilon\mathcal{H}_0$ at $z = 0$ can be written as (prime means derivative with respect to θ)

$$z' = \varepsilon(N + \mu^2\varepsilon^{2p+1}A(\theta))z. \tag{9}$$

Lemma 4.4. *There exists a canonical linear change of variables $\mathcal{C}_1$ that transforms* (9) *into*

$$w' = \varepsilon \begin{pmatrix} 0 & 1 \\ \mu^2\varepsilon^{2p+1}b(\mu, \varepsilon) & 0 \end{pmatrix} w,$$

where $b(\mu, \varepsilon) = (2/\pi)\int_0^{2\pi} b_{11}(\theta)B_{20}(\theta)\,d\theta + O(\mu, \varepsilon)$. Moreover $\mathcal{C}_1$ is continuous, C^1 and 2π-periodic in θ, analytic in μ and $\mathcal{C}_1 = \mathrm{Id} + O(\mu^2\varepsilon^{2p+1})$.

Proof. Let $\phi(\theta)$ be the fundamental solution of (9) such that $\phi(0) = \mathrm{Id}$. It is clear that there exists $a > 0$ such that $\|\phi(\theta) - \mathrm{Id}\| \leqslant a\varepsilon$ for $\theta \in [0, 2\pi]$. Moreover

$$\phi(\theta) = \mathrm{Id} + \varepsilon N\theta + \mu^2\varepsilon^{2p+2}\int_0^\theta A(\zeta)\,d\zeta + O(\mu^2\varepsilon^{2p+3}).$$

Indeed, if we introduce $\psi(\theta) = \phi(\theta) - \mathrm{Id} - \varepsilon N\theta - \mu^2\varepsilon^{2p+2}\int_0^\theta A(\zeta)\,d\zeta$, we have that

$$\psi' = \varepsilon N\psi + \mu^2\varepsilon^{2p+2}U(\theta), \qquad \psi(0) = 0,$$

with $U(\theta) = A(\theta)\phi - A(\theta) + \varepsilon N\int_0^\theta A(\zeta)\,d\zeta = O(\varepsilon)$. By the variation of constants formula we get $\psi(\theta) = \mu^2\varepsilon^{2p+2}\int_0^\theta e^{\varepsilon N(\theta-\zeta)}U(\zeta)\,d\zeta$. Then $\psi(\theta) = O(\mu^2\varepsilon^{2p+3})$.

By Floquet's theory there exist a constant matrix M and a 2π-periodic matrix $P(\theta)$, such that $\phi(\theta) = P(\theta)e^{M\theta}$, with $M = M_{\mu,\varepsilon} = \frac{1}{2\pi}\log(\phi(2\pi))$. Moreover, the change of coordinates $z = P(\theta)w$ transforms Eq. (9) into

$$w' = Mw. \tag{10}$$

Since (9) is Hamiltonian, $\det\phi(\theta) = 1$. Therefore $\mathrm{tr}\, M = 0$ and $\det P(\theta) = 1$.

This implies that the change $z = P(\theta)w$ is canonical and then the transformed system will also be Hamiltonian. Since $N^2 = 0$, it is not difficult to see that $M = \frac{1}{2\pi}\log(\phi(2\pi)) = \varepsilon N + O(\mu^2\varepsilon^{2p+2})$, and thus

$$P(\theta) = \phi(\theta)e^{-M\theta} = \mathrm{Id} + O(\mu^2\varepsilon^{2p+2}). \tag{11}$$

To estimate the eigenvalues of $\phi(2\pi)$, we write

$$\int_0^{2\pi} A(\zeta)\,d\zeta = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + O(\varepsilon), \tag{12}$$

where $A_{11} = -A_{22} = -4\int_0^{2\pi} b_{02}(\theta)B_{20}(\theta)\,d\theta$, $A_{12} = -4\int_0^{2\pi} b_{02}(\theta)B_{11}(\theta)\,d\theta$ and $A_{21} = 4\int_0^{2\pi} b_{11}(\theta)B_{20}(\theta)\,d\theta$. Note that $\int_0^{2\pi} b_{11}(\theta)B_{11}(\theta)\,d\theta = 0$.

If we write $\phi(2\pi)$ as $\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$, the condition $\det\phi(2\pi) = 1$ becomes $a+d = -(ad-cb)$. Therefore $\operatorname{tr}\,\phi(2\pi) = 2+a+d = 2+2\pi\mu^2\varepsilon^{2p+3}A_{21} + O(\mu^2\varepsilon^{2p+4})$. The characteristic equation of $\phi(2\pi)$ is $\lambda^2 - (2+2\pi\mu^2\varepsilon^{2p+3}A_{21} + O(\mu^2\varepsilon^{2p+4}))\lambda + 1 = 0$ and hence the eigenvalues of $\phi(2\pi)$ are

$$\lambda_\pm = 1 \pm \sqrt{2\pi A_{21}}\,\mu\varepsilon^{p+3/2} + O(\mu\varepsilon^{p+5/2})$$

and the eigenvalues of M are

$$\alpha_\pm = \frac{1}{2\pi}\log(\lambda^\pm) = \pm\mu\varepsilon^{p+3/2}\sqrt{A_{21}/(2\pi)} + O(\mu\varepsilon^{p+5/2}).$$

Let $M = (a_{ij})$ and $C = (c_{ij})$ be defined by $c_{11} = \sqrt{a_{12}/\varepsilon}$, $c_{12} = 0$, $c_{21} = -a_{11}/\sqrt{\varepsilon a_{12}}$ and $c_{22} = \sqrt{\varepsilon/a_{12}}$. C has the form $\mathrm{Id} + O(\mu^2\varepsilon^{2p+1})$ and the change $z = Cw$ transforms Eq. (10) to

$$w' = \varepsilon \begin{pmatrix} 0 & 1 \\ \mu^2\varepsilon^{2p+1}b(\mu,\varepsilon) & 0 \end{pmatrix} w,$$

where $b(\mu,\varepsilon) = A_{21}/(2\pi) + O(\mu,\varepsilon)$. We take $\mathcal{C}_1 z = CP(\theta)z$. $\square$

Since $\mathcal{C}_1$ is area preserving the transformed Hamiltonian becomes

$$\varepsilon\mathcal{H}_1 = \varepsilon y^2/2 - \mu^2\varepsilon^{2p+2}bx^2/2 + \varepsilon V(x) + \mu\varepsilon^{p+9}F_4 + \mu^2\varepsilon^{2p+2}R_3, \qquad R_3 \in \mathcal{P}_3.$$

Finally, we will remove all cubic terms of R_3 but one. We observe that, if $\mu \neq 0$, standard normal form calculations give that all cubic terms of R_3 can be eliminated, but, in general, the corresponding change of variables is not regular at $\mu = 0$. However we have Lemma 4.5.

Let us write

$$R_3(x,y,\theta) = a_{30}(\theta)x^3 + a_{21}(\theta)x^2y + a_{12}(\theta)xy^2 + a_{03}(\theta)y^3 + r_4(x,y,\theta)$$

with $r_4 \in \mathcal{P}_4$. We will also denote by $\overline{a_{ij}}$ the average of a_{ij}.

Lemma 4.5. *There exists a change of variables $\mathcal{C}_2$ defined in a neighborhood of the origin which transforms the Hamiltonian system $\varepsilon\mathcal{H}_1$ into*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \varepsilon \begin{pmatrix} y \\ \mu^2\varepsilon^{2p+1}(bx - cx^2) - V'(x) \end{pmatrix} + \mu\varepsilon^{p+9} \begin{pmatrix} \partial_y F_4(x,y,\theta) \\ -\partial_x F_4(x,y,\theta) \end{pmatrix}$$
$$+\mu^2\varepsilon^{2p+2}s_3(x,y,\theta),$$

with $c = c(\mu, \varepsilon) = 3\overline{a_{30}} + O(\mu^2\varepsilon^{2p+1})$ *and* $s_3 \in \mathcal{P}_3$*. Moreover the change* $\mathcal{C}_2$ *is continuous,* C^1 *and* 2π*-periodic with respect to* θ*, analytic with respect to* (x, y, μ) *and it satisfies* $\mathcal{C}_2 = \mathrm{Id} + O(\mu^2\varepsilon^{2p+1})$.

Proof. We look for a change of variables of the form

$$\mathcal{C}_2(u, v, \theta) = (u, v) + \mu^2\varepsilon^{2p+1}(f(u, v, \theta), g(u, v, \theta)), \tag{13}$$

where f and g are 2π-periodic with respect to θ and have the form

$$f(u, v, \theta) = f_{20}(\theta)u^2 + f_{11}(\theta)uv + f_{02}(\theta)v^2,$$
$$g(x, y, \theta) = g_{20}(\theta)u^2 + g_{11}(\theta)uv + g_{02}(\theta)v^2.$$

A direct computation shows that

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \varepsilon \begin{pmatrix} v \\ \mu^2\varepsilon^{2p+1}bu - V'(u) \end{pmatrix} + \mu^2\varepsilon^{2p+1}(u^2B_{20}(\theta, \varepsilon) + uvB_{11}(\theta, \varepsilon) + v^2B_{02}(\theta, \varepsilon))$$
$$+\mu\varepsilon^{p+9}\begin{pmatrix} \partial_v F_4(u, v, \theta) \\ -\partial_u F_4(u, v, \theta) \end{pmatrix} + \mu^2\varepsilon^{2p+2}s_3(u, v, \theta),$$

where $s_3 \in \mathcal{P}_3$ and

$$B_{20} = \begin{pmatrix} -f_{20}' + \varepsilon a_{21} + \varepsilon g_{20} - b\mu^2\varepsilon^{2p+2}f_{11} \\ -g_{20}' - 3\varepsilon a_{30} + b\mu^2\varepsilon^{2p+2}(f_{20} - g_{11}) \end{pmatrix},$$
$$B_{11} = \begin{pmatrix} -f_{11}' + 2\varepsilon a_{12} + \varepsilon(g_{11} - 2f_{20}) - 2b\mu^2\varepsilon^{2p+2}f_{02} \\ -g_{11}' - 2\varepsilon a_{21} - 2\varepsilon g_{20} + b\mu^2\varepsilon^{2p+2}(f_{11} - 2g_{02}) \end{pmatrix},$$
$$B_{02} = \begin{pmatrix} -f_{02}' + 3\varepsilon a_{03} + \varepsilon g_{02} - \varepsilon f_{11} \\ -g_{02}' - \varepsilon a_{12} - \varepsilon g_{11} + b\mu^2\varepsilon^{2p+2}f_{02} \end{pmatrix}.$$

We ask B_{ij} to satisfy $B_{11} = B_{02} = 0$ and $B_{20} = (0, \varepsilon d)^T$ with $d = d(\mu, \varepsilon)$ independent of θ to be determined later.

First of all we observe that, by imposing the above conditions on B_{ij}, f_{ij} and g_{ij} satisfy a linear system with constant coefficients and periodic non-homogeneous terms. For the functions $h_1 = f_{11} + 2g_{02}$ and $h_2 = g_{11} + 2f_{20}$ we have

$$h_1' = -\varepsilon h_2, \qquad h_2' = -b\mu^2\varepsilon^{2p+2}h_1. \tag{14}$$

The only periodic solution of (14) is $h_1(\theta) = h_2(\theta) = 0$. Therefore, $f_{11} = -2g_{02}$ and $g_{11} = -2f_{20}$. This permits to reduce the number of equations.

We introduce $\eta = \mu^2\varepsilon^{2p+1}$, $Z = (f_{02}, g_{02}, f_{20}, g_{20})^T$, $A = (3a_{03}, -a_{12}, a_{21}, -(3a_{30}+d))^T$ and

$$C = \begin{pmatrix} 0 & 3 & 0 & 0 \\ b\eta & 0 & 2 & 0 \\ 0 & 2b\eta & 0 & 1 \\ 0 & 0 & 3b\eta & 0 \end{pmatrix}.$$

With this notation the conditions we impose on f_{ij} and g_{ij} become

$$Z' = \varepsilon C Z + \varepsilon A(\theta). \tag{15}$$

We want to prove the existence of periodic solutions of system (15) being analytic with respect to η. Let $Z(\theta, Z_0)$ be the solution of (15) such that $Z(0, Z_0) = Z_0$. Z is a 2π-periodic solution of (15) if and only if

$$Z_0 = -\varepsilon(\mathrm{Id} - e^{-2\pi\varepsilon C})^{-1} \int_0^{2\pi} e^{-s\varepsilon C} A(s)\, ds. \tag{16}$$

We notice that, if $\eta = \mu^2\varepsilon^{2p+1} \neq 0$, $(\mathrm{Id} - e^{-2\pi\varepsilon C})$ is invertible. Indeed, it follows from the fact that, if $\eta \neq 0$, C is invertible, and therefore

$$\begin{aligned} (\mathrm{Id} - e^{-2\pi\varepsilon C})^{-1} &= \frac{1}{2\pi\varepsilon} C^{-1} \left(\mathrm{Id} + \sum_{k \geqslant 2} \frac{1}{k!} (-1)^{k-1} (2\pi\varepsilon C)^{k-1} \right)^{-1} \\ &= \frac{1}{2\pi\varepsilon} C^{-1} (\mathrm{Id} + 2\pi\varepsilon C f(2\pi\varepsilon C)), \end{aligned}$$

where f is an analytic function. Also we can write $e^{-s\varepsilon C} = \mathrm{Id} - s\varepsilon C g(s\varepsilon C)$ with g analytic. Then Eq. (16) takes the form

$$\begin{aligned} Z_0 &= -\frac{1}{2\pi} C^{-1} (\mathrm{Id} + 2\pi\varepsilon C f(2\pi\varepsilon C)) \int_0^{2\pi} (\mathrm{Id} - s\varepsilon C g(s\varepsilon C)) A(s)\, ds \\ &= -\frac{1}{2\pi} C^{-1} \int_0^{2\pi} A(s)\, ds + O(\varepsilon). \end{aligned}$$

Now we are going to determine d. We observe that

$$C^{-1} = \frac{1}{(3b\eta)^2} \begin{pmatrix} 0 & 9b\eta & 0 & -6 \\ 3(b\eta)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3b\eta \\ -6(b\eta)^3 & 0 & (3b\eta)^2 & 0 \end{pmatrix},$$

thus, by definition of A,

$$Z_0 = -\begin{pmatrix} -\frac{1}{b\eta}\,\overline{a_{12}} + \frac{2}{3(b\eta)^2}\,(3\overline{a_{30}} + d) \\ \overline{a_{03}} \\ -\frac{1}{3b\eta}\,(3\overline{a_{30}} + d) \\ -2b\eta\overline{a_{03}} + \overline{a_{21}} \end{pmatrix} + O(\varepsilon).$$

Choosing $d = -3\overline{a_{30}} + (3b\eta/2)\overline{a_{12}}$ we get that $Z_0 = -(0, \overline{a_{03}}, -\overline{a_{12}}/2, \overline{a}_{21}) + O(\varepsilon)$. Therefore with this choice of d we get that the unique 2π-periodic solution of (15) is analytic in μ, and the change (13) is $\mu^2\varepsilon^{2p+1}$ close to the identity. $\square$

We define $F_3^1 = \varepsilon^4\partial_v F_4 + \mu\varepsilon^{p-3}s_3^1$ and $F_3^2 = -\varepsilon^4\partial_u F_4 + \mu\varepsilon^{p-3}s_3^2$; recall that $p \geqslant 3$.

Finally, we scale back to the original time. Let $\mathcal{C}$ be the composition of all changes. It is not difficult to see that $\mathcal{C}$ has form (6). This ends the proof of Proposition 4.2.

4.2. Estimates for the Poincaré map

In this section, we provide an expression of the Poincaré map associated to Eq. (5). We introduce $\rho > 0$ such that $\rho^2 = \mu^2\varepsilon^{2p+1}b(\mu, \varepsilon)$, $\theta = t/\varepsilon$ and $\theta_0 = t_0/\varepsilon$.

We write the right-hand side of (5) as

$$X_{\mu,\varepsilon}(z, \theta) = Y_{\mu,\varepsilon}(z) + \mu\varepsilon^{p+4}F_3(z, \theta),$$

where $Y_{\mu,\varepsilon}$ is the auxiliary vector field defined by

$$Y_{\mu,\varepsilon}(x, y) = (y, \mu^2\varepsilon^{2p+1}(bx - cx^2) - V'(x))^T \tag{17}$$

with $b = b(\mu, \varepsilon)$ and $c = c(\mu, \varepsilon)$ and we introduce the matrix (solution of the linearized vector field at the origin)

$$A(\theta) = A_{\mu,\varepsilon}(\theta) = \begin{pmatrix} \cosh(\rho\theta) & \rho^{-1}\sinh(\rho\theta) \\ \rho\sinh(\rho\theta) & \cosh(\rho\theta) \end{pmatrix}. \tag{18}$$

For any fixed $t_0 \in \mathbb{R}$, we consider the Poincaré maps

$$P^{t_0}_{\mu,\varepsilon}(z) = \varphi_{\mu,\varepsilon}(t_0 + 2\pi\varepsilon, t_0, z) \tag{19}$$

and

$$\hat{P}_{\mu,\varepsilon}(z) = \phi_{\mu,\varepsilon}(2\pi\varepsilon, 0, z), \tag{20}$$

where $\varphi_{\mu,\varepsilon}(t, t_0, z)$ is the solution of equation $\dot{z} = X_{\mu,\varepsilon}(z, t/\varepsilon)$ such that $\varphi_{\mu,\varepsilon}(t_0, t_0, z) = z$ and $\phi_{\mu,\varepsilon}(t, t_0, z)$ is the solution of equation $\dot{z} = Y_{\mu,\varepsilon}(z)$ such that $\phi_{\mu,\varepsilon}(t_0, t_0, z) = z$. We will denote them by $\varphi_{\mu,\varepsilon}(t)$ and $\phi_{\mu,\varepsilon}(t)$, respectively, if the initial conditions do not play an important role.

The goal of this section is to prove the following result:

Proposition 4.6. *The Poincaré maps* $\hat{P}_{\mu,\varepsilon}$ *and* $P^{t_0}_{\mu,\varepsilon}$ *have the form,*

$$\hat{P}_{\mu,\varepsilon}(z) = A(2\pi\varepsilon)z + 2\pi\varepsilon(0, -V'(x))^T + \varepsilon^2 G^1_2(z,\varepsilon) + \mu^2\varepsilon^{2p+2}G^2_2(z,\mu,\varepsilon)$$

and

$$P^{t_0}_{\mu,\varepsilon}(z) = \hat{P}_{\mu,\varepsilon}(z) + \mu\varepsilon^{p+5}T_3(z, t_0/\varepsilon),$$

where $G^1_2, G^2_2 \in \mathcal{P}_2$ *and* $T_3 \in \mathcal{P}_3$. *All functions are* C^0, C^1 *and* 2π*-periodic with respect to* t_0/ε *and analytic with respect to* (z,μ).

We will need a technical lemma which is a small variation of Lemma 3.6 in [BF] and it is proved exactly in the same way.

To deal with the regularity conditions it is more convenient to work with the scaled equations $z' = \varepsilon X_{\mu,\varepsilon}(z,\theta)$ and $z' = \varepsilon Y_{\mu,\varepsilon}(z)$, respectively, where here prime means derivative with respect to θ. Let $\tilde{\varphi}_{\mu,\varepsilon}(\theta) = \tilde{\varphi}_{\mu,\varepsilon}(\theta,\theta_0,z)$ be the solution of $z' = \varepsilon X_{\mu,\varepsilon}(z,\theta)$, with $\tilde{\varphi}_{\mu,\varepsilon}(\theta_0) = z$ and $\tilde{\phi}_{\mu,\varepsilon}(\theta) = \tilde{\phi}_{\mu,\varepsilon}(\theta,\theta_0,z)$ the solution of $z' = \varepsilon Y_{\mu,\varepsilon}(z)$, with $\tilde{\phi}_{\mu,\varepsilon}(\theta_0) = z$.

Clearly $P^{t_0}_{\mu,\varepsilon}$ and $\hat{P}_{\mu,\varepsilon}$ can also be expressed as $\tilde{\varphi}_{\mu,\varepsilon}(t_0/\varepsilon + 2\pi, t_0/\varepsilon, z)$ and $\tilde{\phi}_{\mu,\varepsilon}(2\pi, 0, z)$, respectively.

Lemma 4.7. *With the above-introduced notation, there exist some constants* C, μ_0 *and* ε_0 *such that for all* $\theta \in [\theta_0, \theta_0 + 2\pi]$ *and z belonging to a neighborhood of the origin,* $|\mu| \leqslant \mu_0$ *and* $|\varepsilon| \leqslant \varepsilon_0$ *the following bounds hold*:

(1) $\|\tilde{\varphi}_{\mu,\varepsilon}(\theta)\| \leqslant C\|z\|, \quad \|\tilde{\phi}_{\mu,\varepsilon}(\theta)\| \leqslant C\|z\|.$
(2) $\|\tilde{\varphi}_{\mu,\varepsilon}(\theta) - z\| \leqslant \varepsilon C\|z\|, \quad \|\tilde{\phi}_{\mu,\varepsilon}(\theta) - z\| \leqslant \varepsilon C\|z\|.$
(3) *The solutions* $\tilde{\varphi}_{\mu,\varepsilon}(\theta)$ *and* $\tilde{\phi}_{\mu,\varepsilon}(\theta)$ *can be expressed as*

$$\tilde{\phi}_{\mu,\varepsilon}(\theta) = \varphi_0(\theta) + \mu^2\varepsilon^{2p+2}\Phi_{\mu,\varepsilon}(\theta,\theta_0,z) \quad \textit{and}$$

$$\tilde{\varphi}_{\mu,\varepsilon}(\theta) = \tilde{\phi}_{\mu,\varepsilon}(\theta) + \mu\varepsilon^{p+5}\Psi_{\mu,\varepsilon}(\theta,\theta_0,z)$$

with $\|\Psi_{\mu,\varepsilon}(\theta,\theta_0,z)\| \leqslant C\|z\|^3$.

Furthermore, $\Phi_{\mu,\varepsilon}$ *and* $\Psi_{\mu,\varepsilon}$ *are* C^0, C^1 *with respect to* θ *and* θ_0 *and analytic with respect to* μ *and the initial condition z.*

(4) *The functions*

$$T_1(z,\theta_0) := \Phi_{\mu,\varepsilon}(\theta_0 + 2\pi, \theta_0, z) = \Phi_{\mu,\varepsilon}(2\pi, 0, z) \quad \textit{and}$$

$$T_3(z,\theta_0) := \Psi_{\mu,\varepsilon}(\theta_0 + 2\pi, \theta_0, z)$$

are 2π*-periodic in* θ_0 *and satisfy that* $T_j \in \mathcal{P}_j$.

Proof of Proposition 4.6. It is a simple consequence of Lemma 4.7. Indeed we write $\tilde{\phi}_{\mu,\varepsilon} = (\tilde{\phi}^1_{\mu,\varepsilon}, \tilde{\phi}^2_{\mu,\varepsilon})$ and we note that the solution $\tilde{\phi}_{\mu,\varepsilon}(\theta)$ can be expressed as

$$\tilde{\phi}_{\mu,\varepsilon}(\theta) = A(\varepsilon\theta)\left(z + \varepsilon\int_0^\theta A^{-1}(\varepsilon s)\begin{pmatrix} 0 \\ -\mu^2\varepsilon^{2p+1}c(\tilde{\phi}^1_{\mu,\varepsilon}(s))^2 - V'(\tilde{\phi}^1_{\mu,\varepsilon}(s))\end{pmatrix} ds\right). \tag{21}$$

Since $\rho^2 = O(\mu^2\varepsilon^{2p+1})$, we have that for $\theta \in [0, 2\pi]$, $A(\theta) = \mathrm{Id} + \theta N + O(\mu^2\varepsilon^{2p+1})$. Using the last equality, conclusion (2) of Lemma 4.7 and formula (21) for $\theta = 2\pi$ we obtain

$$\tilde{\phi}_{\mu,\varepsilon}(2\pi) = A_{\mu,\varepsilon}(2\pi\varepsilon)z + 2\pi\varepsilon(0, -V'(x))^T + \varepsilon^2 G^1_2(z,\varepsilon) + \mu^2\varepsilon^{2p+2}G^2_2(z,\mu,\varepsilon),$$

with $G^1_2, G^2_2 \in \mathcal{P}_2$. The conclusion for $P^{t_0}_{\mu,\varepsilon}$ follows from

$$\begin{aligned} P^{t_0}_{\mu,\varepsilon}(z) &= \tilde{\varphi}_{\mu,\varepsilon}(t_0/\varepsilon + 2\pi, t_0/\varepsilon, z) \\ &= \tilde{\phi}_{\mu,\varepsilon}(t_0/\varepsilon + 2\pi, t_0/\varepsilon, z) + \mu\varepsilon^{p+5}\Psi_{\mu,\varepsilon}(t_0/\varepsilon + 2\pi, t_0/\varepsilon, z). \qquad \square \end{aligned}$$

4.3. The homoclinic orbit of the auxiliary system

In this subsection we prove that the auxiliary system $\dot{z} = Y_{\mu,\varepsilon}(z)$ has a homoclinic connection and that is $O(\mu^2\varepsilon^{2p+1})$ close to the homoclinic connection of the unperturbed system. This is the contents of the following result:

Proposition 4.8. *Let γ_0 be the homoclinic orbit for the unperturbed system. Then there exists a parameterization, $\hat{\gamma}(u)$, of the stable invariant manifold of $\dot{z} = Y_{\mu,\varepsilon}(z)$ and there exist $T, M > 0$ independent of ε, such that*

$$\|\hat{\gamma}(u) - \gamma_0(u)\| \leqslant \mu^2\varepsilon^{2p+1}M$$

for all u such that $\mathrm{Re}\, u \geqslant T$ *and* $|\operatorname{Im} u| \leqslant \sqrt{2}$.

Proof. By direct substitution it is immediately checked that $\hat{\gamma}(u) = (\hat{\alpha}(u), \hat{\beta}(u))$ defined by

$$\hat{\alpha}(u) = \frac{k_1\rho^2}{k_2\cosh(\rho u) - 1}, \qquad \hat{\beta}(u) = -\frac{k_1k_2\rho^3\sinh(\rho u)}{(k_2\cosh(\rho u) - 1)^2}, \tag{22}$$

where

$$k_1 = \frac{3}{3 - \mu^2\varepsilon^{2p+1}c} = 1 + \frac{c}{3b}\rho^2 + O(\rho^4) \qquad \text{and} \qquad k_2 = \sqrt{1 + 2\rho^2 k_1^2}$$

is a homoclinic solution of equation $\dot{z} = Y_{\mu,\varepsilon}(z)$. Since $\hat{\alpha}(u) - \alpha_0(u)$ and $\hat{\beta}(u) - \beta_0(u)$ go to 0 as $\mathrm{Re}\, u \to \infty$, by the maximum principle, it is clear that the maximum values of $|\hat{\alpha}(u) - \alpha_0(u)|$ and $|\hat{\beta}(u) - \beta_0(u)|$ on the set $\{u \in \mathbb{C} : \ \mathrm{Re}\, u \geqslant T, |\operatorname{Im} u| \leqslant \sqrt{2}\}$ are taken at points of its boundary. Since the functions are real analytic it is enough to bound them in the boundary intersected with $\{u \in \mathbb{C} : \ \operatorname{Im} u \geqslant 0\}$. We consider the larger domain $\{u \in \mathbb{C} : \operatorname{Re} u \geqslant T, |\operatorname{Im} u| \leqslant \sqrt{2} + v\}$, $0 \leqslant v \leqslant 1/2$, and the following segments of its boundary:

$$I_1^1 = \{u \in \mathbb{C} : T \leqslant \operatorname{Re} u \leqslant \rho^{-1}, \operatorname{Im} u = \sqrt{2} + v\},$$
$$I_1^2 = \{u \in \mathbb{C} : \operatorname{Re} u \geqslant \rho^{-1}, \operatorname{Im} u = \sqrt{2} + v\},$$
$$I_2 = \{u \in \mathbb{C} : \operatorname{Re} u = T, 0 \leqslant \operatorname{Im} u \leqslant \sqrt{2} + v\}.$$

We introduce $c^* = \cos((\sqrt{2}+v)\rho)$ and $s^* = \sin((\sqrt{2}+v)\rho)$. If $u = t + (\sqrt{2}+v)i$, then

$$\hat{\alpha}(u) - \alpha_0(u)$$
$$= \frac{k_1\rho^2(t^2 + 2(\sqrt{2}+v)ti - 2\sqrt{2}v - v^2) - 2k_2c^* \cosh(\rho t) - i2k_2s^* \sinh(\rho t) + 2}{[k_2 \cosh(\rho t + \rho(\sqrt{2}+v)i) - 1]\,[2 + (t + (\sqrt{2}+v)i)^2]}.$$

We decompose the numerator as $-(h_1^1 + h_1^2 + h_1^3) + ih_2$, where

$$h_1^1(t) = 2k_2c^*(\cosh(\rho t) - 1 - \rho^2t^2/2),$$
$$h_1^2(t) = 2k_2c^* - 2 + (2\sqrt{2}v + v^2)\rho^2,$$
$$h_1^3(t) = \rho^2((k_2c^* - k_1)t^2 + (k_1 - 1)(2\sqrt{2}v + v^2)),$$
$$h_2(t) = 2(\sqrt{2}+v)k_1\rho^2 t - 2k_2s^* \sinh(\rho t)$$

and we write the denominator as g_1g_2 where

$$g_1(t) = k_2 \cosh(\rho t + \rho(\sqrt{2}+v)i) - 1,$$
$$g_2(t) = 2 + (t + (\sqrt{2}+v)i)^2.$$

We have to bound the corresponding quotients on the segments I_1^1, I_1^2 and I_2.

For that we use the inequalities $x \cosh x \geqslant \sinh x$ for all $x \geqslant 0$, $|z - \sinh z| \leqslant |z|^2 \sinh |z|$ for all $z \in \mathbb{C}$ and $|\cosh z - 1 - z^2/2| \leqslant |z|^4 \cosh |z|$ for all $z \in \mathbb{C}$ as well as the following simple but tedious lemmas:

Lemma 4.9. *Let $\chi_1(t) = \operatorname{Re} g_1(t) = k_2c^* \cosh(\rho t) - 1$. Given $T > \sqrt{3}$ there exists $\rho_0 > 0$ such that if $\rho \in (0, \rho_0)$ then χ_1 is strictly increasing. Therefore*

$$\chi_1(t) \geqslant \chi_1(T) > 0 \quad \textit{for } t \in [T, \infty) \quad \textit{and } \chi_1(T) \geqslant \frac{T^2 - 3}{2}\rho^2 + O(\rho^4).$$

Lemma 4.10. *Let $\chi_2(x) = x^2 \cosh x(k_2c^* \cosh x - 1)^{-1}$. Given $T > 0$ there exists ρ_0 with $\rho_0 T < 1$, such that if $\rho \in (0, \rho_0)$ then*

$$0 < \chi_2(x) \leqslant \max(\chi_2(\rho T), \chi_2(1)) \leqslant C, \qquad x \in [\rho T, 1]$$

with C independent of ρ and T.

Lemma 4.11. *Let* $\chi_3(x) = \sinh x(k_2 c^* \cosh x - 1)^{-1}$. *We have that* $0 < \chi_3(x) \leqslant C\rho^{-1}$ *for all* $x \geqslant \rho T$.

After some calculations we get that $|\hat{\alpha}(u) - \alpha_0(u)| \leqslant M\rho^2$, for $u \in I_1^1 \cup I_1^2$. In an analogous but simpler way we also obtain $|\hat{\alpha}(u) - \alpha_0(u)| \leqslant M\rho^2$, for $u \in I_2$.

To bound $\hat{\beta} - \beta_0$ we use that $\hat{\beta} - \beta_0 = \hat{\alpha}' - \alpha_0'$. Given T, let $D_v^{\mathrm{s}} = \{u \in \mathbb{C} : \operatorname{Re} u \geqslant T - v,\ |\operatorname{Im} u| \leqslant \sqrt{2} + v\}$ with $0 \leqslant v \leqslant 1/2$. Applying Cauchy's theorem with some $v > 0$ we get that, for $u \in D^{\mathrm{s}}$

$$|\hat{\beta}(u) - \beta_0(u)| \leqslant \frac{1}{v} \sup_{v \in D_v^{\mathrm{s}}} |\hat{\alpha}(v) - \alpha_0(v)| \leqslant \frac{1}{v} \mu^2 \varepsilon^{2p+1} M. \qquad \square$$

The next result is proved using analogous estimates.

Proposition 4.12. *We have that*

$$|u^2 e^{(2/3)\rho u} \hat{\alpha}(u)| \leqslant C, \qquad |u^3 e^{(2/3)\rho u} \hat{\beta}(u)| \leqslant C, \qquad u \in D^{\mathrm{s}},$$

with C independent of μ, ε *and T, and*

$$|u^2 \hat{\alpha}(u)| \leqslant 2 + O(\rho^2) + O(1/T^2), \qquad u \in D^{\mathrm{s}}.$$

4.4. The operator $\mathcal{B}$

The Banach spaces we use in this section were defined at the beginning of Section 3. For every $\varepsilon > 0$ we define the operator $\mathcal{B} : \mathcal{X}_k^l \times \mathcal{X}_k^l \to \mathcal{X}_k^l \times \mathcal{X}_k^l$ by the expression

$$(\mathcal{B}\sigma)(t,s) = \sigma(t + 2\pi\varepsilon, s) - A(2\pi\varepsilon)\sigma(t,s),$$

where $\sigma = (\sigma_1, \sigma_2)$ and $A(\theta)$ is defined in (18).

Let k_1, k_2, l_1 and l_2 be positive real numbers. We endow the product space $\mathcal{X} = \mathcal{X}_{k_1}^{l_1} \times \mathcal{X}_{k_2}^{l_2}$ with the norm

$$\|\psi\|_{\mathcal{X}} = \alpha_1 \|\psi_1\|_{k_1,l_1} + \alpha_2 \|\psi\|_{k_2,l_2} \tag{23}$$

with α_1, $\alpha_2 > 0$ to be chosen later on. We note that the product space becomes a Banach space and that the operator $\mathcal{B}$ is a well defined linear continuous operator.

We look for a formal right inverse of $\mathcal{B}$. For that we rewrite the condition $\mathcal{B}\sigma = \psi$ as

$$\sigma(t,s) = -A^{-1}(2\pi\varepsilon)\psi(t,s) + A^{-1}(2\pi\varepsilon)\sigma(t + 2\pi\varepsilon, s). \tag{24}$$

Applying (24) iteratively we obtain

$$\sigma(t,s) = -\sum_{j=0}^{N} A^{-(j+1)}(2\pi\varepsilon)\psi(t + 2\pi\varepsilon j, s) + A^{-(N+1)}(2\pi\varepsilon)\sigma(t + 2\pi\varepsilon(N+1), s). \tag{25}$$

Since

$$A^{-j}(2\pi\varepsilon) = \begin{pmatrix} \cosh(2\pi\varepsilon\rho j) & -\rho^{-1} \sinh(2\pi\varepsilon\rho j) \\ -\rho \sinh(2\pi\varepsilon\rho j) & \cosh(2\pi\varepsilon\rho j) \end{pmatrix} = A(-2\pi\varepsilon j),$$

if $\sigma \in \mathcal{X}_{k_1}^{l_1} \times \mathcal{X}_{k_2}^{l_2}$ with $l_1, l_2 \geqslant 1$ and $k_1, k_2 > 0$, then $A^{-(N+1)}(2\pi\varepsilon)\sigma(t+2\pi\varepsilon(N+1), s) \to 0$ as $N \to \infty$ and thus from (25) we obtain a formal expression for $\mathcal{B}^{-1}$:

$$\sigma(t, s) = (\mathcal{B}^{-1}\psi)(t, s) = -\sum_{j=0}^{\infty} A^{-(j+1)}(2\pi\varepsilon)\psi(t + 2\pi\varepsilon j, s).$$

The following lemma establishes useful bounds for the right inverse of the operator $\mathcal{B}$. From now on we will simply write $A = A(2\pi\varepsilon)$.

Lemma 4.13. *Let $k > 2$ and $l \geqslant 1$. The operator $\mathcal{B}$ has a right inverse $\mathcal{B}^{-1} : \mathcal{X}_k^l \times \mathcal{X}_k^l \to \mathcal{X}_{k-2}^l \times \mathcal{X}_{k-1}^l$ with*

$$\|[\mathcal{B}^{-1}\psi]_1\|_{k-2,l} \leqslant \frac{e^{2\pi\varepsilon\rho}}{2\pi\varepsilon}\left[\frac{1}{(k-1)T}\,\|\psi_1\|_{k,l} + \frac{1}{(k-1)(k-2)}\,\|\psi_2\|_{k,l}\right] + \frac{K}{T}\,\|\psi\|_{\mathcal{X}}$$

and

$$\|[\mathcal{B}^{-1}\psi]_2\|_{k-1,l} \leqslant \frac{1}{2\pi\varepsilon}\,\frac{e^{2\pi\varepsilon\rho}}{(k-1)}\left[\frac{\rho}{2}\|\psi_1\|_{k,l} + \|\psi_2\|_{k,l}\right] + \frac{K}{T}\,\|\psi\|_{\mathcal{X}}$$

for any choice of α_1, α_2 in the definition of $\|\cdot\|_{\mathcal{X}}$, where K is independent of ε.

Proof. We define $\psi_N(t, s) = -\sum_{j=0}^{N} A^{-(j+1)}\psi(t+2\pi\varepsilon j, s)$ and hence $(\mathcal{B}^{-1}\psi)(t, s) = \lim_{N\to\infty} \psi_N(t, s)$. First we claim that if $\psi \in \mathcal{X}_k^l \times \mathcal{X}_k^l$, ψ_N converges uniformly on $D^{\mathrm{s}}(T, \sqrt{2})$. Indeed, from

$$|[A^{-(j+1)}\psi(t + 2\pi\varepsilon j, s)]_1| \leqslant \frac{e^{-\rho l T}}{(T + 2\pi\varepsilon j)^k}\left(\|\psi_1\|_{k,l} + \frac{1}{2\rho}\,\|\psi_2\|_{k,l}\right)$$

and

$$|[A^{-(j+1)}\psi(t + 2\pi\varepsilon j, s)]_2| \leqslant \frac{e^{-\rho l T}}{(T + 2\pi\varepsilon j)^k}\left(\frac{\rho}{2}\,\|\psi_1\|_{k,l} + \|\psi_2\|_{k,l}\right)$$

the claim follows from the M-test of Weierstrass. As a consequence $[\mathcal{B}^{-1}\psi]_1$ and $[\mathcal{B}^{-1}\psi]_2$ satisfy the first three conditions which define $\mathcal{X}_{k-2}^l$ and $\mathcal{X}_{k-1}^l$, respectively.

For $u > 0$ we introduce the auxiliary functions

$$S_1^k(u) = \sum_{j=0}^{\infty} \frac{u^{k-1}}{(u + 2\pi\varepsilon j)^k} = \frac{1}{2\pi\varepsilon}\sum_{j=0}^{\infty} \frac{2\pi\varepsilon}{u}\,\frac{1}{(1 + \frac{2\pi\varepsilon j}{u})^k},$$

$$S_2^k(u) = \sum_{j=0}^{\infty} \frac{2\pi\varepsilon j u^{k-2}}{(u + 2\pi\varepsilon j)^k} = \frac{1}{2\pi\varepsilon}\sum_{j=0}^{\infty} \frac{2\pi\varepsilon}{u}\,\frac{1}{(1 + \frac{2\pi\varepsilon j}{u})^k}\,\frac{2\pi\varepsilon j}{u}$$

and we observe that, for $k > 1$ we have that

$$S_1^k(u) \leqslant \frac{1}{2\pi\varepsilon}\left[\frac{2\pi\varepsilon}{u} + \int_0^{\infty} \frac{1}{(1+x)^k}\,dx\right] \leqslant \frac{1}{u} + \frac{1}{2\pi\varepsilon(k-1)} \qquad (26)$$

and for $k > 2$

$$S_2^k(u) \leqslant \frac{1}{2\pi\varepsilon}\left[\frac{2\pi\varepsilon}{u}\,\frac{1}{(k-1)e} + \int_0^\infty \frac{x}{(1+x)^k}\,dx\right]$$
$$= \frac{1}{(k-1)eu} + \frac{1}{2\pi\varepsilon(k-1)(k-2)}. \tag{27}$$

We write $t + \mathrm{Re}\, s = u$. Let $\psi \in \mathcal{X}_k^l \times \mathcal{X}_k^l$. We have that:

$$\|[\mathcal{B}^{-1}\psi]_1\|_{k-2,l} \leqslant \sup_{u \geqslant T} \sum_{j=0}^{\infty} \frac{u^{k-2}}{(u+2\pi\varepsilon j)^k}\, e^{-2\pi\varepsilon\rho l j} \cosh(2\pi\varepsilon\rho(j+1))\|\psi_1\|_{k,l}$$
$$+ \sup_{u \geqslant T} \sum_{j=0}^{\infty} \frac{u^{k-2}}{(u+2\pi\varepsilon j)^k}\, e^{-2\pi\varepsilon\rho l j}\,\frac{1}{\rho}\,\sinh(2\pi\varepsilon\rho(j+1))\|\psi_2\|_{k,l}.$$

Using that for $x \geqslant 0$, $e^{-x}\cosh x \leqslant 1$ and $\sinh x \leqslant x \cosh x$, and bounds (26) and (27) we obtain

$$\|[\mathcal{B}^{-1}\psi]_1\|_{k-2,l} \leqslant e^{2\pi\varepsilon\rho} \sup_{u \geqslant T}\left[\frac{1}{u}\, S_1^k(u)(\|\psi_1\|_{k,l} + 2\pi\varepsilon\|\psi_2\|_{k,l}) + S_2^k(u)\|\psi_2\|_{k,l}\right]$$
$$\leqslant \frac{e^{2\pi\varepsilon\rho}}{2\pi\varepsilon}\left[\frac{1}{(k-1)T}\,\|\psi_1\|_{k,l} + \frac{1}{(k-1)(k-2)}\,\|\psi_2\|_{k,l}\right] + \frac{K}{T}\,\|\psi\|_{\mathcal{X}},$$

where K depends on α_1, α_2, but can be chosen independently of ε. Analogously we obtain

$$\|[\mathcal{B}^{-1}\psi]_2\|_{k-1,l} \leqslant \sup_{u \geqslant T} \sum_{j=0}^{\infty} \frac{u^{k-1}}{(u+2\pi\varepsilon j)^k}\, e^{-2\pi\varepsilon\rho l j}\rho \sinh(2\pi\varepsilon\rho(j+1))\|\psi_1\|_{k,l}$$
$$+ \sup_{u \geqslant T} \sum_{j=0}^{\infty} \frac{u^{k-1}}{(u+2\pi\varepsilon j)^k}\, e^{-2\pi\varepsilon\rho l j} \cosh(2\pi\varepsilon\rho(j+1))\|\psi_2\|_{k,l}$$

and using that for $x \geqslant 0$, $e^{-x}\sinh x \leqslant 1/2$ and $e^{-x}\cosh x \leqslant 1$ we obtain

$$\|[\mathcal{B}^{-1}\psi]_2\|_{k-1,l} \leqslant e^{2\pi\varepsilon\rho} \sup_{u \geqslant T} S_1^k(u)((\rho/2)\|\psi_1\|_{k,l} + \|\psi_2\|_{k,l})$$
$$\leqslant \frac{1}{2\pi\varepsilon}\left[\frac{e^{2\pi\varepsilon\rho l}}{k-1}\,((\rho/2)\|\psi_1\|_{k,l} + \|\psi_2\|_{k,l})\right] + \frac{K}{T}\,\|\psi\|_{\mathcal{X}},$$

where K depends on α_1, α_2, but can be chosen independently of ε. $\square$

4.5. The fixed point equation

We look for a parameterization $\chi^{s}_{\mu,\varepsilon}(t,s)$ of the stable manifold of Eq. (5) such that $t \in \mathbb{R}$ is the time and $s \in \mathbb{C}$ is a complex parameter. For this we will look for a parameterization of the stable manifold of the Poincaré map $P^{t}_{\mu,\varepsilon}$, which we will denote by $\tilde{\gamma}^{s}_{\mu,\varepsilon}$ by means of imposing the invariance condition:

$$P^{t}_{\mu,\varepsilon}(\tilde{\gamma}^{s}_{\mu,\varepsilon}(t,s)) = \tilde{\gamma}^{s}_{\mu,\varepsilon}(t+2\pi\varepsilon,s). \tag{28}$$

Let $\phi_{\mu,\varepsilon}(t,t_0,w)$ be the flow of the auxiliary system $\dot{z} = Y_{\mu,\varepsilon}(z)$. The following remarks are elementary but provide useful properties of $\hat{\gamma}(u) = (\hat{\alpha}(u),\hat{\beta}(u))$. Since the auxiliary system is autonomous, we have that

$$\hat{P}_{\mu,\varepsilon}(\hat{\gamma}(t+s)) = \phi_{\mu,\varepsilon}(2\pi\varepsilon,0,\hat{\gamma}(t+s)) = \hat{\gamma}(t+s+2\pi\varepsilon). \tag{29}$$

We consider $\hat{\gamma}$ as a first approximation of $\tilde{\gamma}^{s}_{\mu,\varepsilon}$ and therefore we look for $\tilde{\gamma}^{s}_{\mu,\varepsilon}$ of the form,

$$\tilde{\gamma}^{s}_{\mu,\varepsilon}(t,s) = \hat{\gamma}(t+s) + \mu\varepsilon^{p+2}\sigma(t,s)$$

with $\sigma = (\sigma_1,\sigma_2) \in \mathcal{X}^1_4 \times \mathcal{X}^1_5$ and satisfying $\tilde{\gamma}^{s}_{\mu,\varepsilon}(t+2\pi\varepsilon,s) = \tilde{\gamma}^{s}_{\mu,\varepsilon}(t,s+2\pi\varepsilon)$. From condition (28) we will derive a fixed point equation for σ.

In order to simplify the exposition we introduce

$$B(z) = (0, 3x^2 - 4x^3)^T, \qquad Q_2(z) = G^1_2(z,\varepsilon) + \mu^2\varepsilon^{2p}G^2_2(z,\mu,\varepsilon), \tag{30}$$

thus, by Proposition 4.6

$$\hat{P}_{\mu,\varepsilon}(z) = A(2\pi\varepsilon)z + 2\pi\varepsilon B(x) + \varepsilon^2 Q_2(z)$$

and

$$P^{t}_{\mu,\varepsilon}(z) = \hat{P}_{\mu,\varepsilon}(z) + \mu\varepsilon^{p+5}T_3(z,t/\varepsilon).$$

By Taylor's theorem

$$\begin{aligned} P^{t}_{\mu,\varepsilon}(\tilde{\gamma}^{s}_{\mu,\varepsilon}(t,s)) \;=\; & \hat{P}_{\mu,\varepsilon}(\hat{\gamma}(t+s)) + \mu\varepsilon^{p+5}T_3(\hat{\gamma}(t+s),t/\varepsilon) \\ & +\mu\varepsilon^{p+2}D\hat{P}_{\mu,\varepsilon}(\hat{\gamma}(t+s))\sigma(t,s) + \mu^2\varepsilon^{2p+7}DT_3(\hat{\gamma}(t+s),t/\varepsilon)\sigma(t,s) \\ & +\mu^2\varepsilon^{2p+4}R(\sigma)(t,s), \end{aligned} \tag{31}$$

where $R(\sigma)(t,s)$ is defined by (31) and, taking into account that the second derivatives of $\hat{P}_{\mu,\varepsilon}$ and T_3 are bounded independently of μ,ε we get that$|R(\sigma)(t,s)| \leqslant M|\sigma(t,s)|^2$.

Using (29), the condition $P^{t}_{\mu,\varepsilon}(\tilde{\gamma}^{s}_{\mu,\varepsilon}(t,s)) = \tilde{\gamma}^{s}_{\mu,\varepsilon}(t+2\pi\varepsilon,s)$ can be rewritten as

$$\begin{aligned} \sigma(t+2\pi\varepsilon,s) \;=\; & A(2\pi\varepsilon)\sigma(t,s) + 2\pi\varepsilon DB(\hat{\gamma}(t+s))\sigma(t,s) \\ & +\varepsilon^2 DQ_2(\hat{\gamma}(t+s))\sigma(t,s) + \varepsilon^3 T_3(\hat{\gamma}(t+s),t/\varepsilon) \\ & +\mu\varepsilon^{p+5}DT_3(\hat{\gamma}(t+s),t/\varepsilon)\sigma(t,s) + \mu\varepsilon^{p+2}R(\sigma)(t,s). \end{aligned}$$

We introduce the notation

$$\mathcal{G}(\sigma)(t,s) = DQ_2(\hat\gamma(t+s))\sigma(t,s) + \varepsilon T_3(\hat\gamma(t+s),t/\varepsilon) + \mu\varepsilon^{p+3}DT_3(\hat\gamma(t+s),t/\varepsilon)\sigma(t,s) + \mu\varepsilon^p R(\sigma)(t,s) \tag{32}$$

and

$$\mathcal{F}(\sigma) = 2\pi\varepsilon DB(\hat\gamma)\sigma + \varepsilon^2\mathcal{G}(\sigma). \tag{33}$$

We can reduce the problem to finding σ such that

$$\sigma = \mathcal{B}^{-1}\mathcal{F}(\sigma). \tag{34}$$

In the remaining part of this section we endow the product space $\mathcal{X}_{k_1}^{l_1}\times\mathcal{X}_{k_2}^{l_2}$ with the norm

$$\|\psi\|_{\mathcal{X}_{k_1}^{l_1}\times\mathcal{X}_{k_2}^{l_2}} = \|\psi_1\|_{k_1,l_1} + \frac{1}{7}\|\psi\|_{k_2,l_2}. \tag{35}$$

We introduce $\mathcal{X}^* = \mathcal{X}_4^1\times\mathcal{X}_5^1$ and $B(r)\subset\mathcal{X}^*$ the closed ball of radius r of $\mathcal{X}^*$. We look for $\sigma\in\mathcal{X}^*$ satisfying (34).

Lemma 4.14. *If T is big and μ,ε are small, there exists $r>0$ such that the operator $\mathcal{N}$ given by*

$$\mathcal{N}(\sigma) = \mathcal{B}^{-1}\mathcal{F}(\sigma) \tag{36}$$

sends $B(r)$ into $B(r)$ and is a contraction.

Proof. We recall that $\hat\gamma\in\mathcal{X}_2^{2/3}\times\mathcal{X}_3^{2/3}$ and that the norm of $\hat\gamma$ in this space is bounded independently of μ,ε. From Proposition 4.12 we know that $\|\hat\gamma\|_{\mathcal{X}_2^{2/3}\times\mathcal{X}_3^{2/3}}\leqslant C$ with C independent of μ,ε, and $\|\hat\alpha\|_{2,0}\leqslant 2+O(1/T^2)+O(\rho^2)$.

Let $\sigma=(\sigma_1,\sigma_2)\in B(r)\subset\mathcal{X}^*$. Then

$$\begin{aligned}
\|[DB(\hat\gamma)\sigma]_2\|_{6,1} &= \sup_{(t,s)\in D^{\mathrm{s}}}(t+\operatorname{Re}s)^6 e^{\rho(t+\operatorname{Re}s)}|6\hat\alpha(t+s)-12\hat\alpha^2(t+s)|\,|\sigma_1(t,s)| \\
&= 6\sup_{(t,s)\in D^{\mathrm{s}}}(t+\operatorname{Re}s)^2|\hat\alpha(t+s)|\,|1-2\hat\alpha(t+s)| \\
&\quad\times(t+\operatorname{Re}s)^4 e^{\rho(t+\operatorname{Re}s)}|\sigma_1(t,s)| \\
&\leqslant 6(2+O(1/T^2)+O(\rho^2))(1+O(1/T^2))\|\sigma_1\|_{4,1}.
\end{aligned}$$

Therefore $DB(\hat\gamma)\sigma\in\{0\}\times\mathcal{X}_6^1$ and $\|DB(\hat\gamma)\sigma\|_{\mathcal{X}_6^1\times\mathcal{X}_6^1}\leqslant(12+O(1/T^2)+O(\rho^2))\|\sigma\|_{\mathcal{X}^*}$. Proceeding in the same way, using that $Q_2\in\mathcal{P}_2$ and $T_3\in\mathcal{P}_3$, we get that

$$DQ_2(\hat\gamma(t+s))\sigma(t,s)\in\mathcal{X}_6^{5/3}\times\mathcal{X}_6^{5/3},$$

$$T_3(\hat\gamma(t+s),t/\varepsilon)\in\mathcal{X}_6^2\times\mathcal{X}_6^2,\qquad DT_3(\hat\gamma(t+s),t/\varepsilon)\sigma(t,s)\in\mathcal{X}_8^{7/3}\times\mathcal{X}_8^{7/3}$$

and

$$R(\sigma)(t,s) \in \mathcal{X}_8^2 \times \mathcal{X}_8^2.$$

Hence, from definition (33) of $\mathcal{F}$ we have that $\mathcal{F}(\sigma) \in \mathcal{X}_6^{5/3} \times \mathcal{X}_6^{5/3} \subset \mathcal{X}_6^1 \times \mathcal{X}_6^1$. Moreover, the norms of all the previous functions in the corresponding spaces are bounded independently of μ, ε. By Lemma 4.13, $\mathcal{B}^{-1}\mathcal{F}(\sigma) \in \mathcal{X}^*$ and therefore $\mathcal{N}(\sigma) \in \mathcal{X}^*$.

Next we prove that $\|\mathcal{N}(\sigma)\|_{\mathcal{X}^*} < r$ if $\|\sigma\|_{\mathcal{X}^*} \leqslant r$. Indeed, let $\sigma \in B(r) \subset \mathcal{X}^*$, with r small enough, but independent of μ, ε. By definitions (30), (32) and (33) of B, $\mathcal{G}$ and $\mathcal{F}$, respectively, and the previous estimates we have that

$$\begin{aligned}
\|[\mathcal{F}(\sigma)]_1\|_{6,1} &\leqslant M\varepsilon^2\|\sigma\|_{\mathcal{X}^*} + M\varepsilon^3,\\
\|[\mathcal{F}(\sigma)]_2\|_{6,1} &\leqslant 2\pi\varepsilon[12 + O(1/T^2) + O(\rho^2)]\|\sigma\|_{\mathcal{X}^*} + M\varepsilon^2\|\sigma\|_{\mathcal{X}^*} + M\varepsilon^3.
\end{aligned}$$

Therefore by Lemma 4.13 with $k=6$ and $l=1$,

$$\begin{aligned}
\|\mathcal{B}^{-1}\mathcal{F}(\sigma)\|_{\mathcal{X}^*} &= \|[\mathcal{B}^{-1}\mathcal{F}(\sigma)]_1\|_{4,1} + \frac{1}{7}\|[\mathcal{B}^{-1}\mathcal{F}(\sigma)]_2\|_{5,1}\\
&\leqslant \frac{e^{2\pi\varepsilon\rho}}{2\pi\varepsilon}\left[\frac{1}{5T}\|[\mathcal{F}(\sigma)]_1\|_{6,1} + \frac{1}{20}\|[\mathcal{F}(\sigma)]_2\|_{6,1}\right]\\
&\quad + \frac{1}{7}\frac{1}{2\pi\varepsilon}\frac{e^{2\pi\varepsilon\rho}}{5}\left[\frac{\rho}{2}\|[\mathcal{F}(\sigma)]_1\|_{6,1} + \|[\mathcal{F}(\sigma)]_2\|_{6,1}\right] + \frac{K}{T}\|\mathcal{F}(\sigma)\|_{\mathcal{X}_6^1\times\mathcal{X}_6^1}\\
&= \left[\frac{33}{35} + O\left(\frac{\varepsilon}{T}\right) + O(\varepsilon)\right]\|\sigma\|_{\mathcal{X}^*} + O(\varepsilon^2).
\end{aligned}$$

Therefore,

$$\|\mathcal{N}(\sigma)\|_{\mathcal{X}^*} \leqslant \left[\frac{33}{35} + O\left(\frac{\varepsilon}{T}\right) + O(\varepsilon)\right]\|\sigma\|_{\mathcal{X}^*} + O(\varepsilon^2) < r$$

if T is big enough and ε is small enough.

To check that $\mathcal{N}$ is a contraction we have to estimate $\|\mathcal{N}(\bar{\sigma}) - \mathcal{N}(\sigma)\|_{\mathcal{X}^*} = \|\mathcal{B}^{-1}[\mathcal{F}(\bar{\sigma}) - \mathcal{F}(\sigma)]\|_{\mathcal{X}^*}$. The more delicate term to bound is $2\pi\varepsilon\mathcal{B}^{-1}DB(\hat{\gamma})(\bar{\sigma}-\sigma)$.

We have

$$
\begin{aligned}
&\|2\pi\varepsilon\mathcal{B}^{-1}DB(\hat{\gamma})(\bar{\sigma}-\sigma)\|_{\mathcal{X}^*}\\
&\quad = 2\pi\varepsilon\|[\mathcal{B}^{-1}DB(\hat{\gamma})(\bar{\sigma}_1-\sigma_1)]_1\|_{5,1} + \frac{2\pi\varepsilon}{7}\|[\mathcal{B}^{-1}DB(\hat{\gamma})(\bar{\sigma}_1-\sigma_1)]_2\|_{5,1}\\
&\quad \leqslant \frac{e^{2\pi\varepsilon\rho}}{20}\|[DB(\hat{\gamma})(\bar{\sigma}_1-\sigma_1)]_1\|_{6,1} + \frac{1}{7}\frac{e^{2\pi\varepsilon\rho}}{5}\|[DB(\hat{\gamma})(\bar{\sigma}_1-\sigma_1)]_2\|_{6,1}\\
&\qquad + O(\varepsilon/T)\|DB(\hat{\gamma})(\bar{\sigma}_1-\sigma_1)\|_{\mathcal{X}_6^1\times\mathcal{X}_6^1}\\
&\quad \leqslant \left(\frac{33}{35} + O(\varepsilon) + O(\varepsilon/T)\right)\|\bar{\sigma}-\sigma\|_{\mathcal{X}^*}.
\end{aligned}
$$

Studying the remaining terms we conclude that $\mathcal{N}$ is a contraction. □

4.6. *End of the proof of Theorem 3.1*

By Lemma 4.14 we can apply the fixed point theorem and we obtain that there exists a unique $\sigma \in \mathcal{X}^*$ such that

$$
P^t_{\mu,\varepsilon}(\hat{\gamma}(t+s) + \mu\varepsilon^{p+2}\sigma(t,s)) = \hat{\gamma}(t+2\pi\varepsilon+s) + \mu\varepsilon^{p+2}\sigma(t+2\pi\varepsilon, s).
$$

This provides a parameterization of the local stable manifold of system (5) which, in general, is not a solution with respect to t. To have a parameterization which is a solution with respect to t we follow the same scheme as in [BF]. Let T be big enough such that the previous results hold and let $t_1 = T - 2\pi\varepsilon$. We define

$$
\chi^{\mathrm{s}}_{\mu,\varepsilon}(t,s) = \varphi_{\mu,\varepsilon}(t, t_1, \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1, s)), \qquad t > T - 2\pi\varepsilon, \quad \operatorname{Re} s > 2\pi\varepsilon, \quad |\operatorname{Im} s| \leqslant \sqrt{2},
$$

where here $\varphi_{\mu,\varepsilon}(t, t_1, x, y)$ is the general solution of Eq. (5).

For $t > T - 2\pi\varepsilon, \operatorname{Re} s > 2\pi\varepsilon$ we have

$$
\begin{aligned}
\chi^{\mathrm{s}}_{\mu,\varepsilon}(t, s+2\pi\varepsilon) &= \varphi_{\mu,\varepsilon}(t, t_1, \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1, s+2\pi\varepsilon))\\
&= \varphi_{\mu,\varepsilon}(t+2\pi\varepsilon, t_1+2\pi\varepsilon, \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1+2\pi\varepsilon, s))\\
&= \varphi_{\mu,\varepsilon}(t+2\pi\varepsilon, t_1+2\pi\varepsilon, P^{t_1}_{\mu,\varepsilon}(\tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1, s))) = \chi^{\mathrm{s}}_{\mu,\varepsilon}(t+2\pi\varepsilon, s).
\end{aligned}
$$

This relation permits to extend $\chi^{\mathrm{s}}_{\mu,\varepsilon}$ to D^{s} and moreover the extension is a solution of Eq. (5) with respect to t and it is analytic with respect to s.

Now we will check that for $(t,s) \in D^{\mathrm{s}}$, $\chi^{\mathrm{s}}_{\mu,\varepsilon}(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+2} r(t,s)$ with $r(t,s) = O(|t + \mathrm{Re}\, s|^{-2})$. Indeed, let $k \in \mathbb{Z}$ such that $|t - 2\pi\varepsilon k - T| < 2\pi\varepsilon$. Then

$$
\begin{aligned}
\chi^{\mathrm{s}}_{\mu,\varepsilon}(t,s) &= \varphi_{\mu,\varepsilon}(t - 2\pi\varepsilon k, t_1 - 2\pi\varepsilon k, \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1, s)) \\
&= \varphi_{\mu,\varepsilon}(t - 2\pi\varepsilon k, t_1, \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1 + 2\pi\varepsilon k, s)) \\
&= \varphi_{\mu,\varepsilon}(t - 2\pi\varepsilon k, t_1, \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1, s + 2\pi\varepsilon k)) \\
&= \phi_{\mu,\varepsilon}(t - 2\pi\varepsilon k, t_1, \hat{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t_1 + s + 2\pi\varepsilon k)) \\
&\quad + \mu\varepsilon^{p+2} O(|\sigma|) + \mu\varepsilon^{p+5} O(|\tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}|^3) + \mu^2\varepsilon^{2p+4} O(|\tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}|^2) \\
&= \hat{\gamma}(t+s) + \mu\varepsilon^{p+2} O(|t + \mathrm{Re}\, s|^{-4}) \\
&= \gamma_0(t+s) + \mu\varepsilon^{p+2} O(|t + \mathrm{Re}\, s|^{-2}).
\end{aligned}
$$

Going back to the original variables we obtain the result we have stated in Theorem 3.1. $\square$

5. Proof of Theorem 3.2 and Corollary 3.4

Once we have proved Theorem 3.1, Theorem 3.2 and Corollary 3.4 follow from the results in [BF]. For the convenience of the reader we provide with a sketch of the proofs.

5.1. Basic results

The next theorem is proved in [BF] in a more general case. It ensures the existence of flow-box coordinates in a neighborhood of a piece of the homoclinic connection γ_0.

Theorem 5.1 (*Flow-box coordinates*). *There exist a neighborhood U independent of μ, ε of a piece of the stable manifold of the unperturbed system and a canonical change of variables*

$$(x, y, \theta = t/\varepsilon) \in U \mapsto (S, E, \theta) = (\mathcal{S}(x, y, \theta), \mathcal{E}(x, y, \theta), \theta) \in \mathcal{U}$$

of class C^1, 2π-periodic in θ and analytic in the x, y variables, such that it transforms the equations associated to (2) into

$$\dot{S} = 1, \quad \dot{E} = 0$$

and satisfies

$$\mathcal{S}(x, y, \theta) = \mathcal{S}_0(x, y) + O(\mu\varepsilon^{p+1}), \qquad \mathcal{E}(x, y, \theta) = h_0(x, y) + O(\mu\varepsilon^{p+1}).$$

Moreover, given $t_0 \in \mathbb{R}$ and $T \geqslant 0$ big enough, U and $(\mathcal{S}, \mathcal{E})$ can be taken such that for all (t, s) such that $T \leqslant |t + \mathrm{Re}\, s| \leqslant 2T$ and $|\mathrm{Im}\, s| < \sqrt{2}$, the parameterization

$\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t,s)$ *of the local stable manifold belongs to* U *and*

$$\mathcal{S}(\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t,s),t/\varepsilon)=t-t_0+s+\mu\varepsilon^{p+1}\mathcal{X}(s) \qquad \textit{and} \qquad \mathcal{E}(\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t,s),t/\varepsilon)=0$$

with $\mathcal{X}(s_0)=0$ *for some* s_0, *which we can choose freely, depending on initial conditions on the stable curve. Moreover* $\mathcal{X}(s)$ *is analytic and* $2\pi\varepsilon$*-periodic.*

In addition the change $(x,y,\theta)\mapsto(S,E,\theta)$ *is continuous in* $(x,y,\theta,\mu,\varepsilon)$ *and analytic in* (x,y,μ).

The goal of the next theorem is to extend the domain of the parameterization of the unstable manifold until it enters into the domain of the flow-box coordinates. It is proved in [DS2] and applies in our case. Let

$$D^{\mathrm{ext}}_{\varepsilon}=\{(t,s)\in\mathbb{R}\times\mathbb{C}:|t+\mathrm{Re}\, s|\leqslant 2T,\ \ |\mathrm{Im}\, s|\leqslant\sqrt{2}-\varepsilon\}.$$

Theorem 5.2 (*Extension theorem*). *Let* $z(t,s)=(x(t,s),y(t,s))$ *be a family of solutions of*

$$\begin{aligned}\dot{x} &= y+\mu\varepsilon^{p}\partial_y h_1(x,y,t/\varepsilon,\mu,\varepsilon),\\ \dot{y} &= -V'(x)-\mu\varepsilon^{p}\partial_x h_1(x,y,t/\varepsilon,\mu,\varepsilon)\end{aligned}$$

defined for $t_0+\mathrm{Re}\, s=-2T$, *for some* $T>0$, *such that*

$$z(t_0,s)-\gamma_0(t_0+s)-\mu\varepsilon^{p+1}G_{\mu,\varepsilon}(\gamma_0(t_0+s),t_0/\varepsilon)=O(\mu\varepsilon^{p+2}),$$

where $G_{\mu,\varepsilon}$ *is the function such that*

$$\partial_\theta G_{\mu,\varepsilon}(x,y,\theta)=(\partial_y h_1(x,y,\theta,\mu,\varepsilon),-\partial_x h_1(x,y,\theta,\mu,\varepsilon))$$

and has zero average with respect to θ, *and* $(t_0,s)\in D^{\mathrm{ext}}_{\varepsilon}$ *verifies* $t_0+\mathrm{Re}\, s=-2T$.

Let ℓ *be defined by* (4). *We assume hypotheses* H1–H4. *Then, there exist* ε_0, μ_0 *and* K *such that the solution* $z(t,s)$ *can be extended to values of* $t\in[t_0,2T-\mathrm{Re}\, s]$, *with the bound*

$$|z(t,s)-\gamma_0(t+s)|\leqslant K\mu\varepsilon^{p-\ell}$$

for $(t,s)\in D^{\mathrm{ext}}_{\varepsilon}$, $0<\varepsilon\leqslant\varepsilon_0$ *and* $|\mu|\leqslant\mu_0$.

Moreover, if $(t,s)\in D^{\mathrm{ext}}_{\varepsilon}\cap\mathbb{R}^2$, *then* $z(t_0,s)-\gamma_0(t_0+s)=O(\mu\varepsilon^{p+1})$.

5.2. Sketch of the proof of Theorem 3.2

We assume hypotheses H1–H5. By Theorem 5.2, it is clear that the unstable manifold can be extended until it enters the domain of the flow-box coordinates. Therefore for all $t_0\in\mathbb{R}$, the expressions

$$\mathcal{S}^{\mathrm{u}}(s)=\mathcal{S}(\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t,s),t/\varepsilon)-(t-t_0), \qquad \mathcal{E}^{\mathrm{u}}(s)=\mathcal{E}(\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t,s),t/\varepsilon) \tag{37}$$

are well defined for $s\in\mathbb{C}$ such that $T\leqslant t+\mathrm{Re}\, s\leqslant 2T$ and $|\mathrm{Im}\, s|\leqslant\sqrt{2}-\varepsilon$. Moreover, as a consequence of Theorem 5.1, they do not depend on time. We choose t in such a way that $T\leqslant t+\mathrm{Re}\, s\leqslant 2T$. The proof of the following result can be found in [BF,DS2].

Lemma 5.3. *The functions $\mathcal{S}^{\mathrm{u}}$ and $\mathcal{E}^{\mathrm{u}}$ satisfy the following properties*:

(a) *The functions $\mathcal{S}^{\mathrm{u}}(s) - s$ and $\mathcal{E}^{\mathrm{u}}(s)$ are $2\pi\varepsilon$-periodic with respect to s. Hence $\mathcal{S}^{\mathrm{u}}$ and $\mathcal{E}^{\mathrm{u}}$ can be analytically extended for all $s \in \mathbb{C}$ such that $|\operatorname{Im} s| \leqslant \sqrt{2} - \varepsilon$.*

(b) *Moreover, for $s \in \mathbb{R}$, $S = \mathcal{S}^{\mathrm{u}}(s)$ is real analytic and invertible, and its inverse $s = s^{\mathrm{u}}(S)$ satisfies that $s^{\mathrm{u}}(S) - S$ is $O(\mu\varepsilon^{p+1})$ and $2\pi\varepsilon$-periodic in S.*

By Theorem 5.1, in the (S, E) coordinates the local stable manifold can be written as

$$(S, E) = (\mathcal{S}(\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t, s), t/\varepsilon), \mathcal{E}(\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t, s), t/\varepsilon)) = (t - t_0 + s + \mu\varepsilon^{p+1}\mathcal{X}(s), 0) \tag{38}$$

and the local unstable manifold as

$$(S, E) = (\mathcal{S}(\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t, s), t/\varepsilon), \mathcal{E}(\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t, s), t/\varepsilon)) = (t - t_0 + \mathcal{S}^{\mathrm{u}}(s), \mathcal{E}^{\mathrm{u}}(s))$$

for (t, s) such that $|\operatorname{Im} s| \leqslant \sqrt{2} - \varepsilon$ and $T \leqslant t + \operatorname{Re} s \leqslant 2T$.

We consider the Poincaré map $P^{t_0}_{\mu,\varepsilon}(x,y) = \varphi_{\mu,\varepsilon}(2\pi\varepsilon + t_0, t_0, x, y)$, where $\varphi_{\mu,\varepsilon}(t, t_0, x, y)$ is the solution of system (2). Let C^{u} be the restriction to U of the unstable curve of $P^{t_0}_{\mu,\varepsilon}$. It is not difficult to see that C^{u} is parameterizated by $\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t_0, s)$ for $s \in \mathbb{C}$ such that $T \leqslant t_0 + \operatorname{Re} s \leqslant 2T$ and $|\operatorname{Im} s| \leqslant \sqrt{2} - \varepsilon$. Moreover, in the (S, E) coordinates, C^{u} is represented by

$$(S, E) = (\mathcal{S}(\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t_0, s), t_0/\varepsilon), \mathcal{E}(\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t_0, s), t_0/\varepsilon)) = (\mathcal{S}^{\mathrm{u}}(s), \mathcal{E}^{\mathrm{u}}(s)).$$

Next we write C^{u} as a graph of a function which will be called the splitting function. We note that, by property (b) of Lemma 5.3, the relation $S = \mathcal{S}^{\mathrm{u}}(s)$ can be inverted for values of s such that $|\operatorname{Im} s| < \sqrt{2} - \varepsilon$. Let $s = s^{\mathrm{u}}(S)$ be its inverse. Thus the equation

$$\phi(S) = \mathcal{E}^{\mathrm{u}}(s^{\mathrm{u}}(S)) \tag{39}$$

defines C^{u} as the graph of a function ϕ. We note that it is $2\pi\varepsilon$-periodic and hence its domain extends to $\mathbb{R}$.

Since $s^{\mathrm{u}}(S) - S$ is $O(\mu\varepsilon^{p+1})$ and $2\pi\varepsilon$-periodic in S we can introduce the new parameterization for the unstable manifold $\tilde{\gamma}^{\mathrm{u}}_{\mu,\varepsilon}(t, S) = \gamma^{\mathrm{u}}_{\mu,\varepsilon}(t, s^{\mathrm{u}}(S))$ which satisfies the same properties as $\gamma^{\mathrm{u}}_{\mu,\varepsilon}$ does. After this change of parameter, the splitting function defined in (39) can also be represented in the form

$$\phi(S) = \mathcal{E}(\tilde{\gamma}^{\mathrm{u}}_{\mu,\varepsilon}(t, S), t/\varepsilon). \tag{40}$$

Finally, we show that the function ϕ given in (39) can be used to measure some magnitudes related to the splitting and then we will prove the formulas in Theorem 3.2. In the next proposition we prove the existence of primary homoclinic points and we relate the angle between the invariant manifolds and the area of the lobes with the splitting function.

Proposition 5.4. *The function* $\phi : \mathbb{R} \to \mathbb{R}$ *is* $2\pi\varepsilon$*-periodic, real analytic and satisfies the following properties*:

(a) *There exists* $h^{\mathrm{u}} \in \mathbb{R}$ *such that* $\gamma^{\mathrm{u}}_{\mu,\varepsilon}(t, h^{\mathrm{u}}) = \gamma^{\mathrm{s}}_{\mu,\varepsilon}(t, h^{\mathrm{s}})$, *for all t* (*giving a homoclinic orbit*), *with* $h^{\mathrm{s}} = \mathcal{S}^{\mathrm{u}}(h^{\mathrm{u}})$. *For* $n \in \mathbb{N}$, *we define* $h^{\mathrm{s}}_n = h^{\mathrm{s}} + 2\pi\varepsilon n$ *which give homoclinic points. Clearly, for all n,* $\phi(h^{\mathrm{s}}_n) = 0$. *Moreover,* $\phi'(h^{\mathrm{s}}_n)$ *is independent of n, and*

$$\begin{aligned}\phi'(h^{\mathrm{s}}_n) &= \partial_S \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}_n) \wedge \partial_S \tilde{\gamma}^{\mathrm{u}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}_n)(1 + O(\mu\varepsilon^{p+1})) \\ &= \|\partial_S \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}_n)\| \|\partial_S \tilde{\gamma}^{\mathrm{u}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}_n)\| \sin \vartheta(t, h^{\mathrm{s}}_n)(1 + O(\mu\varepsilon^{p+1}))\end{aligned}$$

for all t, where $\wedge$ *denotes the exterior product on* $\mathbb{R}^2$, *and* $\vartheta(t, h^{\mathrm{s}}_n)$ *is the angle between* $\partial_s \tilde{\gamma}^{\mathrm{u}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}_n)$ *and* $\partial_s \tilde{\gamma}^{\mathrm{s}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}_n)$.

(b) *The area of the lobe between the invariant curves is given by*

$$A = \left| \int_h^{\bar{h}} \phi(S)\, dS \right|,$$

where h and $\bar{h}$ *are two consecutive zeros of* $\phi(S)$.

(c) $\phi_0 = \int_{h_n}^{h_n+2\pi\varepsilon} \phi(S)\, dS = 0$.

(d) *For* $S \in \mathbb{R}$, $\phi(S)$ *satisfies the estimate*

$$\phi(S) \equiv \mathcal{E}^{\mathrm{u}}(s^{\mathrm{u}}(S)) = \mu\varepsilon^p M(S, \varepsilon) + O(\mu^2\varepsilon^{2\nu+1}, \mu^2\varepsilon^{\nu+p}, \mu\varepsilon^{p+1})e^{-\sqrt{2}/\varepsilon}.$$

Proof of Proposition 5.4 (*Sketch*)**.** Let $t_0 \in \mathbb{R}$. Since $P^{t_0}_{\mu,\varepsilon}$ is area preserving and it is a perturbation of $P_{0,\varepsilon}$, a map which has a homoclinic connection, $P^{t_0}_{\mu,\varepsilon}$ must have primary homoclinic points in $U \cap \mathbb{R}^2$. Let $h^{\mathrm{u}}, h^{\mathrm{s}} \in \mathbb{R}$ be such that $T \leqslant h^{\mathrm{u}} + t_0, h^{\mathrm{s}} + t_0 \leqslant 2T$ and

$$z^h = \gamma^{\mathrm{s}}_{\mu,\varepsilon}(t_0, h^{\mathrm{s}}) = \gamma^{\mathrm{u}}_{\mu,\varepsilon}(t_0, h^{\mathrm{u}}).$$

By Theorem 5.1, we can choose $s_0 = h^{\mathrm{s}}$ and then $h^{\mathrm{s}} = \mathcal{S}(\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}), t/\varepsilon) - (t - t_0) = \mathcal{S}^{\mathrm{u}}(h^{\mathrm{u}})$, for $t \in \mathbb{R}$ such that $T \leqslant h^{\mathrm{u}} + t, h^{\mathrm{s}} + t \leqslant 2T$. Consequently,

$$\phi(h^{\mathrm{s}}) = \mathcal{E}^{\mathrm{u}}(h^{\mathrm{u}}) = \mathcal{E}^{\mathrm{u}}(\gamma^{\mathrm{s}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}), t/\varepsilon) = 0.$$

Differentiating expressions (40) and (38) with respect to S, using that $\tilde{\gamma}^{\mathrm{u}}_{\mu,\varepsilon}(t, h^{\mathrm{s}}) = \gamma^{\mathrm{u}}_{\mu,\varepsilon}(t, h^{\mathrm{u}})$ and making some elementary computations we get the formula stated in (a).

Property (b) follows from the fact that the change given in Theorem 5.1, which transforms the initial coordinates into the flow-box coordinates, (S, E), is canonical and the Poincaré map is orientation preserving.

We note that, since $P^{t_0}_{\mu,\varepsilon}$ is area preserving, the area of two consecutive lobes (one inner and the other outer) coincide. Therefore, (c) follows from (b).

Finally we prove (d). Estimating the Fourier coefficients of $\mathcal{E}^{\mathrm{u}}$ we can prove that, for $s \in \mathbb{R}$,

$$\mathcal{E}^{\mathrm{u}}(s) - \mathcal{E}^{\mathrm{u}}_0(\varepsilon) = \mu\varepsilon^p M(s, \varepsilon) + O(\mu^2\varepsilon^{2\nu+1}, \mu\varepsilon^{p+1})e^{-\sqrt{2}/\varepsilon},$$

where $\mathcal{E}^{\mathrm{u}}_0(\varepsilon) = \frac{1}{2\pi\varepsilon} \int_0^{2\pi\varepsilon} \mathcal{E}^{\mathrm{u}}(s)\, ds$.

On the other hand, it is clear that the Melnikov function, $M(s,\varepsilon)$, is $2\pi\varepsilon$-periodic with respect to s. We denote by $M_k(\varepsilon)$ its Fourier's coefficients. Using residue theory as in [DS2], or more generally as in [BF], we can prove that

$$\mu\varepsilon^p M_k(\varepsilon) = \mu\varepsilon^\nu 2\pi i J^-_{-k,0}(-i)^\ell |k|^\ell \frac{1}{\ell!} e^{-|k|\sqrt{2}/\varepsilon}(1+O(\varepsilon)) \tag{41}$$

for $k \in \mathbb{Z}\backslash\{0\}$, thus $\mu\varepsilon^p \frac{dM}{dS}(S,\varepsilon) = O(\mu\varepsilon^{\nu-1})e^{-\sqrt{2}/\varepsilon}$. Then, by Taylor's theorem,

$$\begin{aligned}\phi(S) &= \mathcal{E}_0^{\mathrm{u}}(\varepsilon) + \mu\varepsilon^p M(s^{\mathrm{u}}(S),\varepsilon) + O(\mu^2\varepsilon^{2\nu+1}, \mu\varepsilon^{p+1})e^{-\sqrt{2}/\varepsilon} \\ &= \mathcal{E}_0^{\mathrm{u}}(\varepsilon) + \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{2\nu+1}, \mu\varepsilon^{p+1}, \mu^2\varepsilon^{\nu+p})e^{-\sqrt{2}/\varepsilon}.\end{aligned} \tag{42}$$

Since the average of h_1 is zero, $M_0(\varepsilon) = 0$ and by (c), $\phi_0 = 0$. Therefore $\mathcal{E}_0^{\mathrm{u}}(\varepsilon) = O(\mu^2\varepsilon^{2\nu+1}, \mu\varepsilon^{p+1})e^{-\sqrt{2}/\varepsilon}$ and (d) follows from (42). □

The proof of Theorem 3.2 is an immediate consequence of Proposition 5.4. Corollary 3.4 can be proved using (41) and Theorem 3.2.

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