Non-smooth pitchfork bifurcations in a quasi-periodically forced piecewise-linear map

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Abstract

We study a class of one-dimensional family of quasiperiodically forced maps $F_{a,b}(x,\theta) = (f_{a,b}(x,\theta), \theta+\omega)$, where x is real, θ is an angle, and ω is an irrational frequency, such that $f_{a,b}(x,\theta)$ is a real piecewise linear map with respect to x of certain kind. The family depends on two real parameters, $a > 0$ and $b > 0$. For this family, we prove the existence of non-smooth pitchfork bifurcations. For $a < 1$ and any b there is only a continuous invariant curve. For $a > 1$ there exists a smooth map $b = b_0(a)$ such that: a) For $b < b_0(a)$, $f_{a,b}$ has two continuous attracting invariant curves and one continuous repelling one; b) For $b = b_0(a)$ it has one continuous repelling invariant curve and two semicontinuous (non-continuous) attracting invariant curves that intersect the unstable one in a zero-Lebesgue measure set of angles; c) For $b > b_0(a)$ it has one continuous attracting invariant curve. The case $a = 1$ is a degenerate case that is also discussed in the paper. It is interesting to note that this family is a simplified version of the smooth family $G_{a,b}(x,\theta) = (\arctan(ax) + b\sin(\theta), \theta+\omega)$ for which there is numerical evidence of a non-smooth pitchfork bifurcation. Finally, we also discuss the limit case when $a \to \infty$.

Contents

1 Introduction

In this paper we study the dynamics of the quasiperiodically forced map $(\bar{x}, \bar{\theta}) = F_{a,b}(x, \theta)$, where

$$\begin{cases} \bar{x} = h_a(x) + b \sin\theta, \\ \bar{\theta} = \theta + \omega \mod 2\pi, \end{cases} \tag{1}$$

for $(x, \theta) \in \mathbb{R} \times \mathbb{T}^1$, $\omega \notin 2\pi\mathbb{Q}$ and h_a is the continuous piecewise linear map

$$h_a(x) = \begin{cases} -\dfrac{\pi}{2} & \text{if} \quad x \le -\dfrac{\pi}{2a}, \\ ax & \text{if} \quad -\dfrac{\pi}{2a} < x < \dfrac{\pi}{2a}, \\ \dfrac{\pi}{2} & \text{if} \quad x \ge \dfrac{\pi}{2a}. \end{cases} \tag{2}$$

This dynamical system depends on two parameters: b is a real value and a is a real value strictly larger than 0.

The motivation to study this map comes from [JMAT18], that studies the map

$$\begin{cases} \bar{x} = \arctan(ax) + b \sin\theta, \\ \bar{\theta} = \theta + \omega \mod 2\pi, \end{cases} \tag{3}$$

which is a rescaled version of a map already studied in [Jäg03]. First, assume that $a > 1$ and that $b = 0$. In this case, it is clear that the map $x \mapsto \arctan(ax)$ has exactly three fixed points, that are seen in (3) as three (constant) invariant curves, one repelling and two attracting. When b becomes different from zero (but small), the three invariant curves can be continued w.r.t. b (for instance, using the Implicit Function Theorem) and they become three non-constant invariant curves, one repelling and two attracting. Note that (3) is invariant by the symmetry $S : (x, \theta) \mapsto (-x, \theta + \pi)$ and, therefore, if $\theta \mapsto \varphi(\theta)$ is an invariant curve of (3),

$$\varphi(\theta + \omega) = \arctan(a\varphi(\theta)) + b \sin\theta,$$

then $\theta \mapsto -\varphi(\theta + \pi)$ is also an invariant curve of (3). As the system cannot have more than three invariant curves (see also [Jäg03]), the repelling curve is self-symmetric and the attracting curves are one the image of the other by the symmetry S. If a is small enough (but larger than 1), when the value of b increases the three invariant curves meet in a pitchfork bifurcation and, after that, only one self-symmetric attracting curve exists ([Jäg03, JMAT18]). The situation seems to be different when a is large. In this case, when b is increased and the three invariant curve approach, they start to wrinkle and they seem (numerically) to become a strange set when they merge for a critical value b^*. After the merging, if we increase again the value of b, the strange set becomes a smooth (but very wrinkled) attracting curve, see [JMAT18].

The map $F_{a,b}$ given by (1) is a "simplified" version of (3) that aims to approximate the main dynamical features of (1), specially for large values of a. We also note that (1) has the same symmetry $S : (x, \theta) \mapsto (-x, \theta + \pi)$ as (3).

Figure 1 shows the attracting sets of (1) for several values of the parameters. For these plots we have chosen ω to be the golden mean ($\omega = \pi(\sqrt{5} - 1)$) and $a = 4$. The first plot ($b = 1.8$) shows two attracting curves that are 'mirror' images under the symmetry S. As we will show

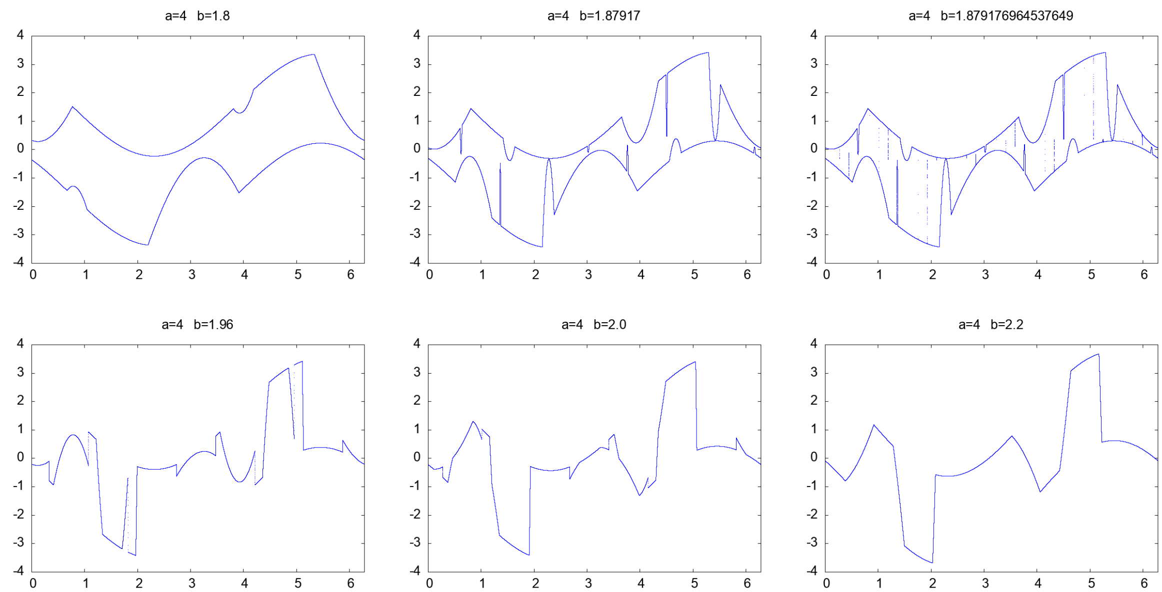

Figure 1: Attracting sets of (1) for different values of the parameters. The horizontal axis is θ and the vertical is x. The frequency ω is the golden mean.

later, there is a repelling self-symmetric invariant curve between these two. When the parameter b is increased, the two attracting curves start to wrinkle and to approach the repelling curve and it seems that they merge into a single self-symmetric attracting curve as it happens in the pitchfork bifurcation but, in this case, the merging process seems to happen through a strange attracting set.

In this paper we prove that, for each $\omega \notin 2\pi\mathbb{Q}$ and for $a > 1$, there exists a critical value $b^* = b^*(a, \omega)$, such that when $0 < b < b^*$ the map $F_{a,b}$ has three continuous invariant curves, two attracting (that are mirror images under S) and one repelling which is self-symmetric. When $b = b^*$ the map has only one continuous curve (which is repelling and self-symmetric), plus two semicontinuous invariant curves that are mirror images under S. Each semicontinuous curve intersects the continuous repelling curve in a dense set of points, and the θ coordinates of these points have zero measure in $[0, 2\pi]$. The semicontinuous curve above the repelling one is upper semicontinuous and the one below the repelling curve is lower semicontinuous. Finally, when $b > b^*$, we show that there exists a unique self-symmetric invariant curve. In other words, we show the existence of a non-smooth pitchfork bifurcation, in the sense that for $b < b^*$ there are three continuous invariant graphs, for $b > b^*$ there is only one continuous graph and for the critical value $b = b^*$ there exists a strange invariant set that can be described as follows: the upper boundary is given by an attracting upper semicontinuous graph and the lower boundary is given by an attracting lower semicontinuous graph. Moreover, the set also contains a continuous repelling curve, and the two semicontinuous attracting graphs intersect the continuous repelling curve on a dense set of values of θ. This set is also invariant by the symmetry S.

The case $0 < a \leq 1$ is also considered. We show that, for $a < 1$ there is only one invariant curve which is attracting. The case $a = 1$ is a bit more involved due to the neutral character of the map around the origin. We show that for $b < b^*(1, \omega)$ there is a one parametric family of invariant curves (parametrized over a nontrivial closed interval) and, for $b \geq b^*(1, \omega)$, there exists

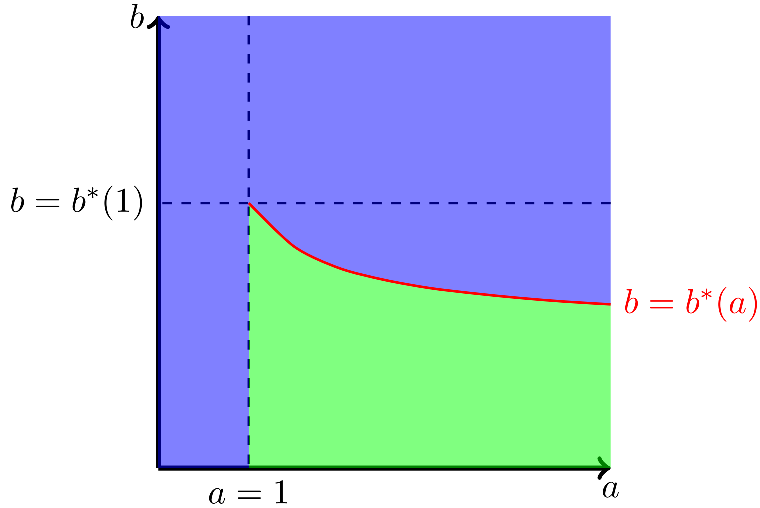

Figure 2: The parameter space (a, b) for the model (1). The blue zone corresponds to a single attracting self symmetric invariant curve, the green zone to three invariant curves, one repelling and self symmetric and two attracting curves that are mirror images under S. The red curve corresponds to SNAs. Crossing the red line when $a > 1$ results in a non smooth pitchfork bifurcation.

a unique attracting invariant curve. All this is summarized in Figure 2 (see also Section 2.4 for another discussion of this figure).

A limit case is when $a \to +\infty$. To study this case we introduce the map

$$h(x) = \begin{cases} -\dfrac{\pi}{2} & \text{if} \quad x < 0, \\ \;\;0 & \text{if} \quad x = 0, \\ \;\;\dfrac{\pi}{2} & \text{if} \quad x > 0, \end{cases} \tag{4}$$

which is the pointwise limit of $h_a(x)$ when $a \to \infty$ (and also the pointwise limit of the $\arctan(ax)$ function that appears in (3)). Figure 3 shows the attracting sets for different values of b. This case is simpler, but it still has some similarities with the previous one. Here, the critical value for b is $\pi/2$ and, as we will see, it is the limit of the previous value $b^*(a, \omega)$ when a tends to infinity. When $h_a = h$ and $b < \pi/2$ the map (1) has two continuous invariant curves,

$$\varphi_0(\theta) = \frac{\pi}{2} + b\sin(\theta - \omega), \qquad \gamma_0(\theta) = -\frac{\pi}{2} + b\sin(\theta - \omega).$$

Both are, in fact, superattracting since all initial conditions land on one of these curves in a finite number of iterates. As before, the curves are mirror images by the symmetry S. When b reaches $\pi/2$ these curves touch $x = 0$ (at $\theta = 3\pi/2 + \omega$ for φ_0 and $\theta = \pi/2 + \omega$ for γ_0). There are two different situations, depending on the sign of $\sin(\pi/2 + \omega)$. If this sign is positive, there are two disjoint attracting curves that are discontinuous on one point, and if it is negative (this is the case when ω is the golden mean), the attracting set is the union of two attracting curves, with a dense set of discontinuities. It is remarkable that, in both cases, none of the attracting curves is invariant. Finally, when $b > \pi/2$, there is only one invariant curve which is attracting

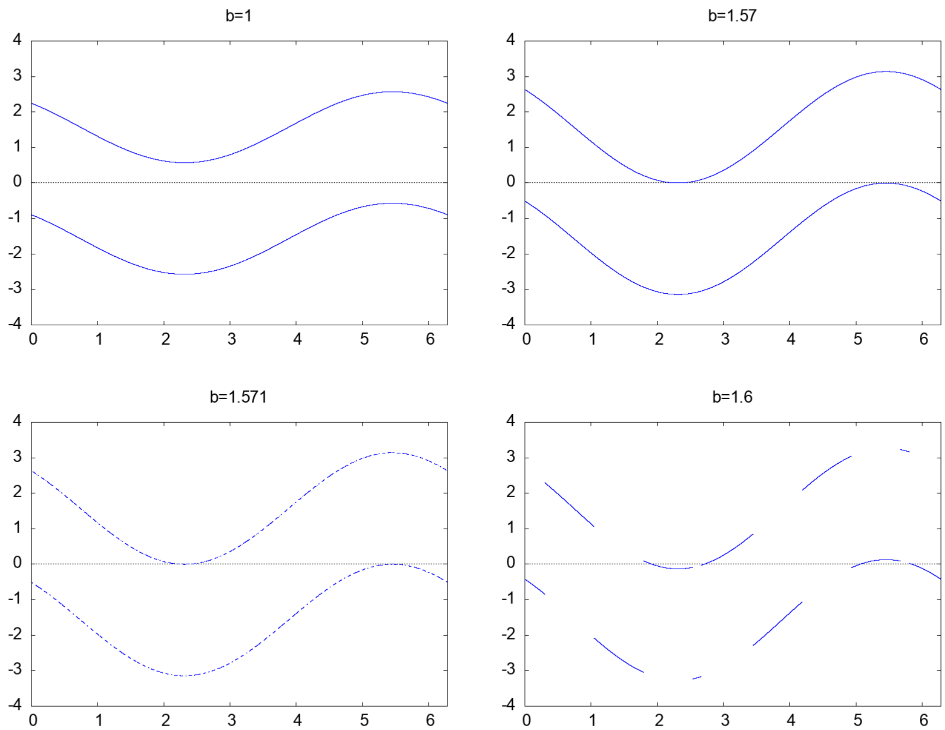

Figure 3: Attracting sets of (4) for different values of the parameters. The horizontal axis is θ and the vertical is x. The frequency ω is the golden mean.

and has a finite number of discontinuities. The number of discontinuities goes to infinity when b tends to $\pi/2$ from above.

Finally, Figure 4 shows the attracting sets of the different maps corresponding to (3), (1) and (4). Each column refers to one of the maps, the first row corresponds to $b = 1$ and the second row to $b = 10$. We have used the value $a = 10$ to show the strong similarities between these maps when a is large. This similarity is one of the motivations to study the piecewise linear map of this paper as an intermediate step to understand (3).

The existence of SNAs in quasiperiodically forced skew-product is known since the examples of [GOPY84] and [Kel96] (see also [JNOT07, LYL$^+$20] and references therein). It is worth noting that numerical methods are quite limited to study SNAs, as pointed out in [JT08]. Studies of non-smooth saddle-node bifurcations in quite general situations can be found in [NDJS11, Fuh16, FGJ18]. A first investigation of non-smooth pitchfork bifurcations in a concrete model is contained in [Gle04].

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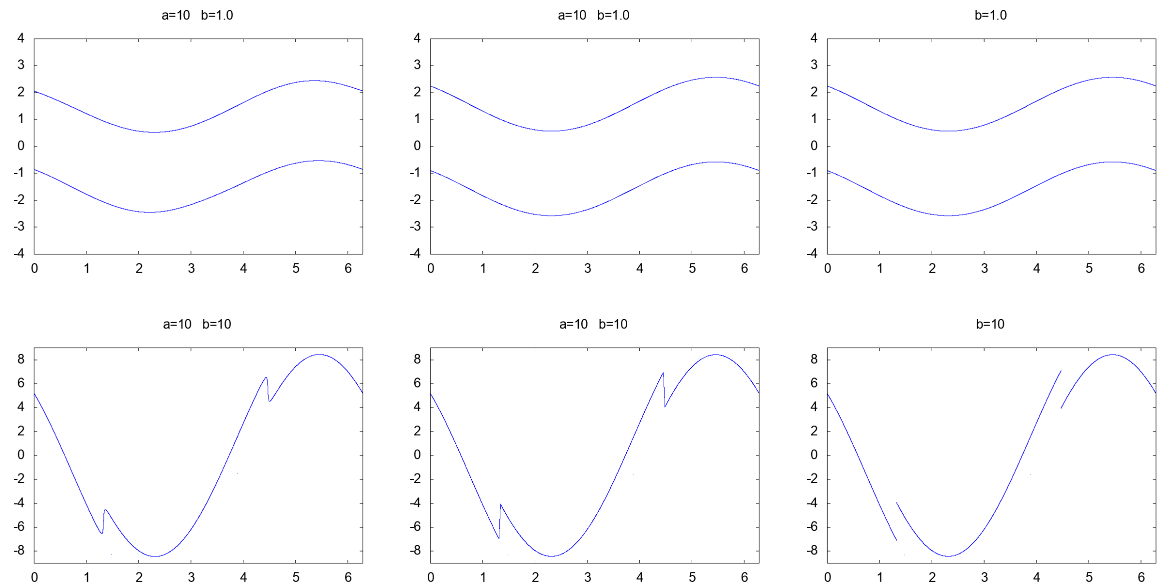

Figure 4: The first, second and third columns contain the attracting sets for the maps (3), (1) and the discontinuous map given by (4), respectively, for different values of the parameters. The horizontal axis is θ and the vertical is x. The frequency ω is the golden mean.

2 A piecewise linear quasiperiodically forced system

In this section we focus on the invariant curves of (1) with h_a defined as in (2), with $\omega \notin 2\pi\mathbb{Q}$. We consider $a > 0$ but, to simplify the reading, the presentation has been split in the cases $a > 1$ (Sections 2.1 and 2.2) and $a \geq 1$ (Section 2.3). As we will see, the values $a = 1$, $b = b^*$ (b^* is defined below) is a critical point in the parameter space (a, b) and it will be discussed in Section 2.4.

Now, let us assume that $a \neq 1$ (and $a > 0$). Suppose that this map has an invariant curve η such that $\eta(\theta) \in [-\pi/(2a), \pi/(2a)]$ for all $\theta \in \mathbb{T}^1$. Imposing the invariant condition,

$$\eta(\theta + \omega) = h_a(\eta(\theta)) + b \sin\theta = a\eta(\theta) + b\sin\theta, \tag{5}$$

we obtain a closed expression for η,

$$\eta(\theta) = -\frac{b\sin\omega}{(\cos\omega - a)^2 + \sin^2\omega}\cos\theta + \frac{b(\cos\omega - a)}{(\cos\omega - a)^2 + \sin^2\omega}\sin\theta. \tag{6}$$

The condition $\eta(\theta) \in [-\pi/(2a), \pi/(2a)]$ implies a restriction for the values of a and b: it is easy to see that the restriction is

$$|b| \leq b^* = b^*(a) = \frac{\pi}{2a}\sqrt{1 - 2a\cos\omega + a^2}. \tag{7}$$

Moreover, there exist two angles $\theta_\pm$ such that $\eta'(\theta_\pm) = 0$. We have that

$$\cos\theta_\pm = \pm\frac{\sin\omega}{\sqrt{1 + a^2 - 2a\cos\omega}}, \qquad \sin\theta_\pm = \mp\frac{\cos\omega - a}{\sqrt{1 + a^2 - 2a\cos\omega}}.$$

and

$$\eta(\theta_\pm) = \mp \frac{b}{\sqrt{1+a^2-2a\cos\omega}}.$$

In the limit case $b = b^*$, we have that $\eta(\theta_\pm) = \mp\frac{\pi}{2a}$ and $\eta'(\theta_\pm) = 0$. This implies this curve is no longer invariant if $|b| > b^*$. Note that, for $a = 1$ and $b < b^*$, η is still an invariant curve but it is not the only invariant curve, and it is unique again for $a = 1$ and $b = b^*$. The details are discussed in Section 2.3.

For any $a > 0$, it follows that the image of the set $\{(x,\theta)$ such that $x \geq \pi/(2a),\ \theta \in \mathbb{T}^1\}$ is the curve $\varphi_0(\theta) = \pi/2 + b\sin(\theta-\omega)$.

Lemma 1. *If $a > 0$ and $0 \leq b \leq b^*$, we have that*

$$\min_{\theta\in\mathbb{T}}(\varphi_0(\theta)-\eta(\theta)) = \frac{\pi}{2} - \frac{ab}{\sqrt{1+a^2-2a\cos\omega}} = a\min_{\theta\in\mathbb{T}}\left(\frac{\pi}{2a}-\eta(\theta)\right).$$

Moreover, the first minimum is attained at the single point $\theta_0 = \theta_- + \omega$ and the second minimum is attained at the single point $\theta_- = \theta_0 - \omega$.

Proof: By taking derivatives, it is easy to see that the angle θ_0 for which we have the minimum distance satisfies:

$$\cos\theta_0 = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}, \qquad \sin\theta_0 = -\frac{\alpha}{\sqrt{\alpha^2+\beta^2}},$$

where

$$\alpha = \frac{ab(1+a\cos\omega-2\cos^2\omega)}{1-2a\cos\omega+a^2}, \qquad \beta = \frac{ab\sin\omega(a-2\cos\omega)}{1-2a\cos\omega+a^2}.$$

Moreover,

$$\varphi_0(\theta_0)-\eta(\theta_0) = \frac{\pi}{2} - \sqrt{\alpha^2+\beta^2},$$

and then

$$\varphi_0(\theta_0)-\eta(\theta_0) = \frac{\pi}{2} - \frac{ab}{\sqrt{1+a^2-2a\cos\omega}}.$$

To see that $\theta_0 = \theta_- + \omega$ we note that (5) implies

$$\varphi_0(\theta)-\eta(\theta) = a\left(\frac{\pi}{2a}-\eta(\theta-\omega)\right), \quad \forall\theta\in\mathbb{T}. \tag{8}$$

As the maximum of η is attained at θ_-, the minimum of $a(\pi/(2a)-\eta(\theta))$ is attained at θ_-, and using (8) we deduce that $\theta_- = \theta_0 - \omega$. □

As a conclusion, we have that $\varphi(\theta_0) = \eta(\theta_0)$ if and only if $b = b^*$, and that the curves are disjoint if $b < b^*$. If $b > b^*$ the two curves meet transversally in two points. Finally, η is self-symmetric by the symmetry of the map $S : (x,\theta) \mapsto (-x, \theta+\pi)$. This symmetry also implies that if we replace $\varphi_0(\theta)$ by $\gamma_0(\theta) = -\frac{\pi}{2} + b\sin(\theta-\omega)$ the distance between γ_0 and η is the same as the distance between φ_0 and η.

Lemma 2. *We recall that $\varphi_0(\theta) = \pi/2 + b\sin(\theta-\omega)$ and that $\gamma_0(\theta) = -\frac{\pi}{2} + b\sin(\theta-\omega)$. Let us define $\lambda_0 = \varphi_0 - \eta$, and*

$$\begin{aligned}
\varphi_{n+1}(\theta) &= h_a(\varphi_n(\theta-\omega)) + b\sin(\theta-\omega),\\
\gamma_{n+1}(\theta) &= h_a(\gamma_n(\theta-\omega)) + b\sin(\theta-\omega),\\
\lambda_{n+1}(\theta) &= h_a(\lambda_n(\theta-\omega)+\eta(\theta-\omega)) - a\eta(\theta-\omega).
\end{aligned}$$

for all $n \geq 0$. Then, if $a > 0$, we have that

1. *$\lambda_n = \varphi_n - \eta$ for any $n \geq 0$,*
2. *if $0 \leq b \leq b^*$, then $\lambda_n \geq 0$,*
3. *the sequences $\{\varphi_n\}_{n\geq 0}$ and $\{\lambda_n\}_{n\geq 0}$ are decreasing and $\{\gamma_n\}_{n\geq 0}$ is increasing.*

Proof:

1. By definition it is true for $n = 0$, so we will proceed by induction: we assume that it is true for $n-1$, and we show it holds for n.

$$\begin{aligned}\lambda_n(\theta) &= h_a(\lambda_{n-1}(\theta-\omega) + \eta(\theta-\omega)) - a\eta(\theta-\omega) \\ &= h_a(\varphi_{n-1}(\theta-\omega)) - \eta(\theta) + b\sin(\theta-\omega) = \varphi_n(\theta) - \eta(\theta),\end{aligned}$$

   where we have used that $\eta(\theta) = a\eta(\theta-\omega) + b\sin(\theta-\omega)$.

2. This follows from Lemma 1.
3. Let us start by showing that $\varphi_1 \leq \varphi_0$:

$$\varphi_1(\theta) = h_a(\varphi_0(\theta-\omega)) + b\sin(\theta-\omega) \leq \frac{\pi}{2} + b\sin(\theta-\omega) = \varphi_0(\theta).$$

   Now, using that h_a is an increasing function we can easily check that if $\varphi_{n-1} \leq \varphi_{n-2}$ then $\varphi_n \leq \varphi_{n-1}$ and the result on $\{\varphi_n\}_n$ follows by induction. To show that $\{\gamma_n\}_n$ is increasing we can use a similar proof, or simply to recall that γ_n is the mirror image of φ_n by the symmetry S. Finally, as $\lambda_n = \varphi_n - \eta$ we have that $\{\lambda_n\}_n$ is decreasing. □

For each $b \in [0,\, b^*]$ we consider the following subsets of $\mathbb{R} \times \mathbb{T}^1$,

$$\begin{aligned}A_+ &= \{(x,\theta) \in \mathbb{R} \times \mathbb{T}^1 \mid \eta(\theta) \leq x \leq \pi/2 + b\sin(\theta-\omega)\}, \\ A_- &= \{(x,\theta) \in \mathbb{R} \times \mathbb{T}^1 \mid -\pi/2 + b\sin(\theta-\omega) \leq x \leq \eta(\theta)\}.\end{aligned}$$

Note that $S(A_+) = A_-$ and viceversa. In what follows we focus on the set A_+, and the results will be translated to A_- by the symmetry S.

Lemma 3. *If $a > 0$ and $|b| \leq b^*$, then $F_{a,b}(A_+) \subset A_+$.*

Proof: If $(x,\theta) \in A_+$ we can distinguish two cases:

1. If $x \geq \pi/(2a)$ then $F_{a,b}(x,\theta) = (\pi/2 + b\sin(\theta), \theta+\omega) \in A_+$.
2. If $\eta(\theta) \leq x \leq \pi/(2a)$ then $\eta(\theta+\omega) = a\eta(\theta) + b\sin(\theta) \leq ax + b\sin(\theta) \leq \pi/2 + b\sin(\theta)$, and this also implies that $F_{a,b}(x,\theta) \in A_+$. □

Then, we have a compact invariant set given by

$$\Lambda_+ = \bigcap_{n=0}^{\infty} F_{a,b}^n(A_+).$$

As, for each θ, $\{\varphi_n(\theta)\}_n$ is a decreasing sequence that is bounded from below (by $\eta(\theta)$), it has a limit. Therefore, the pointwise limit of $\{\varphi_n\}_n$ is an at least upper semicontinuous invariant curve φ_∞ contained in Λ_+. Moreover, $(x,\theta) \in \Lambda_+$ if and only if $\eta(\theta) \leq x \leq \varphi_\infty(\theta)$.

As $\{\lambda_n(\theta)\}_n$ is decreasing and bounded from below by 0, it has a limit $\lambda(\theta) = \lim_{n\to\infty} \lambda_n(\theta)$ that is at least upper semi-continuous. We have

$$\lambda_{n+1}(\theta) = \begin{cases} \pi/2 - a\eta(\theta-\omega) & \text{if} \quad \lambda_n(\theta-\omega) + \eta(\theta-\omega) \geq \pi/(2a), \\ a\lambda_n(\theta-\omega) & \text{if} \quad \lambda_n(\theta-\omega) + \eta(\theta-\omega) \leq \pi/(2a). \end{cases} \tag{9}$$

Now, we define

$$I_n = \{\theta \in \mathbb{T}^1 \mid \lambda_n(\theta-\omega) + \eta(\theta-\omega) \geq \pi/(2a)\}.$$

Note that if $\theta \in I_{n+1}$ then

$$\lambda_n(\theta-\omega) + \eta(\theta-\omega) \geq \lambda_{n+1}(\theta-\omega) + \eta(\theta-\omega) \geq \pi/(2a),$$

which implies that $I_{n+1} \subset I_n$. Then $I = \bigcap_{i=0}^{\infty} I_n \neq \emptyset$ is a compact set such that if $\theta \in I$ then

$$\lambda(\theta) = \frac{\pi}{2} - a\eta(\theta-\omega).$$

2.1 A strange non-chaotic attractor

In this section we assume that $a > 1$ is fixed and that $b = b^*$. As a side comment, it easily follows from (7) that $b^* \to \pi/2$ when $a \to \infty$. The goal here is to show that the repelling continuous invariant curve η and the two attracting invariant curves $\varphi_\infty(\theta)$ and $\gamma_\infty(\theta)$,

$$-\frac{\pi}{2} + b\sin(\theta-\omega) \leq \gamma_\infty(\theta) \leq \eta(\theta) \leq \varphi_\infty(\theta) \leq \frac{\pi}{2} + b\sin(\theta-\omega),$$

are such that $\gamma_\infty(\theta)$ is lower semicontinuous, $\varphi_\infty(\theta)$ is upper semicontinuous and all three curves are different. Each of the curves φ_∞, γ_∞ intersects η on a dense set of zero measure of values of θ. This means that the curves φ_∞, γ_∞ are not continuous everywhere.

Lemma 4. *Let us define θ_0 as the only zero of $\lambda_0 = \varphi_0 - \eta$. Then, λ_n has exactly $n+1$ zeros at $\theta_0, \theta_0+\omega, \ldots, \theta_0+n\omega$.*

Proof: Given any value $n \geq 1$, we note that

$$F^{n-1}_{a,b^*(a)}(\eta(\theta_0-n\omega), \theta_0-n\omega) = (\eta(\theta_0-\omega), \theta_0-\omega),$$
$$F^{n-1}_{a,b^*(a)}(\varphi_0(\theta_0-n\omega), \theta_0-n\omega) = (\varphi_{n-1}(\theta_0-\omega), \theta_0-\omega).$$

As $\eta(\theta_0-\omega) = \pi/(2a)$ and $\varphi_{n-1}(\theta_0-\omega) \geq \eta(\theta_0-\omega)$ we have that

$$F_{a,b^*(a)}(\eta(\theta_0-\omega), \theta_0-\omega) = F_{a,b^*(a)}(\varphi_{n-1}(\theta_0-\omega), \theta_0-\omega)$$

and this implies that $\lambda_n(\theta_0) = \varphi_n(\theta_0) - \eta(\theta_0)$ is zero. To complete the existence of the $n+1$ zeros note that, if θ is a zero of λ_n then from (9) it follows that $\lambda_{n+1}(\theta+\omega) = a\lambda_n(\theta) = 0$.

To see that these are the only zeroes, we start from the fact that λ_0 has only one zero and we proceed by induction. If the result is true for $n \geq 0$ and we assume that λ_{n+1} has an extra zero θ_1 then from (9) there are two possibilities: i) $\pi/2 - a\eta(\theta_1-\omega) = 0$, or ii) $\lambda_n(\theta_1-\omega) = 0$. For the first option, note that $\pi/2 - a\eta(\theta_1-\omega) = \lambda_0(\theta_1)$ which contradicts the fact that λ_0 has

θ_0 as its only zero. The second option implies that λ_n has an extra zero. We can repeat this reasoning until we show (either using i) or ii)) that λ_0 has an extra zero. $\square$

Moreover, we have that $\lambda_0(\theta_0) = 0$ and $\lambda_0(\theta) > 0$, for all $\theta \in \mathbb{T}^1 \setminus \{\theta_0\}$. Then, $\lambda_n(\theta)$ has $n+1$ double zeros at $\theta_0, \theta_0 + \omega, \ldots, \theta_0 + n\omega$. This implies that $\lambda(\theta)$ is equal to zero in all the forward orbit of θ_0. Note that, when $\lambda_n(\theta)$ is very small it means that $\varphi_n(\theta)$ is very close to $\eta(\theta)$ so it implies that $\lambda_{n+1}(\theta + \omega) = a\lambda_n(\theta)$. This justifies to define, by continuity, that $\lambda_{n+1}(\theta + \omega)/\lambda_n(\theta) = a$ when $\lambda_n(\theta) = 0$. Therefore, we define

$$\psi_n(\theta) = \begin{cases} \dfrac{\lambda_{n+1}(\theta+\omega)}{\lambda_n(\theta)} & \text{if } \lambda_n(\theta) \neq 0, \\ a & \text{if } \lambda_n(\theta) = 0. \end{cases} \tag{10}$$

We note that $\psi_n(\theta) > 0$ for all $\theta \in \mathbb{T}^1 \setminus \{\theta_0 - \omega\}$.

Lemma 5. *For $a \geq 1$ and every $n \geq 1$,*

$$0 \leq \psi_{n-1}(\theta) \leq \psi_n(\theta) \leq a,$$

for all $\theta \in \mathbb{T}^1$.

Proof: It is clear that

$$\psi_n(\theta) = \begin{cases} \dfrac{\lambda_0(\theta+\omega)}{\lambda_n(\theta)} & \text{if } \lambda_n(\theta) + \eta(\theta) \geq \pi/(2a), \\ a & \text{if } \lambda_n(\theta) + \eta(\theta) \leq \pi/(2a). \end{cases} \tag{11}$$

Now we have to distinguish several cases, depending on the pieces of the map h_a.

1) $\lambda_{n-1}(\theta) + \eta(\theta) \geq \pi/(2a)$.

   1.a) $\lambda_n(\theta) + \eta(\theta) \geq \pi/(2a)$. In this case,

   $$\psi_{n-1}(\theta) = \frac{\lambda_0(\theta+\omega)}{\lambda_{n-1}(\theta)} \quad \text{and} \quad \psi_n(\theta) = \frac{\lambda_0(\theta+\omega)}{\lambda_n(\theta)}.$$

   Then, as $\lambda_n(\theta) \leq \lambda_{n-1}(\theta)$ we have that $\psi_n(\theta) \geq \psi_{n-1}(\theta)$.

   1.b) $\lambda_n(\theta) + \eta(\theta) \leq \pi/(2a)$. Here,

   $$\psi_{n-1}(\theta) = \frac{\lambda_0(\theta+\omega)}{\lambda_{n-1}(\theta)} \quad \text{and} \quad \psi_n(\theta) = a.$$

   Now, because $\lambda_{n-1}(\theta) + \eta(\theta) \geq \pi/(2a)$ we obtain

   $$\lambda_{n-1}(\theta) \geq \frac{\pi}{2a} - \eta(\theta) = \frac{1}{a}\left(\frac{\pi}{2} - a\eta(\theta)\right) = \frac{1}{a}\lambda_0(\theta+\omega),$$

   and, therefore, $\psi_n(\theta) \geq \psi_{n-1}(\theta)$.

2) $\lambda_{n-1}(\theta) + \eta(\theta) \leq \pi/(2a)$. In this situation, as we also have $\lambda_n(\theta) + \eta(\theta) \leq \pi/(2a)$, we conclude that $\psi_n(\theta) = \psi_{n-1}(\theta) = a$.

To complete the proof, we use (11):

$$\psi_n(\theta) = \frac{\lambda_0(\theta+\omega)}{\lambda_n(\theta)} \le \frac{\lambda_0(\theta+\omega)}{\frac{\pi}{2a} - \eta(\theta)} = \frac{\lambda_0(\theta+\omega)}{\lambda_0(\theta+\omega)/a} = a.$$

□

Lemma 6. *The sequence $\{\psi_n\}_n$ converges pointwise to a measurable and integrable function ψ, and*

$$\int_0^{2\pi} \log \psi(\theta)\, d\theta \le 0.$$

Proof: Let us see first that the functions $\log \psi_n$ are integrable. As Lemma 5 shows that $0 \le \psi_n \le a$ and $\psi_0 \le \psi_n$, we only have to see that $\log \psi_0$ is integrable. This follows from the fact that

$$\psi_0(\theta) = \begin{cases} \dfrac{\pi/2 - a\eta(\theta)}{\pi/2 - a\eta(\theta-\omega)} & \text{if } \lambda_0(\theta) + \eta(\theta) \ge \pi/(2a), \\ a & \text{if } \lambda_0(\theta) + \eta(\theta) \le \pi/(2a), \end{cases}$$

that is, ψ_0 is well defined, continuous and it only takes the value 0 for a single value of $\theta = \theta_0$, which is a zero of multiplicity two. This implies that $\log \psi_0$ is integrable and, therefore, all $\log \psi_n$ are integrable. Moreover, as $\{\lambda_n(\theta)\}_n$ is a decreasing sequence for all θ we have that, for a set of values of θ of full measure (those for which $\lambda_n(\theta) \ne 0$),

$$\psi_n(\theta) \le \frac{\lambda_n(\theta+\omega)}{\lambda_n(\theta)}.$$

Hence, for those values of θ,

$$\log \psi_n(\theta) \le \log \lambda_n(\theta+\omega) - \log \lambda_n(\theta).$$

As $\log \lambda_n$ is an integrable function (the zeroes of λ_n are of multiplicity 2), we have that

$$\int_0^{2\pi} \log \psi_n(\theta)\, d\theta \le \int_0^{2\pi} \log \lambda_n(\theta+\omega)\, d\theta - \int_0^{2\pi} \log \lambda_n(\theta)\, d\theta = 0.$$

Finally, using the dominated convergence theorem, the statement follows. □

Theorem 2.1. *For each $\theta \in \mathbb{T}^1$, let us consider the sequence $\{\varphi_n(\theta)\}_n$.*

a) The sequence $\{\varphi_n(\theta)\}_n$ has a limit for each θ, and the function φ_∞ defined as the point limit

$$\varphi_\infty(\theta) = \lim_{n\to\infty} \varphi_n(\theta),$$

is upper semicontinuous and invariant.

b) The set $A = \{\theta \in \mathbb{T}^1$ such that $\varphi_\infty(\theta) = \eta(\theta)\}$ is dense and it has zero Lebesgue measure.

Proof:

a) We note that φ_∞ is well-defined since, for each θ, the sequence $\{\varphi_n(\theta)\}_n$ is decreasing and bounded from below. This also implies that the function φ_∞ is upper semicontinuous. The invariance follows from the definition of the sequence φ_n,

$$\varphi_{n+1}(\theta) = h_a(\varphi_n(\theta - \omega)) + b^* \sin(\theta - \omega).$$

b) As θ_0 belongs to A, all the points $\theta_0 + k\omega$, $k \in \mathbb{N}$, belong to A. Since $\omega/(2\pi)$ is irrational, A is dense in $\mathbb{T}^1$. To show that A has zero Lebesgue measure we will show first that

$$A \setminus \{\theta_0 - \omega\} \subset \{\theta \in \mathbb{T}^1 \text{ such that } \psi(\theta) = a\}.$$

To see it, let us select an arbitrary point $\theta \in A$, $\theta \neq \theta_0 - \omega$. Then, $\varphi_\infty(\theta) = \eta(\theta) < \pi/(2a)$. Therefore, there exists a value n_0 (depending on θ) such that $\varphi_n(\theta) < \pi/(2a)$ for $n \geq n_0$. This implies that $\lambda_{n+1}(\theta + \omega) = a\lambda_n(\theta)$ and this is equivalent to $\psi_n(\theta) = a$ for all $n \geq n_0$ and, hence, $\psi(\theta) = a$. To complete this part, let us assume that the Lebesgue measure of A is not zero. Then, by ergodicity, A has to have total measure. This implies that the measure of the set of values of θ such that $\psi(\theta) = a$ has total measure and then, since $a > 1$,

$$\int_0^{2\pi} \log \psi(\theta)\, d\theta = \log a > 0,$$

which contradicts Lemma 6. Therefore, A has zero measure. $\square$

Let us now focus on Lyapunov exponents. It is well known that the Lyapunov exponent of an invariant curve ϕ of a smooth map is

$$\Lambda = \frac{1}{2\pi} \int_0^{2\pi} \log(|\partial_x f(\phi(\theta), \theta)|)\, d\theta. \tag{12}$$

In our case, $f(x, \theta) = h_a(x) + b \sin\theta$ and h_a is piecewise linear, and its derivative is well defined except in two points, $x = \pm\pi/(2a)$. Moreover, the attracting curve φ_∞ crosses $x = \pi/(2a)$, which means that we cannot directly write the formula (12) since the derivative w.r.t. x is not well defined at $x = \pi/(2a)$. On the other hand, h_a has left and right derivatives at $x = \pi/(2a)$ equal to a and 0 respectively. The value a corresponds to an expansion ($a > 1$) while 0 corresponds to a compression. For the moment being, let us define the derivative of h_a at $x = \pi/(2a)$ as a and let us compute the Lyapunov exponent of φ_∞. We note that, with this assumption, the curve φ_∞ will be considered "as repelling as possible".

Proposition 1. *The Lyapunov exponent of φ_∞ is $-\infty$.*

Proof: Let us first show that the measure of the set

$$B = \{\theta \in \mathbb{T}^1 \text{ such that } \varphi_\infty(\theta) > \pi/(2a)\},$$

is strictly positive. To see it, let us assume that it is zero. Then, we consider $\lambda = \varphi_\infty - \eta$ and let us show that all the Fourier coefficients of λ, $\lambda^{(k)}$, are zero:

$$\lambda^{(k)} = \frac{1}{2\pi} \int_0^{2\pi} \lambda(\theta) e^{-k\omega\mathbf{i}}\, d\theta = \frac{1}{2\pi} \int_{\mathbb{T}^1 \setminus B} \lambda(\theta) e^{-k\omega\mathbf{i}}\, d\theta.$$

As $\varphi_\infty(\theta) \leq \pi/(2a)$ for $\theta \in \mathbb{T}^1 \setminus B$, the functions φ_∞ and η take values on $[-\pi/(2a), \pi/(2a)]$ and, as they are invariant, we have that $\lambda(\theta+\omega) = a\lambda(\theta)$, which forces that $\lambda^{(k)}e^{k\omega\mathbf{i}} = a\lambda^{(k)}$ and, as $a > 1$, we conclude $\lambda^{(k)} = 0$ and then $\lambda = 0$ a.e., which contradicts *b)*.

Now,

$$\begin{aligned}\Lambda &= \frac{1}{2\pi}\int_0^{2\pi} \log\left(h_a'(\varphi_\infty(\theta))\right)\, d\theta \\ &= \frac{1}{2\pi}\left(\int_B \log\left(h_a'(\varphi_\infty(\theta))\right)\, d\theta + \int_{\mathbb{T}^1\setminus B} \log\left(h_a'(\varphi_\infty(\theta))\right)\, d\theta\right) \\ &= \frac{1}{2\pi}\left((-\infty)\mathrm{meas}\,(B) + (\log a)\mathrm{meas}\left(\mathbb{T}^1 \setminus B\right)\right) = -\infty,\end{aligned}$$

because meas $(B) > 0$. □

Remark 1. *Note that the value assigned to $h'(\pi/(2a))$ is irrelevant as long as it is finite.*

Proposition 2. *There exists a set $E \subset \mathbb{T}^1$ with total Lebesgue measure such that, for any point $(x_0, \theta_0) \in \mathbb{R} \times E$ such that $x_0 > \eta(\theta_0)$, the orbit $(x_{n+1}, \theta_{n+1}) = F_{a,b^*}(x_n, \theta_n)$ satisfies that there exists a natural value $m = m(x_0, \theta_0)$ such that $x_m = \varphi_\infty(\theta_m)$. That is, the orbit starting at (x_0, θ_0) arrives to the attracting curve φ_∞ in a finite number of iterates.*

Proof: Let us choose an arbitrary point (x_0, θ_0) (for the moment being, $\theta_0 \in \mathbb{T}^1$). As $x_0 > \eta(\theta_0)$, we have that $\eta(\theta_n) < x_n$ $(n \geq 0)$. Let us define the rotation $T_\omega : \mathbb{T}^1 \to \mathbb{T}^1$ as $T_\omega(\theta) = \theta + \omega$. We recall that we have defined the set B as

$$B = \{\theta \in \mathbb{T}^1 \text{ such that } \varphi_\infty(\theta) > \pi/(2a)\},$$

Then, we define the set C as

$$C = \bigcup_{n\in\mathbb{N}} T_\omega^{-n}(B).$$

As the Lebesgue measure is invariant and ergodic by T_ω (recall that ω is irrational), using Theorem 1.5 in [Wal82] we obtain that C has total measure. Now, we distinguish several cases.

a) Case $\varphi_\infty(\theta_0) \leq x_0$, $\theta_0 \in C$.

As $\theta_0 \in C$, there exists a value $n_0 \in \mathbb{N}$ such that $\theta_{n_0} = T_\omega^{n_0}(\theta_0) \in B$. Then, we have that $\pi/(2a) < \varphi_\infty(\theta_{n_0}) \leq x_{n_0}$, and this implies that $\varphi_\infty(\theta_{n_0+1}) = x_{n_0+1}$.

b) Case $\eta(\theta_0) < x_0 \leq \varphi_\infty(\theta_0)$, $\theta_0 \in D = \{\theta \in \mathbb{T}^1 \text{ such that } \varphi_\infty(\theta) \neq \eta(\theta)\}$.

We note that this is a total measure set. First, let us note that there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} > \pi/(2a)$ (otherwise, we would have that $x_n - \eta(\theta_n) = a^n(x_0 - \eta(\theta_0))$ and then x_n would become unbounded). Then, $x_{n_0+1} = \varphi_0(\theta_{n_0+1}) \leq \varphi_\infty(\theta_{n_0+1})$ and, on the other hand, $\varphi_0(\theta_{n_0+1}) \geq \varphi_\infty(\theta_{n_0+1})$, which implies $x_{n_0+1} = \varphi_\infty(\theta_{n_0+1})$.

To complete the proof we define $E = C \cap D$, which is also of full measure. □

Remark 2. *As we have mentioned before, we can use the symmetry S to transport these results to the set A_-. So, we can define $\varphi^-(\theta) = -\varphi_\infty(\theta+\pi)$ and then it is a lower semicontinuous function such that the set $\{\theta \in \mathbb{T}^1$ such that $\gamma_\infty(\theta) = \eta(\theta)\}$ is dense and it has zero Lebesgue measure. Moreover, the Lyapunov exponent of γ_∞ is $-\infty$ and the orbit starting at an arbitrary point (x_0, θ_0) such that $x_0 < \eta(\theta_0)$ arrives to γ_∞ in a finite number of iterates.*

2.2 Non-smooth pitchfork bifurcation

In this section we still assume that $a > 1$. The goal here is to show that, for $0 < b < b^*$, the map $F_{a,b}$ has three continuous invariant curves, one repelling and two attracting, and for $b > b^*$, the map $F_{a,b}$ has only one invariant curve which is continuous and attracting. We have already seen in Section 2.1 that, when $b = b^*$, the two attracting curves that existed for $0 < b < b^*$ collide with the repelling curve creating a strange non-chaotic attractor.

2.2.1 Three invariant curves

Let us start with the case $b < b^*$. The existence of a repelling curve η has already been shown at the beginning of Section 2. Note that we only need to show the existence of an attracting invariant curve above η, since by means of the symmetry S we will obtain immediately the existence of another attracting invariant curve below η.

Proposition 3. *If $0 < b < b^*$ there exists a unique continuous attracting invariant curve φ_∞ such that $\eta(\theta) < \varphi_\infty(\theta) \le \varphi_0(\theta) = \pi/2 + b\sin(\theta - \omega)$ for all $\theta \in \mathbb{T}^1$.*

Proof: Let us define $\delta_b > 0$ as (see Lemma 1)

$$\delta_b = \min_{\theta\in\mathbb{T}} \left(\frac{\pi}{2a} - \eta\right) = \frac{\pi}{2a} - \frac{b}{\sqrt{1 + a^2 - 2a\cos\omega}} > 0,$$

and let us also define $\eta_0(\theta) = \eta(\theta) + \delta_b$. Obviously, $\eta_0(\theta) \in [-\pi/(2a), \pi/(2a)]$. Moreover, Lemma 1 also implies that $\eta_0(\theta) \le \varphi_0(\theta)$. Next, let us define the sequences

$$\eta_n(\theta) = h_a(\eta_{n-1}(\theta - \omega)) + b\sin(\theta - \omega), \qquad \varphi_n(\theta) = h_a(\varphi_{n-1}(\theta - \omega)) + b\sin(\theta - \omega),$$

for $n \ge 1$ and for all $\theta \in \mathbb{T}$. Let us see that $\eta_0(\theta) \le \eta_1(\theta)$:

$$\begin{aligned} \eta_1(\theta) &= h_a(\eta_0(\theta - \omega)) + b\sin(\theta - \omega) = a\eta_0(\theta - \omega) + b\sin(\theta - \omega) \\ &= a\eta(\theta - \omega) + a\delta_b + b\sin(\theta - \omega) = \eta(\theta) + \delta_b + (a-1)\delta_b > \eta_0(\theta). \end{aligned}$$

The monotonicity of h_a implies that $\eta_n(\theta) \ge \eta_{n-1}(\theta)$ and $\eta_n(\theta) \le \varphi_n(\theta)$ for all $n \ge 1$ and $\theta \in \mathbb{T}$. Therefore, the sequence $\{\eta_n\}_{n\ge0}$ is increasing and bounded from above, which implies that it is pointwise convergent to a lower semicontinuous invariant curve η_∞. We have seen in Lemma 2 that the sequence $\{\varphi_n\}_{n\ge0}$ is decreasing and bounded from below, which implies that it is pointwise convergent to an upper semicontinuous invariant curve φ_∞.

To complete the proof, we will show that $\eta_\infty = \varphi_\infty$. To this end, let us choose a fixed $\theta_0 \in \mathbb{T}$ and let us consider the images of $(\eta_0(\theta_0), \theta_0)$ by the map $F_{a,b}$. It is clear that $\eta_0(\theta_0) \in [-\pi/(2a), \pi/(2a)]$, and the iterates of this point are defined as

$$F^n_{a,b}(\eta_0(\theta_0), \theta_0) = (\eta_n(\theta_0 + n\omega), \theta_0 + n\omega).$$

Let us see that, after a suitable number (n_0) of iterates of the map $F_{a,b}$, the first component satisfies $\eta_{n_0}(\theta_0 + n_0\omega) > \pi/(2a)$. So, assume that this is not true. This would imply that, for all n, we have $\eta_n(\theta_0 + n\omega) = \eta(\theta + n\omega) + a^n\delta_b$ which is absurd since $a^n\delta_b$ is not bounded. Therefore, by continuity of the maps η_n, there exists a small open neighbourhood I_0 of θ_0 such that, for all $\theta \in I_0$, we have that $\eta_{n_0}(\theta + n_0\omega) > \pi/(2a)$. The definition of the map $F_{a,b}$ implies that the next image of these points satisfies that $\eta_{n_0+1}(\theta + (n_0+1)\omega) = \varphi_0(\theta + (n_0+1)\omega)$ for all $\theta \in I_0$ which means that, on the set I_0, the two limit curves η_∞ and φ_∞ coincide. Finally, as the two curves are invariant and coincide on an open set, they must be equal. Moreover, this attracting invariant curve is continuous. $\square$

2.2.2 One invariant curve

Here we consider the case $b > b^*$. We proceed in a similar way as in the previous section.

Proposition 4. *If $a \geq 1$ and $b > b^*$ there exists a unique, self-symmetric continuous attracting invariant curve.*

Proof: Let us consider the sequences $\{\varphi_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ that have been defined in Lemma 2, with pointwise limits φ_∞ and γ_∞ respectively. We recall that φ_∞ and γ_∞ are related by the symmetry S. To show that $\varphi_\infty = \gamma_\infty$ (this also implies that they are continuous and self-symmetric) we will show that φ_n coincides with γ_n for n large enough. To this end, we will prove first that there exists a point θ_0 such that, for a suitable n_1, φ_{n_1} coincides with γ_{n_1} on an open neighbourhood of θ_0. Then, we will show that this implies that $\varphi_{n_2} = \gamma_{n_2}$ for a suitable $n_2 > n_1$. Let us see the details.

Let θ_0 be the value that minimizes $\varphi_0(\theta) - \eta(\theta)$. According to Lemma 1,

$$\varphi_0(\theta_0) - \eta(\theta_0) = \frac{\pi}{2} - \frac{ab}{\sqrt{1 + a^2 - 2a\cos\omega}} = a \min_{\theta\in\mathbb{T}} \left(\frac{\pi}{2a} - \eta(\theta)\right), \tag{13}$$

and, if $b > b^*$, we have that $\varphi_0(\theta_0) - \eta(\theta_0) < 0$. Let us see that there exists $n_0 \geq 0$ such that $\varphi_{n_0}(\theta_0 + n_0\omega) < -\pi/(2a)$. First, from (13) we see that

$$\varphi_0(\theta_0) - \eta(\theta_0) \leq a\left(\frac{\pi}{2a} - \eta(\theta_0)\right) < \frac{\pi}{2a} - \eta(\theta_0),$$

which implies that $\varphi_0(\theta_0) < \pi/(2a)$. If $\varphi_0(\theta_0) < -\pi/(2a)$ then $n_0 = 0$. Otherwise, let us consider the points $\varphi_n(\theta_0+n\omega)$ and note that they cannot be inside $[-\pi/(2a), \pi/(2a)]$ for all n because, if they are inside this interval they must satisfy that $\varphi_n(\theta_0+n\omega) - \eta(\theta_0+n\omega) = a^n(\varphi_0(\theta_0) - \eta(\theta_0))$ which means that the distance from φ_n to η is unbounded and this is impossible. Therefore, let us define n_0 as the first value of n for which $\varphi_n(\theta_0 + n\omega)$ is outside $[-\pi/(2a), \pi/(2a)]$. Hence, using that $\min(\pi/(2a) - \eta(\theta)) < 0$ we have

$$\varphi_{n_0}(\theta_0 + n_0\omega) - \eta(\theta_0 + n_0\omega) = a^{n_0+1} \min_{\theta\in\mathbb{T}} \left(\frac{\pi}{2a} - \eta(\theta)\right) \leq \min_{\theta\in\mathbb{T}} \left(\frac{\pi}{2a} - \eta(\theta)\right) \leq \frac{\pi}{2a} - \eta(\theta_0 + n_0\omega),$$

which implies that $\varphi_{n_0}(\theta_0 + n_0\omega) \leq \pi/(2a)$ and, as $\varphi_{n_0}(\theta_0 + n_0\omega)$ is outside $[-\pi/(2a), \pi/(2a)]$, we conclude that $\varphi_{n_0}(\theta_0 + n_0\omega) < -\pi/(2a)$.

By continuity, there exists an open neighbourhood I_0 of θ_0 such that $\varphi_{n_0}(\theta+n_0\omega) < -\pi/(2a)$ for all $\theta \in I_0$. This means that $\varphi_{n_0+1}(\theta + (n_0+1)\omega) = \gamma_0(\theta + (n_0+1)\omega)$ for all $\theta \in I_0$. Defining $n_1 = n_0 + 1$, this forces that $\varphi_{n_1}(\theta + n_1\omega) = \gamma_0(\theta + n_1\omega) = \varphi_\infty(\theta + n_1\omega) = \gamma_\infty(\theta + n_1\omega)$ for all $\theta \in I_0$. This implies, as the sequence $\{\gamma_n\}_n$ is increasing and bounded by γ_∞, that $\varphi_{n_1}(\theta + n_1\omega) = \gamma_{n_1}(\theta + n_1\omega)$. Moreover, $\varphi_n(\theta + n\omega) = \gamma_n(\theta + n\omega)$ for $n \geq n_1$ and, as $\omega \notin 2\pi\mathbb{Q}$, we conclude that there exists $n_2 \geq n_1$ such that $\varphi_{n_2} = \gamma_{n_2} = \varphi_\infty = \gamma_\infty$. $\square$

2.3 The case $a \leq 1$

Let us focus first with the case $a < 1$, $b \geq 0$. If $C(\mathbb{T}, \mathbb{R})$ denotes the vector space of continuous functions from $\mathbb{T}$ to $\mathbb{R}$ endowed with the sup norm, it is clear that an invariant curve of $F_{a,b}$ is a fixed point of the map $\mathcal{F} : C(\mathbb{T}, \mathbb{R}) \to C(\mathbb{T}, \mathbb{R})$ defined as

$$\mathcal{F}(\varphi)(\theta) = h_a(\varphi(\theta - \omega)) - b \sin(\theta - \omega).$$

It is immediate to check that $\mathcal{F}$ is a Lipschitz function with Lipschitz constant $a < 1$ so it is contractive. Then, the Banach fixed point theorem implies that there exists a unique invariant curve in $C(\mathbb{T}, \mathbb{R})$.

Now let us consider the case $a = 1$. This is a degenerate case, for instance for $0 < b < b^*$ there is a continuous family of invariant curves: Lemma 1 implies that the invariant curve η defined in (6) is at a finite distance d_b of $x = \pm\pi/2$ and then all the curves $\eta + \delta$ with δ a constant such that $|\delta| \leq d_b$ are also invariant. If $b = b^*$ then $\delta = 0$ and this set of invariant curves reduces to a single one, η. The distance d_b is attained at a some angle θ_0. As $\eta + d_b$ is an invariant curve, we have that $\eta + d_b \leq \varphi_\infty \leq \varphi_0$. Moreover, $\varphi_0(\theta_0) = \eta(\theta_0) + d_b$ and then $\varphi_\infty(\theta_0) = \eta(\theta_0) + d_b$. To see that these are the only invariant curves let us discuss the different options separately.

$a = 1$, $0 < b < b^*$. Let us consider the sequence $\{\lambda_n\}_n$ defined in Lemma 2 and then the sequence $\{\psi_n\}_n$ defined in (10). Lemma 5 shows that the sequence $\{\psi_n\}_n$ is increasing and bounded by 1. Moreover, as $\lambda_n(\theta)$ is never zero, (10) implies that

$$\psi_n(\theta) = \frac{\lambda_{n+1}(\theta + \omega)}{\lambda_n(\theta)} \leq 1, \tag{14}$$

and, as $\lambda(\theta) = \lim_{n\to\infty} \lambda_n(\theta)$, then

$$\lambda(\theta + \omega) \leq \lambda(\theta) \quad \text{for all} \quad \theta \in \mathbb{T}^1. \tag{15}$$

As λ is an upper semicontinuous function, let us see that the previous inequality implies that λ is a constant function. To this end, we choose two arbitrary values $\hat{\theta}$ and $\tilde{\theta}$ to show that $\lambda(\hat{\theta}) \leq \lambda(\tilde{\theta})$: as $\omega \notin 2\pi\mathbb{Q}$ we select an integer sequence $\{k_n\}_n$, $k_n > 0$, such that $\{\hat{\theta} - k_n\omega\}_n$ tends to $\tilde{\theta}$. Using (15) we have that the sequence $\{\lambda(\hat{\theta} - k_n\omega)\}_n$ is increasing and $\lambda(\hat{\theta}) \leq \lambda(\hat{\theta} - k_n\omega)$ for all n. Therefore,

$$\lambda(\hat{\theta}) \leq \lim_{n\to\infty} \lambda(\hat{\theta} - k_n\omega) \leq \lambda(\tilde{\theta}).$$

This scheme can also be used to show that $\lambda(\tilde{\theta}) \leq \lambda(\hat{\theta})$, to conclude that λ is a constant function. Therefore, as we have seen above that $\lambda(\theta_0) = d_b$, we conclude that $\lambda \equiv d_b$. To finish this case, we note that any invariant curve must be either above or below η. If it is above, then it must be contained between φ_n and η for all n, so it must be between φ_∞ and η. Therefore, it must be one of the invariant curves $\eta + \delta$ for $0 \leq \delta \leq d_b$. Finally, if the invariant curve is below η we arrive to the same conclusion by using the symmetry S.

$a = 1$, $b = b^*$. The idea is to prove (15) and then the proof is exactly the same as in the previous item. The main difference to define ψ_n is that we have to take into account that λ can be zero. If θ is a value such that $\lambda(\theta) = 0$, then $\lambda(\theta + \omega) = 0$ and (15) holds. If $\lambda(\theta) \neq 0$ then there exists a n_0 such that if $n \geq n_0$, $\lambda_n(\theta) > 0$. Therefore, (14) holds and then (15) holds.

$a = 1$, $b > b^*$. This case is included in Proposition 4.

2.4 A codimension two bifurcation

It is interesting to compare these results with the ones in [JMAT18] for the model (3). To simplify this comparison, we focus on the case where ω is the golden mean, $\omega = \pi(\sqrt{5} - 1)$. In the model (3), if the parameter $a > 1$ is close enough to 1, there is a smooth pitchfork bifurcation of invariant curves, going from three curves (two attracting and one repelling) for b small that merge into one neutral curve when b reaches a critical value b^* that becomes attracting when $b > b^*$. This picture changes drastically when a is large enough: there is still three smooth curves for b small and one curve for b large, but the neutral bifurcating curve does not look like a continuous curve but a SNA. The paper [JMAT18] contains a numerical computation of the pitchfork bifurcating invariant curve starting from low values of a (where still is a smooth curve) and going up in a from there. It turns out that these bifurcating curves becomes more and more wrinkled until the number of Fourier modes needed to approximate them becomes very large (near 5×10^7) and the computation is stopped. The last computed curve corresponds to $(a, b) = (5.348847, 1.905990)$. The numerical simulations seem to show that, shortly after this point (i.e., for larger values of a), there exists critical values for the parameters, namely $(\hat{a}, \hat{b})$ such that, from this point on, the smooth bifurcating curve is replaced by a SNA.

This situation is very similar to what happens in the model (1), where all the previous calculations are done explicitly in the previous sections. We have seen that

1. the critical value $(\hat{a}, \hat{b})$ is here $(1, \pi\sqrt{2 - 2\cos\omega}/2)$,
2. there exists a curve $(a, b^*(a))$, defined for $a \geq 1$, such that, for $a > 1$, the model has a SNA for each point of this curve,
3. for $a > 1$, if we cross transversally the previous curve there is a nonsmooth pitchfork bifurcation of invariant curves,
4. for $0 < a < 1$ and $b < \hat{b}$, there is only a smooth attracting invariant curve that, if we increase the value of a (and keeping the value of b), this curve becomes three invariant curves after crossing $a = 1$.

These properties are also shown in Figure 2. We note that the critical value $(\hat{a}, \hat{b})$ seems to play the same role in both models. It is remarkable that to unfold all the possible behaviours of the system around this point two parameters are needed.

3 A piecewise constant quasiperiodically forced map

Here we focus on the pointwise limit case when $a \to \infty$ (either in (2) or (3)), that is, the map $(\bar{x}, \bar{\theta}) = F_b(x, \theta)$ defined as

$$\begin{cases} \bar{x} = h(x) + b\sin\theta, \\ \bar{\theta} = \theta + \omega \mod 2\pi, \end{cases} \tag{16}$$

where $b > 0$ and now h is defined as

$$h(x) = \begin{cases} -\dfrac{\pi}{2} & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ \dfrac{\pi}{2} & \text{if} \quad x > 0. \end{cases}$$

As before, we assume ω to be irrational ($\omega \notin 2\pi\mathbb{Q}$). We are interested in the existence of invariant curves, that in this case do not need to be continuous.

It follows from (7) that, when a tends to infinity, the critical value $b^*(a)$ tends to $\pi/2$. It is also immediate to see that, if $0 < b < \pi/2$, the map (16) has two smooth attracting invariant curves given by $\varphi_0(\theta) = \pi/2 + b\sin(\theta - \omega)$ and $\gamma_0(\theta) = -\pi/2 + b\sin(\theta - \omega)$.

3.1 The case $b = \pi/2$

This case is similar to the previous one, in the sense that for a total measure set of values of θ the attractor lies on two disjoint attracting curves, $\varphi_0(\theta) = \pi/2 + (\pi/2)\sin(\theta - \omega)$ and $\gamma_0(\theta) = -\pi/2 + (\pi/2)\sin(\theta - \omega)$. The difference with the case $b < \pi/2$ comes from the orbits for which φ_0 or γ_0 takes the value zero. It is clear that $\gamma_0(\theta)$ is zero when $\theta = \pi/2 + \omega$ and $\varphi_0(\theta)$ is zero when $\theta = 3\pi/2 + \omega$. We distinguish two cases, based on the sign of $\sin(\pi/2 + \omega)$ (note that this value cannot be zero since then ω would belong to $2\pi\mathbb{Q}$).

The first case is when $\sin(\pi/2 + \omega) < 0$ (and, therefore, $\sin(3\pi/2 + \omega) > 0$). In this case there are two disjoint attracting curves. One of these curves coincides with γ_0 except for one point: at $\pi/2 + 2\omega$ it takes the value $(\pi/2)\sin(\pi/2 + \omega)$. The other curve coincides with φ_0 except for the point $3\pi/2 + 2\omega$ where it takes the value $(\pi/2)\sin(3\pi/2 + \omega)$.

The second case is when $\sin(\pi/2 + \omega) > 0$ (and, therefore, $\sin(3\pi/2 + \omega) < 0$). Here, the orbit on the curve γ_0 that reaches the value $x = 0$ for $\theta = \pi/2 + \omega$ is sent to $x = (\pi/2)\sin(\pi/2 + \omega)$ which is negative so the next iterate falls on the curve φ_0 and stays there. Similarly, there is an orbit on φ_0 that ends on γ_0. So the attracting set can be described as the union of two discontinuous curves.

3.2 The case $b > \pi/2$

As we will see, in this case there is only one invariant curve, which is attracting and discontinuous.

Theorem 3.1. *When $b > \frac{\pi}{2}$ the dynamical system* (16) *has a unique discontinuous attracting invariant curve. Moreover,*

a) the number of discontinuities is at least 2,

b) when $b \to (\frac{\pi}{2})^+$ the number of discontinuities tends to infinity.

Proof: As before, we denote $\varphi_0(\theta) = \pi/2 + b\sin(\theta - \omega)$ and $\gamma_0(\theta) = -\pi/2 + b\sin(\theta - \omega)$ that are not invariant since now $b > \pi/2$. It is clear that the set between the two curves,

$$A = \{(x, \theta) \in \mathbb{R} \times \mathbb{T}^1 \mid \gamma_0(\theta) \leq x \leq \varphi_0(\theta)\},$$

is invariant by the map F_b. It is clear that the set

$$\Omega = \bigcap_{n=0}^{\infty} F_b^n(A),$$

is non-empty and satisfies that $F_b(\Omega) = \Omega$. Let us define $\gamma_n(\theta) = h(\gamma_{n-1}(\theta - \omega)) + b\sin(\theta - \omega)$ and $\varphi_n(\theta) = h(\varphi_{n-1}(\theta - \omega)) + b\sin(\theta - \omega)$. Both sequences are bounded, $\{\gamma_n\}_n$ is increasing and

$\{\varphi_n\}_n$ is decreasing. Therefore, we can define the functions γ_∞ and φ_∞ as the pointwise limit of these sequences. Moreover, as γ_0 is the image of φ_0 by the symmetry S (and viceversa), we have that γ_n is also the image of φ_n by S, and then γ_∞ is the image of φ_∞ by S, and viceversa.

If we define the map σ as $\sigma(\theta) = b\sin(\theta-\omega)$ we have that

$$\mathrm{Im}\,(\gamma_\infty) \cup \mathrm{Im}\,(\varphi_\infty) \subset \mathrm{Im}\,(\gamma_0) \cup \mathrm{Im}\,(\varphi_0) \cup \mathrm{Im}\,(\sigma)\,.$$

If θ is such that $\varphi_\infty(\theta) \neq 0$, then we can consider $\varphi_n(\theta) = h(\varphi_{n-1}(\theta-\omega)) + b\sin(\theta-\omega)$ and taking limits to both sides we can obtain $\varphi_\infty(\theta) = h(\varphi_\infty(\theta-\omega)) + b\sin(\theta-\omega)$. Analogously, if $\gamma_\infty(\theta) \neq 0$ we can obtain $\gamma_\infty(\theta) = h(\gamma_\infty(\theta-\omega)) + b\sin(\theta-\omega)$. Let us denote by $\theta_{1,2}^{(0)}$ the two zeros of φ_0 and $\theta_{3,4}^{(0)}$ the two zeros of γ_0. Let us consider the set $\mathcal{T}$ defined as the result of removing the set of values $\{(\theta_j^{(0)} + k\omega) \mod 2\pi \mid k \in \mathbb{N},\ 1 \le j \le 4\}$ from $\mathbb{T}^1$. Let us define the map $\tilde\varphi_\infty$ as follows: if $\theta \in \mathcal{T}$, $\tilde\varphi_\infty(\theta) = \varphi_\infty(\theta)$ and, if $\theta \notin \mathcal{T}$, we define its value recurrently

$$\tilde\varphi_\infty(\theta_j^{(0)} + k\omega) = h(\tilde\varphi_\infty(\theta_j^{(0)} + (k-1)\omega)) + b\sin(\tilde\varphi_\infty(\theta_j^{(0)} + (k-1)\omega)), \qquad k \ge 1.$$

We note that $\tilde\varphi_\infty$ is an invariant, not necessarily continuous, curve. The curve $\tilde\gamma_\infty$ can be defined analogously. On the other hand, there exist an open interval I_1 on which the two functions φ_0 and γ_0 are strictly positive (and hence, due to the symmetry S, they are strictly negative on the interval $I_1+\pi$). This implies that φ_1 and γ_1 coincide on I_1. As

$$\gamma_n(\theta) \le \tilde\gamma_\infty(\theta) \le \tilde\varphi_\infty(\theta) \le \varphi_n(\theta), \qquad \theta \in \mathbb{T}^1, \quad n \ge 0,$$

then $\tilde\gamma_\infty$ and $\tilde\varphi_\infty$ coincide on the interval I_1. As both are invariant, they must coincide. Finally, let us note that $\tilde\gamma_\infty$ and γ_∞ (respectively $\tilde\varphi_\infty$ and φ_∞) are identical except on a finite (and strictly positive) number of values of θ. This implies that γ_∞ and φ_∞ are also identical except on a finite number of points. In summary, we have shown that there exists an attracting invariant curve $\tilde\varphi_\infty$ (or $\tilde\gamma_\infty$ since they are identical) which is unique, and that the number of discontinuities is at least 2, so item a) is proved.

To prove item b), let us define θ_0 as the minimum of φ_0, so $\theta_0 = -\pi/2 + \omega$. The symmetry S implies that $\theta_1 = \pi/2 + \omega$ is the maximum of γ_0. Let us also define the sets

$$\Theta_M^{(i)} = \{\theta_i + k\omega \ (\mathrm{mod}\ 2\pi) \mid k = 1, \ldots, M\}, \quad i = 0, 1.$$

Note that, for all $M \in \mathbb{N}$, the fact $\omega \notin 2\pi\mathbb{Q}$ implies $\Theta_M^{(0)} \cap \Theta_M^{(1)} = \emptyset$ and $0 \notin \Theta_M^{(0)} \cup \Theta_M^{(1)}$ (if not, then $\omega \in 2\pi\mathbb{Q}$). Let us see that, for a given number of discontinuities n_d, there exist a value $b_{n_d} > \pi/2$ such that the attracting curve has at least n_d discontinuities for $b \in (\pi/2, b_{n_d}]$. To this end, we will construct n_d intervals where the attractor is on φ_0 and another n_d intervals, interspersed with the previous intervals, where the attractor is on γ_0. This will guarantee the existence of more than n_d discontinuities. To this end, let us choose a value M_{n_d} such that each of the sets $\Theta_M^{(0)}$ and $\Theta_M^{(1)}$ contain n_d points that are interspersed with points of the other set: in other words, there exist angles $0 < \alpha_1 < \cdots < \alpha_{n_d} < 2\pi$ in $\Theta_M^{(0)}$ and $0 < \beta_1 < \cdots < \beta_{n_d} < 2\pi$ in $\Theta_M^{(1)}$ such that $\alpha_j < \beta_j$ for $j = 1, \ldots, n_d$. The existence of these angles is ensured by the fact that both $\Theta_M^{(0)}$ and $\Theta_M^{(1)}$ fill densely the circle when $M \to \infty$. The final step is to construct small open intervals, I_{α_j}, I_{β_j}, with $\alpha_j \in I_{\alpha_j}$ and $\beta_j \in I_{\beta_j}$ and such that the attractor is on γ_0 for I_{α_j} and on φ_0 for I_{β_j}. Let us start by defining the intervals $I_{\theta_i} = (\theta_i - \delta, \theta_i + \delta)$, $i = 0, 1$. Let $\delta > 0$

be such that all the intervals $I_{\theta_i+k\omega}$ ($i = 0, 1$, $0 \leq k \leq M_{n_d}$) are disjoint. We note that δ goes to zero when n_d goes to infinity. Now, let us choose a value of b, b_{n_d}, such that the interval centered in θ_0 and of radius δ, I_{θ_0}, is the largest open interval centered in θ_0 such that $\varphi_0(I_{\theta_0}) < 0$. By symmetry, $I_{\theta_1} = I_{\theta_0} + \pi$ is the largest open interval centered in θ_1 such that $\gamma_0(I_{\theta_1}) > 0$. Then, the attracting invariant curve $\tilde{\varphi}_\infty$ satisfies that $\tilde{\varphi}_\infty(\theta) < 0$ if $\theta \in I_{\theta_0} + k\omega$ and $\tilde{\varphi}_\infty(\theta) > 0$ if $\theta \in I_{\theta_1} + k\omega$, $1 \leq k \leq M_{n_d}$. In particular, as n_d of these intervals are interspersed, we ensure the existence of n_d discontinuities. We note that this number of discontinuities is ensured for any value of b such that $\pi/2 < b < b_{n_d}$. □

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