Persistence of hyperbolic solutions of ODE's under functional perturbations: Applications to the motion of relativistic charged particles

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Abstract

We rigorously construct a variety of orbits for certain delay differential equations, including the electrodynamic equations formulated by Wheeler and Feynman in 1949. These equations involve delays and advances that depend on the trajectory itself, making it unclear how to formulate them as evolution equations in a conventional phase space. Despite their fundamental significance in physics, their mathematical treatment remains limited.

Our method applies broadly to various functional differential equations that have appeared in the literature, including advanced/delayed equations, neutral or state-dependent delay equations, and nested delay equations, under appropriate regularity assumptions.

Rather than addressing the notoriously difficult problem of proving the existence of solutions for all the initial conditions in a set, we focus on the direct construction of a diverse collection of solutions. This approach is often sufficient to describe physical phenomena. For instance, in certain models, we establish the existence of families of solutions exhibiting symbolic dynamics.

Our method is based on the assumption that the system is, in a weak sense, close to an ordinary differential equation (ODE) with "hyperbolic" solutions as defined in dynamical systems. We then derive functional equations to obtain space-time corrections.

As a byproduct of the method, we obtain that the solutions constructed depend very smoothly on parameters of the model. Also, we show that many formal approximations currently used in physics are valid with explicit error terms. Several of the relations between different orbits of the ODE persist qualitatively in the full problem.

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Contents

1. Introduction

We consider the problem of finding trajectories $x : \mathbb{R} \to \mathbb{R}^n$ solving an equation of the form:

$$\dot{x}(t) = f \circ x(t) + \varepsilon P(t, x_t, \varepsilon, \mu), \tag{1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function (possibly only defined on an open subset in $\mathbb{R}^n$), ε a perturbative parameter, μ an additional parameter, $P\colon \mathbb{R} \times \mathscr{R}[-h, h] \times (0, 1)^2 \to \mathbb{R}^n$ is a smooth map. Here, $\mathscr{R}$ is a space of differentiable functions which will be specified later. The notation x_t is a "history segment" of size $h \geqslant 0$ of the solution which is defined as:

$$x_t(s) \stackrel{\text{def}}{=} x(t + s), \qquad s \in [-h, h]. \tag{2}$$

The term P changes the nature of (1) for $\varepsilon \neq 0$. Hence, in spite of having the small parameter ε, the perturbation is not small. Even for common delays

$$\dot{x}(t) = f \circ x(t) + \varepsilon x(t - 1),$$

the natural phase space for $\varepsilon \neq 0$ is infinite dimensional, but for $\varepsilon = 0$ it is just $\mathbb{R}^n$. Our result can also deal with equations like $\dot{x}(t) = f \circ x(t) + \varepsilon \dot{x}(t - 1)$ which makes the unboundedness even more apparent.

We will take solutions of (1) for $\varepsilon = 0$ which are *"hyperbolic"* in the sense of dynamical systems (see Definition 2.1) and show that one can find solutions that resemble them after corrections (both in the positions occupied and the speed of travel) when $\varepsilon \neq 0$. This allows to show that the equations (1) contain sets of solutions that support symbolic dynamics.

In Section 4.3, we formulate the precise regularity assumptions on P. We anticipate that, roughly, the main assumptions are that applying P to functions with $\ell + 1$ derivatives produce functions with ℓ derivatives and that there are some Lipschitz bounds in C^0 when the arguments lie in spaces of smooth functions.

A case that served as a motivation for us is electrodynamics of point charges. See Section 7.6. In the model of [WF49], the particles move in the Liénard-Wiechert potentials (relativistic analogues of Coulomb-Ampere formulas) generated by the other particles. This leads to advanced/delayed equations with several delays which are obtained solving implicit equations that involve the trajectories. Given the physical importance of these equations, there have been several results establishing existence of solutions in the literature, mainly in the one dimensional case (see later). Other models of electrodynamics (notably several versions of the Post-Newtonian formalism) can be accommodated. The results here provide existence of various solutions for all of them and allow to discuss how approximate are the solutions. We note that the effect of the delays/advances are formally of size inversely to the speed of light, $1/c$, which is much larger than radiation effects that are of size $(1/c)^3$.

Our results also cover other cases in the literature in which the perturbation is clearly singular. Including for an arbitrary $\vartheta \in \mathscr{R}[-h, h]$,

- $P(t, \vartheta, \varepsilon, \mu) = g \circ \vartheta(0)$, an ODE perturbation as $P\big(t, x_t, \varepsilon, \mu\big) = g \circ x(t)$;
- $P(t, \vartheta, \varepsilon, \mu) = \vartheta(-1)$, a perturbation with a constant delay as $P\big(t, x_t, \varepsilon, \mu\big) = x(t - 1)$;
- $P(t, \vartheta, \varepsilon, \mu) = \frac{1}{\varepsilon} f \circ \vartheta(-\varepsilon) - \frac{1}{\varepsilon} f \circ \vartheta(0)$, the small delay system $\dot{x}(t) = f \circ x(t - \varepsilon)$;
- $P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ r \circ \vartheta(0)$, a state-dependent delay perturbation as $P\big(t, x_t, \varepsilon, \mu\big) = x(t + r \circ x(t))$;
- $P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ r \circ \vartheta \circ r_1 \circ \vartheta(0)$ containing nested delays as $P\big(t, x_t, \varepsilon, \mu\big) = x(t + r \circ x(t + r_1 \circ x(t)))$;
- $P(t, \vartheta, \varepsilon, \mu) = (\frac{d}{ds}\vartheta)(0)$, an implicitly defined ODE as $P\big(t, x_t, \varepsilon, \mu\big) = \dot{x}(t)$;
- $P(t, \vartheta, \varepsilon, \mu) = (\frac{d}{ds}\vartheta)(-1)$, a neutral equation with a constant delay as $P\big(t, x_t, \varepsilon, \mu\big) = \dot{x}(t - 1)$;

- $P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ r \circ (\frac{d}{ds}\vartheta)(0)$, a first order neutral equation as $P\big(t, x_t, \varepsilon, \mu\big) = x(t + r \circ \dot{x}(t))$;
- $P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ \tau(t)$ containing an explicit time-dependent delay as $P\big(t, x_t, \varepsilon, \mu\big) = x(t + \tau(t))$;
- $P(t, \vartheta, \varepsilon, \mu)$ obtained by solving an implicit equation of ϑ, e.g. (60) in electrodynamics;
- a bounded time-dependent map P, so that we have a non-autonomous perturbation;
- $P(t, \vartheta, \varepsilon, \mu) = \vartheta(-1) + \vartheta(-2) + \vartheta(+1)$, a perturbation containing several delays or advances.

In the theory of delay differential equations (DDE's) with constant delay, it is customary to take $\mathscr{R}$ to be the space of continuous functions, whereas here we find it useful to consider spaces of more differentiable functions so that the functional P can involve first derivatives (neutral equations) or more complicated forms. Note that we are considering "history segments" that include both the past and the future of the trajectory so that our theory works just as well for delayed, advanced, or mixed expressions. The value h will by default belong to $[0, +\infty)$ even if some existence results may hold for $h = \infty$. The assumption $h < \infty$ seems to be essential for the local uniqueness and for the a-posteriori results.

1.1 Informal main results

Our results show that, if equation (1) admits a set consisting of uniformly hyperbolic solutions (see Definition 2.1) when $\varepsilon = 0$, under appropriate regularity assumptions on the functional P, equation (1) admits a set of solutions which are close to the hyperbolic solutions of the ODE as long as ε is small enough. Our result is similar to structural stability interpreted in the functional analysis formulation of the perturbed problem (1), see Theorem 4.8. We stress that our approach does not need to discuss the phase space of solutions to solve any possible initial value problem of (1), what we do is to search for solutions with a specific structure. In this work we look for solutions of a form based on the uniformly hyperbolic solution of the ODE. We consider a space of functions $x(t)$ of a specific form and formulate equations which imply that such x satisfies (1).

The above strategy bypasses the study of general existence, uniqueness and dependence on initial conditions of the solutions. The solutions of (1) we construct could fail to be surrounded by other solutions. This strategy was used already in [HdlL17, HDlL16, HY20, YGdlL21, YGdlL22]. We also note that the equations considered are numerically well conditioned and can be implemented to produce approximate solutions.

The existence of hyperbolic sets (collections of hyperbolic orbits) in differential equations is a rather common situation. Notably, existence of a transverse homoclinic intersections implies the existence of a uniformly hyperbolic set (the horseshoe) which has a very rich dynamics including an uncountable set of hyperbolic orbits described by symbolic dynamics. Other famous attractors (Lorenz, Rössler, Chua, ...) have also been documented. These attractors include uniformly hyperbolic sets. Our results imply that all these uniformly hyperbolic sets persist when we add a perturbation with a sufficiently small parameter. The perturbations allowed are very general and include perturbations that are singular from a conventional point of view. See an informal presentation in Theorem 1.1. A precise formulation is in Theorem 4.8.

We call attention to [LW95] which uses Poincare returns to establish persistence of hyperbolic sets in C^1 perturbations of Functional Differential Equations that generate a C^1 evolution. In [LWW16], existence of chaotic motion was established for an SDDE analyzing the evolution and finding an analogue of Shilnikov phenomenon. In [WZ05], the authors focused on small constant delay perturbations of an ODE and obtained persistence of topological horseshoes. In contrast, the present method is not based on analysis of evolution and applies to advanced/delayed equations that do not define any evolution. Indeed, we do not need to study regularity properties of the evolution and not even the space where evolution is defined (which may involve the study of solution manifolds).

We also call attention to the papers [HBC+16] and [CHK17]. These papers use numerics and bifurcation analysis to study singularly perturbed state-dependent delay equations, where they also find that state-dependence of the delays can generate very complex dynamics. We think it would be interesting to reformulate our fixed point problems so that they could validate the new solutions found and, specially the bifurcation point (where hyperbolicity is lost).

We are not assuming many properties of the hyperbolic sets beyond requiring that the hyperbolicity constants are uniform (sometimes called Pesin sets [BP13] in non-uniform hyperbolic theory). The hyperbolic sets we considered may fail to be closed or *maximal*.

The proofs are rater explicit and the sizes of the perturbations allowed are computable in concrete examples. (Some related calculations were achieved in [GLMY23].)

One downside of the program presented here is that, by design, we cannot discuss properties of the evolution for all initial data. Nevertheless, one could remark that, even in the qualitative theory of ODE's one often relies only on developing landmarks that organize the behavior of all the solutions. The analogue of the qualitative theory in our program would be to develop a theory indicating that solutions of a certain kind imply the existence of others. In that respect, we hope to come back to the study of stable manifolds and the study of existence of symbolic dynamics from some finite calculations.

Another downside of the present treatment is that we are constrained by the regime of solutions close to the solutions of the ODE's. It is well known that many of the equations we study will have many solutions which do not resemble the solutions of the ODE. Nevertheless, in physical applications (e.g. motions of charged particles) the delays are small so that the effects of the delay are small and hard to observe. (This is why relativity was only discovered in the XX century).

Our results will apply to the space of functions that are finitely differentiable with finite norm in a segment, namely $[-h, h]$, where $h > 0$ is the domain of the "history segment". Notice that if we have a finite differentiable function $x\colon [-\tilde{h}, \tilde{h}] \to \mathbb{R}^n$ in a slightly bigger history segment, $\tilde{h} > h$, then the function x_t in (2) is defined for all $t \in [-(\tilde{h} - h), \tilde{h} - h]$. Therefore, we can apply functionals to all x_t in an open interval of t.

To give in a glimpse of the precise main result on this paper (see Theorem 4.8), let us first provide an informal result omitting many precise formalism: The result shows that under a mild set of hypotheses on the perturbation, the system (1) has solutions nearby the hyperbolic orbit of the unperturbed ODE and such solution will be unique in a suitable neighborhood.

Theorem 1.1 (informal result)**.** *We consider perturbation of an ODE as in* (1) *and let $\ell \geqslant 0$ be an integer. Assume that:*

1. *The unperturbed ODE admits a solution $\{x_0(t)\}_{t\in\mathbb{R}}$ which is uniformly hyperbolic, see Definition 2.1.*
2. *The function f is uniformly bounded as well as its derivatives up to order $\ell + 3$ in a δ-neighborhood of the orbit $\{x_0(t)\}$.*
3. *The perturbation functional P in* (1) *satisfies "propagated bounds" (i.e. when x_t ranges in a ball in a space of $C^{\ell+2}$ functions, P lies in a ball of $C^{\ell+1}$ functions).*
4. *The functional P is Lipschitz in a low regularity, i.e. for all u and v in a $C^{\ell+2}$ ball and all t and s, there are constants B_1, B_2 such that*

$$|P(t, u, \mu) - P(s, v, \mu)| \leqslant B_1|t - s| + B_2\|u - v\|_{C^1}.$$

Then there is an $\varepsilon_0 > 0$ such that for all $|\varepsilon| \leqslant \varepsilon_0$, there exist differentiable maps $\widehat{x}^\varepsilon\colon \mathbb{R} \to \mathbb{R}^n$ and $\phi^\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that $\widehat{x}^\varepsilon$ is in $C^{\ell+1}$, $\mathrm{D}\phi^\varepsilon$ is in C^ℓ, and

$$x(t) = (x_0 + \widehat{x}^\varepsilon) \circ \phi^\varepsilon(t) \tag{3}$$

is a $C^{\ell+1}$ solution of (1)*. Moreover, if* (1) *depends smoothly on parameters in an appropriate sense, so does the new solution x.*

The informal Theorem 1.1 is slightly different from the formal (see Theorem 4.8). We simplified the statement to avoid introducing $C^{\ell+\mathrm{Lip}}$ spaces and we did not include some technical considerations. All of these are discussed in Theorem 4.8 along with the a-posteriori formulation. We also omitted several properties that are used to prove smooth dependence on parameters, see Section 4.3.1.

Remark 1.2 (on perturbative regularity loose). Note that we are allowing that the functional P appearing as a perturbation *looses one derivative*, which means P may have one derivative less than its second argument. This will be key to apply the result to neutral differential equations and to equations with small delay.

Remark 1.3 (on unknown corrections). We have two unknowns in our existence problem: $\widehat{x}$ acting as an additive space correction term and ϕ as a time reparametrization correction term. Both of them depend on the perturbative parameter ε and their regularity properties are derived from a fixed point scheme. Indeed, we will write an equation where $x(t)$ in Theorem 1.1 is a solution and arrive at a fixed point problem after manipulation.

Remark 1.4 (on further conclusions). As a consequence of the fixed point method, we are able to formulate and prove that the solutions we build in the perturbative system depend smoothly on parameters. The parameters can be in the unperturbed ODE or in the perturbative map. Moreover, after a mild change of hypotheses, we will also see how the solution can admit an exponential derivative growth, see Section 6.1.

Remark 1.5 (on interesting applications). Among the applications of our theorem, a particular case is when the perturbed equation is another ODE. The results on persistence of such solutions were studied originally in [Ano69] and later in [Mos69]. We, however, obtain smooth dependence on parameters, which is not true for the formulation in the above references. The formulation we use is slightly different and follows more closely the formulation in [dlLMM86, Appendix A], which also obtained smooth dependence on parameters for the objects considered (the objects considered in [dlLMM86] are roughly, inverses of the objects considered in [Ano69, Mos69]).

Remark 1.6 (on details from previous works). In previous papers [YGdlL21, YGdlL22], we formulated the results in an a-posteriori format, meaning that if we start with an initial guess of the correction whose error is small enough, then the theorems conclude that there is a true solution nearby. The a-posteriori formulation is suitable for performing computer-assisted proofs (the approximate solution is produced by a numerical calculation and the needed estimates are verified using a computer by taking care of truncation and round-off error) [GLMY23]. Here the a-posteriori formulation will be more delicate. Indeed, we will use a different norm to obtain contractions and the a-posteriori argument will be valid on segments of times t, see Section 5.5.

This paper has similarities with [YGdlL22] in that we seek both an embedding and an inner dynamics. However, this paper is significantly more difficult.

The main reason is that, given any vector field in the circle, there is a change of variables that reduce it to a constant, so that, in [YGdlL22] the inner dynamics was just a number. In the present case, vector fields in the line cannot, in general, be reduced to constants (or even be approximated well by periodic; take for example, vector fields who oscillate between two values over longer and longer intervals).

Hence in the present case, rather than dealing with just a number, we have to deal with an infinite dimensional unknown that, furthermore appears in the functional equations as a composition on the right.

If we apply the formalism in this paper to the case of periodic solutions, we will obtain a periodic vector field X, not necessarily constant as it happens when one applies the formalism of [YGdlL22]. Furthermore, the formalism in [YGdlL22] does not satisfy the normalizations (16).

1.2 Organization of the Paper

In Section 2, we describe precisely the assumptions on the unperturbed system. The main part is the (rather standard) definition of hyperbolic orbits, which we use to set the notation. We also present a characterization of the invariant bundles.

Section 3 presents the formalism we use to describe the solutions of perturbed system. We present class of functions we will consider, and perform manipulations to derive a functional equation (called invariance equation, see (27), (29) whose solutions give solutions of (1) when substituted in (3)). As it turns out, this invariance equation has symmetries under changes of variables and we also present normalization conditions that lead to unique solution.

Section 4 is devoted to completing the formulation of the fixed point problem, specifying the fixed point operator, its domain and range, and the rigorous formulation of the main result, Theorem 4.8.

In Section 5, we present the proof of Theorem 4.8 starting with an overview of the strategy. Section 6 contains results not explicitly covered in the main formulation which require cumbersome notations.

Finally, Section 7 provides examples of physical interest where our results apply. In particular, it includes the case in which the delay is small and the motions of charged particles with electromagnetic interactions.

2. The unperturbed ODE

When $\varepsilon = 0$, the unperturbed system (1) is an autonomous ODE. We assume that such a system has a uniformly hyperbolic solution $\{x_0(t)\}_{t\in\mathbb{R}}$, see Definition 2.1.

First, we recall that the variational equation (or equation of variation) of the unperturbed ODE (1), around a solution $\{x_0(t)\}_{t\in\mathbb{R}}$, is the time-dependent linear equation,

$$\dot{\xi}(t) = \mathrm{D}f \circ x_0(t)\xi(t), \tag{4}$$

where $\xi(t) \in \mathbb{R}^n$ has the heuristic meaning of small deviations from the baseline trajectory. The linear equation (4) has a family of fundamental matrix solutions $\{U(v;t)\}_{v,t\in\mathbb{R}}$ such that:

$$\frac{d}{dv}U(v;t) = \mathrm{D}f \circ x_0(v)U(v;t), \qquad U(t;t) = Id_n, \tag{5}$$

where Id_n denotes the $n \times n$ identity matrix. Note that due to the existence and uniqueness of the variational equations, we have

$$U(v;t) = U(v,s)U(s;t). \tag{6}$$

2.1 Uniformly hyperbolic solutions of an ODE and their quality measures

In this section, we present Definition 2.1. The starting point of Theorem 4.8 is precisely that we have a solution of the unperturbed problem satisfying Definition 2.1. We note that this definition has qualitative aspects called *"quality measures of the hyperbolicity"*. The ranges of perturbation parameters that are allowed depend on the values of these numbers.

It is well known that these uniformly hyperbolic orbits often appear together in hyperbolic sets (e.g. horseshoes, Lorenz attractor, etc.) but the quality measures may deteriorate as we consider orbits in the attractors. This is also common in the theory of non-uniformly hyperbolic sets.

Definition 2.1 (Uniformly hyperbolic solution of an ODE)**.** Let $\{x_0(t)\}_{t\in\mathbb{R}}$ be a solution of the unperturbed system (1). We say that $\{x_0(t)\}_{t\in\mathbb{R}}$ is *uniformly hyperbolic* if, and only if, it satisfies:

i.) For each $t \in \mathbb{R}$, there exists a decomposition of the tangent space at $x_0(t)$,

$$\mathbb{R}^n \cong T_{x_0(t)}\mathbb{R}^n = E^c_t \oplus E^s_t \oplus E^u_t, \tag{7}$$

such that

i.a) $E^c_t = \mathrm{Span}\{f \circ x_0(t)\}$ has dimension $n_c = 1$.

i.b) E_t^s and E_t^u have dimensions n_s and n_u respectively. Thus, $n = 1 + n_s + n_u$.

i.c) E_t^σ depends on t continuously for $\sigma \in \{c, s, u\}$.

Moreover, the forward (resp. backward) semiflow of the variational equation is contractive on E_t^s (resp. E_t^u). More precisely, the fundamental matrices $\{U(v;t)\}_{v,t\in\mathbb{R}}$ in (5) admit center, stable, and unstable families of linear operators

$$\begin{aligned} &\{U^c(v;t)\}_{v,t\in\mathbb{R}}, \quad U^c(v;t)\colon E_t^c \to E_v^c,\\ &\{U^s(v;t)\}_{v,t\in\mathbb{R}}, \quad U^s(v;t)\colon E_t^s \to E_v^s,\\ &\{U^u(v;t)\}_{v,t\in\mathbb{R}}, \quad U^u(v;t)\colon E_t^u \to E_v^u,\end{aligned}$$

satisfying for $\sigma \in \{c, s, u\}$,

$$\frac{d}{dv}U^\sigma(v;t) = \mathrm{D}f \circ x_0(v) U^\sigma(v;t), \qquad U^\sigma(t;t) = Id|_{E_t^\sigma}, \tag{8}$$

where $Id|_{E_t^\sigma}$ denotes the identity operator restricted to the linear subspace E_t^σ.

ii.) There exist $C_U, \lambda_s, \lambda_u > 0$ such that

$$\begin{aligned} |U^s(v;t)| &\leqslant e^{-\lambda_s(v-t)}C_U \quad v \geqslant t,\\ |U^u(v;t)| &\leqslant e^{\lambda_u(v-t)}C_U \quad\ \ v \leqslant t,\end{aligned} \tag{9}$$

where $|\cdot|$ is the operator norm.

iii.) There exists $C_\Pi > 0$ such that

$$\sup_{t\in\mathbb{R}} \|\Pi_t^\sigma\| \leqslant C_\Pi, \qquad \sigma \in \{c, s, u\}, \tag{10}$$

where $\Pi_t^\sigma : \mathbb{R}^n \to E_t^\sigma$ denotes the projection corresponding to the splitting (7).

In particular, for the center direction projection, for any given vector $V \in T_{x_0(t)}\mathbb{R}^n$, there exists $A_V \in \mathbb{R}$

$$\Pi_t^c V = A_V f \circ x_0(t).$$

Definition 2.2 (Quality measures of Uniformly Hyperbolic orbit)**.** The quantities C_U, C_Π, λ_s, and λ_u appearing in Definition 2.1 are referred to as *quality measures* of the hyperbolic solution $\{x_0(t)\}_{t\in\mathbb{R}}$.

Remark 2.3 (on the variational on the E_t^σ). The ODE's in (8) are understood as equations for operators on $\mathbb{R}^n$. In particular, the restriction of Id on E_t^σ must be interpreted using the injection from E_t^σ to $\mathbb{R}^n$ for $\sigma \in \{c, s, u\}$.

Remark 2.4 (on the projections). The projections Π_t^σ in Definition 2.1 may not be orthogonal projections on the space and they depend on the decomposition, e.g., Π_t^s could change if E_t^u changes even if E_t^s remains fixed.

The constant C_Π can be interpreted as the inverse of a measure of the angles between the spaces in the decomposition (7).

Remark 2.5 (on hyperbolic regularity). We have formulated Definition 2.1 including continuous dependence of the bundles on the base point along the orbit to keep compatibility with the standard definitions of normally hyperbolic manifolds in [Fen72, HPS77].

These references show that, when the vector field is C^r and bounded and the hyperbolicity is uniform, then the continuous splittings are actually C^{r-1}. We will provide the details in the formal result section.

Remark 2.6 (on the quality measures). Note that the quality measures C_Π and C_U depend on the metric used. In the theoretical literature on hyperbolic systems, it is standard to define a metric (and modify slightly the exponents of contraction) called *adapted metric* so that $C_\Pi = C_U = 1$ and the center, stable, and unstable directions are orthogonal. This adapted metric is equivalent to the original one. Some rigorous proofs get simplified by using the adapted metric.

Nevertheless, we do not use an adapted metric in this work for several reasons. The use of adapted metric would be confusing for us since the formulas for state-dependent delays are affected by the metric as well. Moreover, the strength of the perturbations allowed in this paper depends on the values of the quality measures and the sizes of the derivatives of the perturbations. Changing the metric would require measuring the properties of the perturbing function in the adapted metric.

Besides, the use of an adapted metric also obscures the study of phenomena that happen in the boundary of hyperbolicity. Notably [HdlL06, HdlL07] identified numerically a boundary of hyperbolicity characterized by C_Π blowing up (the angle between the splittings going to zero) while the exponents of contraction remain uniformly bounded away from zero.

2.2 Infinitesimal characterization of the invariant bundles of a hyperbolic orbit

It will be useful for us to characterize trajectories in E_t^σ for $\sigma \in \{c, s, u\}$ as solutions of ODE's. In Lemma 2.7, we use $U^\sigma(0;t)$ to connect $\xi(t) \in E_t^\sigma$ with E_0^σ. This gives an infinitesimal characterization of the invariant bundles.

Lemma 2.7 (Bundle Characterization)**.** *Let $\{x_0(t)\}_{t\in\mathbb{R}}$ be a uniformly hyperbolic orbit and let $\sigma \in \{s, c, u\}$. If f is differentiable enough (at least C^1), then $\xi(t) \in E_t^\sigma$ if, and only if,*

1. *$\xi(0) \in E_0^\sigma$; and*
2. *$\dot\xi(t) = \mathrm{D}f \circ x_0(t)\xi(t) + a(t)$ with $a(t) \in E_t^\sigma$.*

Proof. $\Rightarrow$) Let $\alpha(t) = U^\sigma(0;t)\xi(t)$ or, equivalently, $\xi(t) = U^\sigma(t;0)\alpha(t)$. Note that $\alpha(t) \in E_0^\sigma$ for all t and consequently $\dot\alpha(t)$ belongs to E_0^σ as well. By taking derivatives we obtain

$$\dot\xi(t) = \mathrm{D}f \circ x_0(t)\xi(t) + U^\sigma(t;0)\dot\alpha(t).$$

Thus, $a(t) \stackrel{\text{def}}{=} U^\sigma(t;0)\dot\alpha(t) \in E_t^\sigma$.

$\Leftarrow$) By the variation of parameters formula,

$$\xi(t) = U^\sigma(t;0)\xi(0) + \int_0^t U^\sigma(t;s)a(s)\,ds.$$

And by the invariance of the σ-space; i.e. $U^\sigma(t;s)E_s^\sigma = E_t^\sigma$, then $\xi(t) \in E_t^\sigma$. ☐

2.3 Uniformly Hyperbolic Set

Definition 2.8 (Uniformly Hyperbolic Set)**.** We say that a set $\Sigma \subset \mathbb{R} \times \mathbb{R}^n$ is a *uniformly hyperbolic set* when there exist constants C_Π, C_U, λ_s, and λ_u so that all the orbits in the set Σ are uniformly hyperbolic with the above constants as quality measures.

If the hyperbolic sets considered lie in a subset of $\mathbb{R}^n$, we just need to assume that the derivatives of the vector field f is uniformly bounded in a suitable open set containing the hyperbolic set (it needs to contain all balls of a certain radius centered within the hyperbolic set).

One interesting example is the Lorenz attractor. The Lorenz equations are not bounded in the whole space, but they are bounded in a neighborhood of the Lorenz attractor. The Lorenz attractor is not uniformly hyperbolic but it contains many uniformly hyperbolic sets to which our theory applies.

We do not assume that the set Σ has any particular structure. In particular, we do not need that the set is closed nor that it is *locally maximal*, assumptions that are very common in the theory of hyperbolic systems.

Remark 2.9 (Translation Invariance). By the uniqueness of solutions of ODE, we can identify an orbit with its initial condition. If an orbit $\{x(t)\}$ is hyperbolic, by definition, so are all the translates $\{x(t+\tau)\}$ and they have the same quality measures. Hence, when considering a hyperbolic set for an ODE, we can identify the set of hyperbolic trajectories with a set in $\mathbb{R}^n$ invariant under the flow.

Remark 2.10 (Loss of invariance). When the perturbation P is time-dependent, then the translation invariance of hyperbolic solutions may be lost. For the applications to state-dependent delay or advance equations, where the phase space is not clear, there is no easy way to identify the space of solutions with the space of initial conditions. Hence, for our goal in this paper, it is better to think of a hyperbolic set as a collection of trajectories rather than as a set of initial conditions.

Remark 2.11. In the standard theory of uniformly hyperbolic sets, it is natural to consider the splittings along a trajectory $\{x(t)\}$ not as functions of the time, but as functions of the base point. It is a standard result in hyperbolic systems [Ano69, KH95, FH19] that the stable and unstable bundles depend on the base point in a Hölder way.

For reasons indicated in Remark 2.10, we, instead, choose to study the splittings as functions of time. We will be able to prove some regularity from the space of trajectories of the unperturbed system to the space of trajectories of the perturbed system. The regularity is somewhat technical since it involves weighted spaces.

3. Construction of perturbative solutions

In this section, we introduce the main idea of our result. We will describe the geometric motivations and the manipulations needed to transform the problem considered into a fixed point problem. We postpone a precise discussion of the regularity assumptions and other sophistication. Indeed, those precise assumptions are motivated to make the arguments in this section work.

Of course, readers interested only in precise formulations can move directly to Section 4 and use the present section as a reference for the notations we introduce.

The formalism we present resembles the proof of structural stability for Anosov Flows, which involves a reparametrization of time and a geometric change of the trajectories. These are the two main ideas we apply. Nevertheless, in contrast with many proofs of the structural stability, the reparameterization and the corrections are done differently for each trajectory and we formulate a different functional equation for each trajectory. As mentioned before, given the fact that the functional equations we need involve the composition operator, several formally equivalent equations may have different analytic properties. We have carefully chosen a formulation that leads to smooth dependence on parameters.

3.1 Form of the correction

Let $\{x_0(t)\}$ be a uniformly hyperbolic orbit of the unperturbed system in (1), then for $\varepsilon \neq 0$ we consider a solution of (1) close to $\{x_0(t)\}$ of the form

$$x(t) = (x_0 + \widehat{x}) \circ \phi(t), \tag{11}$$

where $\widehat{x} = O(\varepsilon)$ and $\mathrm{D}\phi(t) = 1 + O(\varepsilon)$ are the correcting unknowns. The term ϕ encodes the internal dynamics of the new solution while $\widehat{x}$ is the displacement from x_0. Note that the form (11) is reminiscent of the Anosov Shadowing Theorem using the functional analysis approach, see Remark 1.5.

We formulate functional equations (invariance equations) for the unknown pairs $\widehat{x}\colon \mathbb{R} \to \mathbb{R}^n$ and $\phi\colon \mathbb{R} \to \mathbb{R}$. These equations require that $x(t)$ in (11) is a solution of the equation (1). We then solve the invariance equations by fixed point methods using geometric assumptions on $\{x_0(t)\}$.

We find it more convenient to use the unknown vector field X

$$\dot{\phi}(t) = X \circ \phi(t), \tag{12}$$

associated to ϕ instead of ϕ itself, since the invariance equations are simpler in terms of X. It is clear that given ϕ we can obtain X by taking derivatives. Conversely, given X, we can recover ϕ using the differential equation (12). After a normalization condition that sets $\phi(0) = 0$, see Section 3.2, we see that if X is C^1, which implies that we can solve the ODE uniquely for all time, a flow $\phi\colon \mathbb{R} \to \mathbb{R}$ is determined. Therefore, we can consider X and ϕ as equivalent unknowns. Although, of course, the natural function spaces for them are different. Going from ϕ to X involves a loss of derivative, but going from X to ϕ gains a derivative. We have collected some of these subtleties in Lemma 5.4.

From here, we adopt the convention that X and ϕ are related as indicated above in (12). When there is a need to discuss the dependence of ϕ on X, we will write

$$\phi = \mathcal{S}[X], \tag{13}$$

to indicate that ϕ is the solution of (12) with a fixed initial condition. We refer to $\mathcal{S}$ as the solution operator. The operator is defined for C^1 vector fields. If X is bounded away from zero, $\mathcal{S}[X]$ will be a diffeomorphism on $\mathbb{R}$.

To find a locally unique pair $(X, \widehat{x})$ from the invariance equations, we require appropriate normalizations, see Section 3.2 later on. As a consequence of that uniqueness, we will be able to discuss smooth dependence on parameters around the initial orbit $\{x_0(t)\}$. However, since we are dealing with functions defined on the whole line, this will involve some subtleties.

The strategy of invariance equations treated by functional analysis is very different from the strategy based on defining an evolution in a space of functions associated to (1) and finding hyperbolic solutions. Notably, we start by fixing the form, (11), and finding functions of this form that satisfy (1). There are cases where invariant objects of systems without globally defined solutions have been studied [dlL09, CdlL20] and we will use some of the techniques developed there.

We also note that the invariance equations in this strategy can be studied numerically or using formal expansions. Numerical treatments of the equations for periodic orbits and their stable manifolds in simple models were done in [GYdlL21]. An interesting problem is to extend the above numerical methods for periodic solutions to the solutions with arbitrary time dependence considered here.

3.2 Non-Uniqueness of the parametrization and normalization conditions

The expressions in (11) is underdetermined. There are many representations of the same function $x(t)$ using different unknown pairs. Indeed, given a solution $(\phi, \widehat{x})$ for the invariance equations and any diffeomorphism w of $\mathbb{R}$,

$$\begin{aligned} x(t) &= ((x_0 + \widehat{x}) \circ w) \circ (w^{-1} \circ \phi)(t) \\ &= \big(x_0 + (x_0 \circ w - x_0 + \widehat{x} \circ w)\big) \circ (w^{-1} \circ \phi)(t), \end{aligned} \tag{14}$$

provides another choice $(\psi, \widehat{y}) \overset{\text{def}}{=} (w^{-1} \circ \phi,\ x_0 \circ w - x_0 + \widehat{x} \circ w)$ for the solution.

The underdeterminacy (14) can be avoided by imposing normalizations that simplify the treatment and lead to local uniqueness of the solution. We choose two normalizing conditions:

$$\widehat{x} \circ \phi(t) \in E^s_{\phi(t)} \oplus E^u_{\phi(t)}, \tag{15}$$

$$\phi(0) = 0. \tag{16}$$

Of course, the fact that (15) and (16) are good normalizations will become apparent when we show that we can find locally unique solution of the invariance equations satisfying them.

The heuristic reason for (15) is that adding a component of $\widehat{x}$ in the direction of the flow is roughly equivalent to adjusting ϕ, which can be seen from (14) and $x_0 \circ w - x_0 \approx x_0'(w - Id)$. Meanwhile, (16) can be justified by choosing the origin of t in the reference line.

3.3 Formulation of the functional equations characterizing a solution of (1)

To derive functional equations for the unknowns $(X, \widehat{x})$, we substitute (11) into (1), yielding

$$X(\phi(t))(x_0 + \widehat{x})'(\phi(t)) = f \circ (x_0 + \widehat{x}) \circ \phi(t) + \varepsilon \mathscr{P}[(x_0 + \widehat{x}) \circ \phi, \varepsilon, \mu](t), \tag{17}$$

where $\phi = \mathcal{S}[X]$ as in (13), and $'$ denotes the derivative, and $\mathscr{P}$ is a functional operator. Explicitly,

$$\mathscr{P}[u, \varepsilon, \mu](t) \stackrel{\text{def}}{=} P(t, u_t, \varepsilon, \mu), \tag{18}$$

where P is the perturbative map in (1).

Note that we are using that $X(\phi(t))$ is a number, so that we can put the product by it either as a prefactor or as a postfactor as would come from the chain rule.

Now, we start to rewrite the equation (17) separating the small terms. We first consider the linear approximation of the vector field f along the hyperbolic orbit x_0 of the ODE

$$f \circ (x_0 + \widehat{x}) \circ \phi(t) = f \circ x_0 \circ \phi(t) + \mathrm{D}f \circ x_0 \circ \phi(t)\widehat{x} \circ \phi(t) + T[x_0, \widehat{x}](\phi(t)),$$

where

$$T[x_0, \widehat{x}](\phi(t)) \stackrel{\text{def}}{=} f \circ (x_0 + \widehat{x}) \circ \phi(t) - f \circ x_0 \circ \phi(t) - \mathrm{D}f \circ x_0 \circ \phi(t)\widehat{x} \circ \phi(t) \tag{19}$$

is the remainder of the first order Taylor expansion.

Using (19), equation (17) is rewritten as

$$\begin{aligned} X(\phi(t))\widehat{x}'(\phi(t)) = (1 - X \circ \phi(t)) f \circ x_0 \circ \phi(t) + \mathrm{D}f \circ x_0 \circ \phi(t)\widehat{x} \circ \phi(t) \\ + T[x_0, \widehat{x}](\phi(t)) + \varepsilon \mathscr{P}[(x_0 + \widehat{x}) \circ \phi, \varepsilon, \mu](t). \end{aligned} \tag{20}$$

We apply the time change $\rho = \phi(t)$, add and subtract $X(\rho)\mathrm{D}f \circ x_0(\rho)\widehat{x}(\rho)$ in (20) to obtain

$$X(\rho)\widehat{x}'(\rho) = X(\rho)\mathrm{D}f \circ x_0(\rho)\widehat{x}(\rho) + (1 - X(\rho)) f \circ x_0(\rho) + \mathcal{B}[X, \widehat{x}](\rho) + \varepsilon\varphi[X, \widehat{x}](\rho) \tag{21}$$

where for typographical reasons, we introduce $\mathcal{B}$ to capture the "quadratically" small terms, i.e.

$$\mathcal{B}[X, \widehat{x}](\rho) \stackrel{\text{def}}{=} (1 - X(\rho))\mathrm{D}f \circ x_0(\rho)\widehat{x}(\rho) + T[x_0, \widehat{x}](\rho), \tag{22}$$

and φ to represent the term from $\mathscr{P}$,

$$\varphi[X, \widehat{x}](\rho) \stackrel{\text{def}}{=} \mathscr{P}[(x_0 + \widehat{x}) \circ \phi, \varepsilon, \mu](\phi^{-1}(\rho)) = P\big(\phi^{-1}(\rho), ((x_0 + \widehat{x}) \circ \phi)_{\phi^{-1}(\rho)}, \varepsilon, \mu\big). \tag{23}$$

Remark 3.1. Note that φ depends on $\widehat{x}$, ϕ, x_0, the perturbation P, the perturbative parameter ε, and the parameter μ in equation (1). By using $\phi = \mathcal{S}[X]$ in (13), we consider φ as a functional which produces a function from $\mathbb{R}$ to $\mathbb{R}^n$ given the vector field X and the deformation $\widehat{x}$. To simplify the notation, we denote $\varphi[X, \widehat{x}]$ without writing explicitly other dependencies.

We first consider the center direction of equation (21) to obtain equation (24). Then we use the uniform hyperbolicity of x_0 and the normalization $\widehat{x} = \widehat{x}^s + \widehat{x}^u$ in (15) to derive (25) and (26). Thus, by Lemma 2.7, (21) is equivalent to the following three equations and that the initial conditions of (25) and (26) are in the corresponding bundles:

$$0 = \Pi^c_\rho (1 - X(\rho)) \cdot f \circ x_0(\rho) + \Pi^c_\rho \big(\mathcal{B}[X, \widehat{x}](\rho) + \varepsilon\varphi[X, \widehat{x}](\rho)\big), \tag{24}$$

$$(\widehat{x}^s)'(\rho) = \mathrm{D}f \circ x_0(\rho)\widehat{x}^s(\rho) + \Pi^s_\rho \frac{1}{X(\rho)} \big(\mathcal{B}[X, \widehat{x}](\rho) + \varepsilon\varphi[X, \widehat{x}](\rho)\big), \tag{25}$$

$$(\widehat{x}^u)'(\rho) = \mathrm{D}f \circ x_0(\rho)\widehat{x}^u(\rho) + \Pi^u_\rho \frac{1}{X(\rho)} \big(\mathcal{B}[X, \widehat{x}](\rho) + \varepsilon\varphi[X, \widehat{x}](\rho)\big). \tag{26}$$

Our goal is to transform (24)–(26) into a fixed point equation for the unknowns $(X, \widehat{x})$. We will define operators Γ_c^ε, Γ_s^ε, and Γ_u^ε of X and $\widehat{x}$ whose fixed point solves (24)–(26).

The operator Γ_c^ε comes from isolating $X(\rho)$ in (24). More explicitly,

$$X(\rho) = \Gamma_c^\varepsilon[X,\widehat{x}](\rho) \stackrel{\text{def}}{=} 1 + \frac{\big\langle \Pi_\rho^c\big(\mathcal{B}[X,\widehat{x}](\rho) + \varepsilon\varphi[X,\widehat{x}](\rho)\big), f\circ x_0(\rho)\big\rangle}{\langle f\circ x_0(\rho), f\circ x_0(\rho)\rangle}, \tag{27}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

To solve equations (25) and (26), we apply the variation of parameters formula on the bundles respectively and take appropriate limits. The procedure is very similar to the method of [Per29, Cot11] in the study of invariant manifolds. Note that, although we are not considering invariant manifolds here, our ideas can be compared to those used for studying normally hyperbolic invariant manifolds.

We first obtain that for $\rho_0 \geqslant \rho \geqslant -\rho_0$,

$$\begin{aligned}
\widehat{x}^s(\rho) &= \int_{-\rho_0}^{\rho} U^s(\rho;v)\Pi_v^s \frac{1}{X(v)}\big(\mathcal{B}[X,\widehat{x}](v) + \varepsilon\varphi[X,\widehat{x}](v)\big)\,dv + U^s(\rho;-\rho_0)\widehat{x}^s(-\rho_0),\\
\widehat{x}^u(\rho) &= -\int_{\rho}^{\rho_0} U^u(\rho;v)\Pi_v^u \frac{1}{X(v)}\big(\mathcal{B}[X,\widehat{x}](v) + \varepsilon\varphi[X,\widehat{x}](v)\big)\,dv + U^u(\rho;\rho_0)\widehat{x}^u(\rho_0).
\end{aligned} \tag{28}$$

Using the bounds (9) on the evolution operators U^s/U^u and assuming that $\widehat{x}^s$ and $\widehat{x}^u$ are bounded (or that if they grow, the growth rate is less than λ_s, λ_u respectively, we let $\rho_0 \to +\infty$ and have

$$\begin{aligned}
\widehat{x}^s(\rho) = \Gamma_s^\varepsilon[X,\widehat{x}](\rho) &\stackrel{\text{def}}{=} \int_{-\infty}^{\rho} U^s(\rho;v)\Pi_v^s \frac{1}{X(v)}\big(\mathcal{B}[X,\widehat{x}](v) + \varepsilon\varphi[X,\widehat{x}](v)\big)\,dv,\\
\widehat{x}^u(\rho) = \Gamma_u^\varepsilon[X,\widehat{x}](\rho) &\stackrel{\text{def}}{=} -\int_{\rho}^{+\infty} U^u(\rho;v)\Pi_v^u \frac{1}{X(v)}\big(\mathcal{B}[X,\widehat{x}](v) + \varepsilon\varphi[X,\widehat{x}](v)\big)\,dv.
\end{aligned} \tag{29}$$

Alternatively, one could check that Γ_s^ε and Γ_u^ε defined in (29) indeed satisfy the equations (25)–(26). Let us provide the details for the stable case: Taking derivatives w.r.t. ρ in $\Gamma_s^\varepsilon[X,\widehat{x}](\rho)$ and using the fundamental theorem of calculus

$$\begin{aligned}
\frac{d}{d\rho}\Gamma_s^\varepsilon[X,\widehat{x}](\rho) &= U^s(\rho;\rho)\Pi_\rho^s\frac{1}{X(\rho)}\big(\mathcal{B}[X,\widehat{x}](\rho) + \varepsilon\varphi[X,\widehat{x}](\rho)\big)\\
&\quad + \int_{-\infty}^{\rho} \mathrm{D}f\circ x_0(\rho)U^s(\rho;v)\Pi_v^s\frac{1}{X(v)}\big(\mathcal{B}[X,\widehat{x}](v) + \varepsilon\varphi[X,\widehat{x}](v)\big)\,dv\\
&= \Pi_\rho^s\frac{1}{X(\rho)}\big(\mathcal{B}[X,\widehat{x}](\rho) + \varepsilon\varphi[X,\widehat{x}](\rho)\big) + \mathrm{D}f\circ x_0(\rho)\Gamma_s^\varepsilon[X,\widehat{x}](\rho).
\end{aligned}$$

To justify the derivative under the integral sign, we observe that the integrand decays exponentially. The previous derivation is a standard argument going back to [Cot11, Per29] and it has the advantage showing that (29) is the only solution of the differential equations (25)–(26) with growth rate smaller than $\lambda_{s,u}$ and, in particular, bounded. In addition, we observe that given the exponential bounds for U^s and U^u in (9), if $\mathcal{B}[X,\widehat{x}]$ and $\varepsilon\varphi[X,\widehat{x}]$ are bounded, the $\widehat{x}^s$ and $\widehat{x}^u$ produced in (29) are bounded and in the corresponding bundles.

Note that to solve (25) and (26), we are not specifying any initial condition for $\widehat{x}^s$ or $\widehat{x}^u$ explicitly, only that the solutions are uniformly bounded by an exponential of time. Indeed, these boundedness requirement fixes the initial condition for (25) and (26): if we specified an initial condition not in the trajectories (29), we would obtain exponential growth solutions with a rate λ_u or λ_s and in particular unbounded.

4. Precise formulation

In this section, we introduce the function spaces and revisit the construction in Section 3.3 with precise formulation. We also provide our main result, see Theorem 4.8.

4.1 The operator

We consider an operator Γ^ε depending on the perturbative parameter ε and whose inputs are:

I1) A non-zero vector field X in $\mathbb{R}$ (whose flow is $\phi = \mathcal{S}[X]$);

I2) A stable correction $\widehat{x}^s$ to the uniformly hyperbolic orbit x_0; and

I3) An unstable correction $\widehat{x}^u$ to the uniformly hyperbolic orbit x_0.

The operator Γ^ε has outputs:

O1) A new vector field $Y = \Gamma^\varepsilon_c[X, \widehat{x}^s, \widehat{x}^u]$ in $\mathbb{R}$, and hence its flow $\psi = \mathcal{S}[Y]$ given by the initial value problem

$$\frac{d}{dt}\psi(t) = Y \circ \psi(t), \qquad \psi(0) = 0;$$

O2) A new stable correction $\widehat{y}^s = \Gamma^\varepsilon_s[X, \widehat{x}^s, \widehat{x}^u]$ to the orbit x_0; and

O3) A new unstable correction $\widehat{y}^u = \Gamma^\varepsilon_u[X, \widehat{x}^s, \widehat{x}^u]$ to the orbit x_0.

Note that in both input and output cases, one could write the unknown orbit correction as a sum of stable and unstable corrections due to (7) and (15). Indeed, $\widehat{x} = \widehat{x}^s + \widehat{x}^u$ and since the operators Γ^ε_s and Γ^ε_u make the corrections on the stable and unstable bundles respectively, the output $\widehat{y}$ satisfies $\widehat{y} = \widehat{y}^s + \widehat{y}^u$ as well.

We define the operator Γ^ε with components, i.e.

$$\Gamma^\varepsilon \equiv \begin{pmatrix} \Gamma^\varepsilon_c \\ \Gamma^\varepsilon_s \\ \Gamma^\varepsilon_u \end{pmatrix}.$$

The operator Γ^ε acts on a function space $\mathcal{X}$, which will be exhaustively detailed in Section 4.2.

Remark 4.1 (Alternatives to the fixed point operator). The operator Γ^ε admits alternative versions, for instance, using the corrected vector field Y instead of X in Γ^ε_s and Γ^ε_u. There are similar variations with the other updated expressions as well.

All these alternative operators may have a convergence impact in a numerical implementation. Nevertheless, to prove the existence and uniqueness of the fixed point, the Γ^ε defined here will give easier inequalities that otherwise can be bounded by simple triangle inequalities.

4.2 The spaces considered

As usual for fixed point problems, one tries to get both existence and uniqueness of fixed point. We look for fixed points of the operator Γ^ε in a product space $\mathcal{X}$ of finitely differentiable maps. The existence results become better when considering a space of more differentiable functions, while the uniqueness results will be better for a bigger space with lower regularity.

The following definitions, although standard, set the notation for the statement of our main results, see Section 4.3.

4.2.1 Spaces of Lipschitz differentiable functions

Let $\ell \geqslant 0$ be a fixed integer, let $I \subset \mathbb{R}$ be an open interval, and let D be the differential operator. The space $C^\ell(I)$ denotes the space of functions defined on I, that are ℓ times differentiable, extend continuously to the closure of I, denoted by $\overline{I}$, and whose derivatives are bounded. More precisely,

$$C^\ell(I) = C^\ell(I, \mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ g\colon I \to \mathbb{R}^n \;\middle|\; \begin{array}{l} g \text{ is } \ell \text{ times continuously differentiable in } I, \\ \text{the derivatives extend continuously to } \overline{I}, \text{ and} \\ \|g\|_{C^\ell} \stackrel{\text{def}}{=} \max\limits_{0\leqslant j\leqslant \ell}\{\sup\limits_{x\in I} |\mathrm{D}^j g(x)|\} < +\infty \end{array} \right\},$$

where $|\cdot|$ denotes a norm in $\mathbb{R}^n$. By our definition, the space $C^\ell(I)$ is a Banach space for each ℓ. If the domain or range of the functions are understood, we will suppress it from the notation. We use the identity $\mathrm{D}^0 = Id$ and define C^0 as the set of continuous functions with bounded C^0-norm.

We denote the space $C^{\ell+\mathrm{Lip}}$ as a subspace of C^ℓ containing functions whose ℓ-th derivative is Lipschitz, and we endow the space with the $\|\cdot\|_{C^{\ell+\mathrm{Lip}}}$ norm. Explicitly,

$$C^{\ell+\mathrm{Lip}}(I) = C^{\ell+\mathrm{Lip}}(I,\mathbb{R}^n) \stackrel{\mathrm{def}}{=} \left\{g \in C^\ell(I,\mathbb{R}^n) \colon \|g\|_{C^{\ell+\mathrm{Lip}}} \stackrel{\mathrm{def}}{=} \max\left\{\|g\|_{C^\ell}, \mathrm{Lip}(\mathrm{D}^\ell g)\right\} < +\infty\right\}.$$

The $C^{\ell+\mathrm{Lip}}$ space is useful for us as it is the C^0 closure of the $C^{\ell+1}$ space.

The Lipschitz constant has some properties that we summarize in the following (known) lemma:

Lemma 4.2. *Let f, g be continuous maps with finite Lipschitz constant and let λ be a scalar. Then*

1. $\mathrm{Lip}(f+g) \leqslant \mathrm{Lip}(f) + \mathrm{Lip}(g)$*;*
2. $\mathrm{Lip}(\lambda f) \leqslant |\lambda| \,\mathrm{Lip}(f)$*;*
3. $\mathrm{Lip}(fg) \leqslant \mathrm{Lip}(f)\|g\|_{C^0} + \|f\|_{C^0}\,\mathrm{Lip}(g) \leqslant 2\|f\|_{C^1}\|g\|_{C^1}$*; and*
4. $\mathrm{Lip}(f \circ g) \leqslant \mathrm{Lip}(f)\,\mathrm{Lip}(g)$*.*

Given $c = (c_0, \ldots, c_\ell, c_\ell^{\mathrm{Lip}}) \in \mathbb{R}_+^{\ell+2}$, we denote the ball centered at $l \in C^{\ell+\mathrm{Lip}}$ with radius c as

$$\mathcal{B}_c^{\ell+\mathrm{Lip}}(l) \stackrel{\mathrm{def}}{=} \left\{g \in C^{\ell+\mathrm{Lip}} \colon |\mathrm{D}^j(g-l)| \leqslant c_j \text{ for } j = 0, \ldots, \ell \text{ and } \mathrm{Lip}(\mathrm{D}^\ell(g-l)) \leqslant c_\ell^{\mathrm{Lip}}\right\}. \tag{30}$$

We will use a similar notation for a ball in C^ℓ space by $\mathcal{B}_c^\ell(l)$. If l is zero, we will sometimes omit the center and write $\mathcal{B}_c^\ell$.

The following straightforward lemma states that the C^ℓ space is a Banach algebra by our definition. With Lemma 4.2, we get that $C^{\ell+\mathrm{Lip}}$ is also a Banach algebra.

Lemma 4.3. *The space C^ℓ is a Banach algebra:*

1. $f \in \mathcal{B}_{c_f}^\ell$ *and* $g \in \mathcal{B}_{c_g}^\ell$ *implies* $f + g \in \mathcal{B}_{c_f+c_g}^\ell$*;*
2. $\lambda \in \mathbb{R}$ *and* $g \in \mathcal{B}_c^\ell$ *implies* $\lambda g \in \mathcal{B}_{|\lambda|c}^\ell$*; and*
3. $f \in \mathcal{B}_{c_f}^\ell$ *and* $g \in \mathcal{B}_{c_g}^\ell$ *implies* $fg \in \mathcal{B}_{\tilde{c}}^\ell$ *with* $\tilde{c}$ *only depending on* c_f *and* c_g*.*

Proof. *1.* and *2.* are straightforward. We define

$$\tilde{c}_j \stackrel{\mathrm{def}}{=} \sum_{k=0}^{j} \binom{j}{k} c_{f,k} c_{g,j-k}$$

so that *3.* is true by Leibnitz rule. □

Hölder space and interpolation inequality For the a-posteriori formulation of our main theorem, we briefly recall the standard notion of Hölder spaces and state the interpolation inequalities in C^r spaces [Ste70] (A short proof of the interpolation inequalities valid in domains in Banach spaces can be found in [dlLO99]).

Given $\alpha \in (0,1)$, the Hölder space $C^{0,\alpha}(I)$ is the set of functions $g \in C^0$ such that the Hölder semi-norm

$$[g]_{C^{0,\alpha}} = [g]_{C^{0,\alpha}(I)} \stackrel{\mathrm{def}}{=} \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|g(x)-g(y)|}{|x-y|^\alpha}$$

is finite. Note that for $\alpha = 1$, this semi-norm is exactly the Lipschitz constant. Similarly, given $k \geqslant 0$ an integer, the Hölder space $C^{k,\alpha}$ is defined by

$$C^{k,\alpha} = C^{k,\alpha}(I) \stackrel{\text{def}}{=} \{g \in C^k(I) \colon \|g\|_{C^{k,\alpha}} \stackrel{\text{def}}{=} \max\{\|g\|_{C^k(I)}, [\mathrm{D}^k g]_{C^{0,\alpha}(I)}\} < +\infty\}.$$

In particular, $C^{k,1} \equiv C^{k+\mathrm{Lip}}$. Notice that this yields the space inclusions

$$C^\infty \subset \cdots \subset C^2 \subset C^{1+\mathrm{Lip}} \subset C^{1,\alpha} \subset C^1 \subset C^{\mathrm{Lip}} \subset C^{0,\alpha} \subset C^0.$$

It is then common to define the C^r space when $r > 0$ is not an integer. That is, $C^r \stackrel{\text{def}}{=} C^{k,\alpha}$ where $r = k + \alpha$, k being an integer and $\alpha \in (0, 1)$.

The space C^r admits interpolation inequalities. We now quote the result (omitting domain assumptions) from [Kol49, Had98, dlLO99] which states that if $0 \leqslant r < t$ and $g \in C^t$, then there is a constant $M_{r,t} > 0$ such that

$$\|g\|_{C^{\theta r+(1-\theta)t}} \leqslant M_{r,t} \|g\|_{C^r}^{\theta} \|g\|_{C^t}^{1-\theta},$$

for any $\theta \in [0, 1]$. This is equivalent to consider $s \in (r, t)$ and $\mu \stackrel{\text{def}}{=} \frac{t-s}{t-r}$ and rewrite the inequality as

$$\|g\|_{C^s} \leqslant M_{r,t} \|g\|_{C^r}^{\mu} \|g\|_{C^t}^{1-\mu}. \tag{31}$$

An interesting remark for the applications is that the unit ball in $C^{k,\alpha}[a, b]$ where $k \in \mathbb{N}$, $0 < \alpha \leqslant \mathrm{Lip}$, $-\infty \leqslant a, b \leqslant \infty$ is compact (and therefore closed) in the C^0 topology.

Arzela-Ascoli theorem shows that the ball is precompact, and we also have that $\mathrm{D}^k u_n$ converges uniformly and are uniformly C^α, then the limit is also C^α with the same constant.

4.2.2 Contraction space for time-dependent perturbation

When the perturbation in (1) depends on time in a bounded manner, we need to bound the difference of time reparametrizations, that is, to bound flows for two vector fields. Hence, for some fixed $\eta > 0$, we introduce the Razumikhin norm for continuous functions on an open interval $I \subset \mathbb{R}$, which is defined as

$$C_\eta = C_\eta(I, \mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ g : I \to \mathbb{R}^n \text{ is continuous: } \|g\|_{C_\eta} = \|g\|_\eta \stackrel{\text{def}}{=} \sup_{\rho \in I} |g(\rho)| e^{-\eta|\rho|} < +\infty \right\}. \tag{32}$$

The parameter η will eventually be chosen to ensure that the operator Γ^ε is a contraction. Note that $\|g\|_{C_\eta} \leqslant \|g\|_{C^0}$ for all $g \in C_\eta$. Moreover, under some assumption on η, the operator $\mathcal{S}$ in (13) can be bounded as:

$$\|\mathcal{S}[X] - \mathcal{S}[Y]\|_{C_\eta} \leqslant \mathfrak{c} \|X - Y\|_{C^0},$$

for a constant $\mathfrak{c}$ depending on η and the bound for the Lipschitz constants of the vector fields X, Y. This is formally proved in the following Lemma 4.4.

Lemma 4.4. *Let ϕ and ψ be flows of the vector fields X and Y in $\mathbb{R}$ in a ball $\mathscr{B}^{\mathrm{Lip}}_{(\mathfrak{t}_0,\mathfrak{t}_1)}(1)$ respectively with zero initial condition at zero. If $\mathfrak{t}_0 \in (0, 1)$ and $\eta > \mathfrak{t}_1 > 0$, then*

$$|\phi(\rho) - \psi(\rho)| e^{-\eta|\rho|} \leqslant \frac{\|X - Y\|_{C^0}}{e(\eta - \mathfrak{t}_1)}.$$

Proof. Let us define the solution operator of the ODE $\dot{\phi} = X \circ \phi$ generated by vector field X as

$$\Upsilon_X(\phi)(\rho) = \begin{cases} \displaystyle\int_0^\rho X \circ \phi(s)\, ds & \rho \geqslant 0 \\ \displaystyle -\int_0^{-\rho} X \circ \phi(-s)\, ds & \rho < 0. \end{cases}$$

We define operator Υ_Y similarly. Let $\widetilde{\phi}\colon \mathbb{R} \to \mathbb{R}$ be another function. Then, for $\rho < 0$ (and similarly for $\rho \geqslant 0$),

$$|\Upsilon_X(\phi)(\rho) - \Upsilon_X(\widetilde{\phi})(\rho)|e^{-\eta|\rho|} \leqslant e^{\eta\rho}\int_0^{-\rho} |X\circ\phi(-s) - X\circ\widetilde{\phi}(-s)|\,ds \leqslant e^{\eta\rho}\operatorname{Lip}(X)\int_0^{-\rho}|\phi(-s)-\widetilde{\phi}(-s)|\,ds$$
$$\leqslant \|\phi-\widetilde{\phi}\|_\eta \operatorname{Lip}(X)e^{\eta\rho}(e^{-\eta\rho}-1) \leqslant \frac{\operatorname{Lip}(X)}{\eta}\|\phi-\widetilde{\phi}\|_\eta.$$

Thus, if $\eta > \mathfrak{t}_1 \geqslant \max\{\operatorname{Lip}(X), \operatorname{Lip}(Y)\}$, then Υ_X and Υ_Y are contractions.

Given $\Upsilon_Y(\psi) = \psi$ a fixed point, the a-posteriori estimates for Υ_Y says:

$$\|\phi-\psi\|_\eta \leqslant \left(1 - \frac{\mathfrak{t}_1}{\eta}\right)^{-1}\|\phi - \Upsilon_Y(\phi)\|_\eta.$$

Now for $\rho < 0$ (and again similarly for $\rho \geqslant 0$)

$$|\phi(\rho) - \Upsilon_Y(\phi)(\rho)|e^{-\eta|\rho|} \leqslant e^{\eta\rho}\int_0^{-\rho}|X\circ\phi(-s) - Y\circ\phi(-s)|\,ds \leqslant e^{\eta\rho}\frac{-\eta\rho}{\eta}\|X-Y\|_{C^0}.$$

By taking supremum and using the fact that $\sup_{u<0} -e^u u = \sup_{u\geqslant 0} e^{-u}u = e^{-1}$, we conclude that

$$\|\phi-\psi\|_\eta \leqslant \left(1-\frac{\mathfrak{t}_1}{\eta}\right)^{-1}\frac{1}{\eta e}\|X-Y\|_{C^0}.$$ □

Interpolation inequality The C_η space also admits interpolation inequalities based on the C^r interpolation inequalities for functions defined on finite intervals. Indeed, given $g\colon [a,b] \to \mathbb{R}^n$, we have

$$\|g|_{[a,b]}\|_{C_\eta} \leqslant \|g|_{[a,b]}\|_{C^0} \leqslant e^{\eta\max\{|a|,|b|\}}\|g|_{[a,b]}\|_{C_\eta}.$$

Therefore, when $r = 0$, we can rewrite the interpolation inequality (31) with $\|\cdot\|_\eta$, yielding

$$\|g|_{[a,b]}\|_{C^{(1-\theta)t}} \leqslant M_{0,t}\,\|g|_{[a,b]}\|_{C^0}^{\theta}\|g|_{[a,b]}\|_{C^t}^{1-\theta} \leqslant M_{0,t}\,e^{\eta\theta\max\{|a|,|b|\}}\,\|g|_{[a,b]}\|_{C_\eta}^{\theta}\|g|_{[a,b]}\|_{C^t}^{1-\theta}.$$

This interpolation property will be used in the a-posteriori formulation of the main result, see Theorem 4.8 for a formal formulation and Section 5.5 for a detailed discussion.

4.2.3 The operator space for Γ^ε

The operator Γ^ε takes values from a space $\mathcal{X}$, which is the product space of three spaces, one for each input of the operator, endowed with the product norm.

The first input of the operator is the vector field X which corrects along the tangent direction of the orbit. We consider a ball centered at 1 of functions $\mathbb{R} \to \mathbb{R}$ that are ℓ times differentiable, with ℓ-th derivative Lipschitz, that is, $X \in \mathscr{B}_{\mathfrak{t}}^{\ell+\operatorname{Lip}}(1)$ for $\mathfrak{t} \in \mathbb{R}_+^{\ell+2}$. We will see in Lemma 5.4 that the flow $\phi = \mathcal{S}[X]$ gains one regularity.

The other two inputs will be taken in a ball centered at the origin of functions $\mathbb{R} \to \mathbb{R}^n$ that are $\ell+1$ times differentiable, with $(\ell+1)$-th derivative Lipschitz, i.e., let $\widehat{x}^s \in \mathscr{B}_{\mathfrak{s}}^{\ell+1+\operatorname{Lip}}$ and $\widehat{x}^s \in \mathscr{B}_{\mathfrak{u}}^{\ell+1+\operatorname{Lip}}$ for $\mathfrak{s},\mathfrak{u} \in \mathbb{R}_+^{\ell+3}$. Because of the normalization (15), $\widehat{x} = \widehat{x}^s + \widehat{x}^u$ unequivocally, therefore $\widehat{x} \in \mathscr{B}_{\mathfrak{s}+\mathfrak{u}}^{\ell+1+\operatorname{Lip}}$.

Then, the space for Γ^ε-inputs consists in product of balls:

$$(X, \widehat{x}^s, \widehat{x}^u) \in \mathcal{X} = \mathcal{X}^\ell_{\mathfrak{t},\mathfrak{s},\mathfrak{u}} \stackrel{\text{def}}{=} \mathscr{B}_{\mathfrak{t}}^{\ell+\operatorname{Lip}}(1) \times \mathscr{B}_{\mathfrak{s}}^{\ell+1+\operatorname{Lip}}(0) \times \mathscr{B}_{\mathfrak{u}}^{\ell+1+\operatorname{Lip}}(0). \tag{33}$$

By construction, the center projection's range can always be identified with $\mathbb{R}$, while the stable and unstable projection ranges are elements in $\mathbb{R}^n$ belonging to subspaces of n_s and n_u dimension respectively, see Definition 2.1 and Remark 2.4. Therefore, the ball $\mathscr{B}_{\mathfrak{t}}^{\ell+\mathrm{Lip}}$ lies in the space of functions $\mathbb{R} \to \mathbb{R}$, while the other two balls in $\mathcal{X}$ are for functions $\mathbb{R} \to \mathbb{R}^n$. Note that the space $\mathcal{X}$ is closed under C^0-norm.

To fix the notations, we specify the components of the vectors $\mathfrak{t}, \mathfrak{s}$, and $\mathfrak{u}$ appearing in $\mathcal{X}$:

$$\begin{aligned} \mathfrak{t} &\stackrel{\text{def}}{=} (\mathfrak{t}_0, \ldots, \mathfrak{t}_\ell, \mathfrak{t}_\ell^{\mathrm{Lip}}) \in \mathbb{R}_+^{\ell+2}, \\ \mathfrak{s} &\stackrel{\text{def}}{=} (\mathfrak{s}_0, \ldots, \mathfrak{s}_\ell, \mathfrak{s}_{\ell+1}, \mathfrak{s}_{\ell+1}^{\mathrm{Lip}}) \in \mathbb{R}_+^{\ell+3}, \\ \mathfrak{u} &\stackrel{\text{def}}{=} (\mathfrak{u}_0, \ldots, \mathfrak{u}_\ell, \mathfrak{u}_{\ell+1}, \mathfrak{u}_{\ell+1}^{\mathrm{Lip}}) \in \mathbb{R}_+^{\ell+3}. \end{aligned} \tag{34}$$

These constants in (34), jointly with the perturbative parameter ε, are the ones that we will constrain in a finite set of inequalities to ensure that the operator Γ^ε, defined in Section 4.1, maps $\mathcal{X}$ into itself and is contractive for some distance.

Remark 4.5 (on the special constant $\mathfrak{t}_0$). There is a crucial requirement on the value $\mathfrak{t}_0$ to be in the interval $(0, 1)$ by Lemma 5.4.

Remark 4.6 (on the absence of the delay in the space definition). We stress that because we are considering a special type of solutions (uniformly hyperbolic solutions) under perturbation, we are able to consider a special space of functions that is not affected by any delay or functional information of the perturbative map P. In particular, the constant $h > 0$ in the "history domain" does not even enter in the space where we apply the fixed point approach as long as the perturbative hypotheses ($\mathrm{H}_\varepsilon 1$)–($\mathrm{H}_\varepsilon 2$), discussed in Section 4.3, hold.

Remark 4.7 (on the regularity). The fact that X belongs to a space with one degree of regularity less than the other functions grants hypothesis ($\mathrm{H}_\varepsilon 1$) in Theorem 4.8. This allows us to consider perturbations P that lose one derivative, a property that will be exploited in the applications, see Section 7. Notably, that property enables the study of neutral equations, equations with small delays and the equations of Wheeler-Feynman electrodynamics.

4.3 The main results

We establish the main results that, under appropriate hypotheses, the operator Γ^ε in Section 4.1 has a fixed point which is locally unique. Then by the construction of Γ^ε in Section 4, this fixed point will be a solution of (1) under the functional perturbation.

The result has two sets of hypotheses: a first set, ($\mathrm{H}_0 1$)–($\mathrm{H}_0 2$), concerning the unperturbed orbit; and a second set, ($\mathrm{H}_\varepsilon 1$)–($\mathrm{H}_\varepsilon 2$), on the perturbation P. The existence of a solution is ensured by ($\mathrm{H}_\varepsilon 1$) and its uniqueness by ($\mathrm{H}_\varepsilon 2$).

As indicated in Section 5.1, the only things to check are the fact that the operator Γ^ε map a smooth ball into itself and that it is a contraction in low regularity for all functions in such a smooth ball. We will show that this follows from some simple hypothesis on (1) and we will verify the hypotheses of Theorem 4.8 in concrete examples of interest. Of course, for each of the models, one could formulate the operator Γ directly and verify the propagated bounds and the low regularity contraction.

Theorem 4.8. *We consider the differential equation* (1). *Let $\ell \geqslant 0$ be an integer, and $\mu_0 \in \mathbb{R}$ a fixed parameter. Assume that the unperturbed system satisfies:*

$H_0 1$) There is a uniformly hyperbolic solution $\{x_0(t)\}_{t\in\mathbb{R}}$, see Definition 2.1.

$H_0 2$) The function f is $C^{\ell+2+\mathrm{Lip}}$ and bounded away from zero in a δ-neighborhood of the orbit $\{x_0(t)\}_{t\in\mathbb{R}}$.

Assume that the perturbative map $P\colon \mathbb{R} \times C^{\ell+1+\mathrm{Lip}}([-h, h], \mathbb{R}^n) \times (0, 1)^2 \to \mathbb{R}^n$ in (1) *defines the operator*

$$\mathscr{P}\colon C^{\ell+1+\mathrm{Lip}}(\mathbb{R}, \mathbb{R}^n) \times (0, 1) \to C^{\ell+\mathrm{Lip}}(\mathbb{R}, \mathbb{R}^n), \qquad \mathscr{P}[u, \varepsilon](t) \stackrel{\text{def}}{=} P(t, u_t, \varepsilon, \mu_0)$$

such that:

H_ε1) For all $\varepsilon \in (0,1)$, $u \in C^{\ell+1+\mathrm{Lip}}(\mathbb{R}, \mathbb{R}^n)$, $t \in \mathbb{R}$,

$$\left|\frac{d^j}{dt^j}\mathscr{P}[u,\varepsilon,\mu](t)\right| \leqslant C_j F_j(\|u\|_{C^{j+1}}) \quad j = 0,\dots,\ell, \ \textit{and} \quad \mathrm{Lip}\left(\frac{d^\ell}{dt^\ell}\mathscr{P}[u,\varepsilon,\mu]\right) \leqslant C_\ell^{\mathrm{Lip}} F_\ell^{\mathrm{Lip}}(\|u\|_{C^{\ell+1+\mathrm{Lip}}}),$$

where the constants C's are positive and the functions F's are continuous and increasing on $\mathbb{R}$.

Then there exists $\varepsilon_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there are differentiable maps ϕ and $\widehat{x}$ such that $\mathrm{D}\phi$ is in $C^{\ell+\mathrm{Lip}}$, $\widehat{x}$ is in $C^{\ell+1+\mathrm{Lip}}$, and

$$x \stackrel{\mathrm{def}}{=} (x_0 + \widehat{x}) \circ \phi \tag{35}$$

is a $C^{\ell+1+\mathrm{Lip}}$ solution of (1).

Moreover, if the history value $h < +\infty$ and $\mathscr{P}$ also satisfies

H_ε2) For all $\varepsilon \in (0,1)$, $u^1, u^2 \in C^{\ell+1+\mathrm{Lip}}(\mathbb{R}, \mathbb{R}^n)$, and $t, s \in \mathbb{R}$, there are constants $\mathfrak{L}_1$ and $\mathfrak{L}_2$ such that

$$|\mathscr{P}[u^2,\varepsilon](s) - \mathscr{P}[u^1,\varepsilon](t)| \leqslant \mathfrak{L}_1|s-t| + \mathfrak{L}_2\|u_s^2 - u_t^1\|_{C^1([-h,h])}.$$

Then there exists $\varepsilon_0' \leqslant \varepsilon_0$ such that the maps ϕ and $\widehat{x}$ in (35) are locally unique for all $\varepsilon \in (0, \varepsilon_0')$.

Furthermore, if (H_ε1)–(H_ε2) hold, we obtain a-posteriori result. Given an initial guess $(X_{(0)}, \widehat{x}^s_{(0)}, \widehat{x}^u_{(0)})$ with errors

$$\mathcal{E}_c \stackrel{\mathrm{def}}{=} \Gamma^\varepsilon_c[X_{(0)}, \widehat{x}^s_{(0)}, \widehat{x}^u_{(0)}] - X_{(0)}, \quad \mathcal{E}_s \stackrel{\mathrm{def}}{=} \Gamma^\varepsilon_s[X_{(0)}, \widehat{x}^s_{(0)}, \widehat{x}^u_{(0)}] - \widehat{x}^s_{(0)}, \quad \mathcal{E}_u \stackrel{\mathrm{def}}{=} \Gamma^\varepsilon_u[X_{(0)}, \widehat{x}^s_{(0)}, \widehat{x}^u_{(0)}] - \widehat{x}^u_{(0)},$$

on any bounded interval $[a,b] \subset \mathbb{R}$, we have

$$\begin{aligned} \|(X - X_{(0)})|_{[a,b]}\|_{C^j} &\leqslant \mathfrak{c} E_\eta^{\frac{\ell+1-j}{\ell+1}}, && \textit{for } 0 \leqslant j \leqslant \ell \\ \|(\widehat{x}^\sigma - \widehat{x}^\sigma_{(0)})|_{[a,b]}\|_{C^j} &\leqslant \mathfrak{c} E_\eta^{\frac{\ell+2-j}{\ell+2}}, && \textit{for } \sigma = s,u, \textit{ and } 0 \leqslant j \leqslant \ell+1, \end{aligned}$$

where $E_\eta \stackrel{\mathrm{def}}{=} \|\mathcal{E}_c\|_\eta + \|\mathcal{E}_s\|_\eta + \|\mathcal{E}_u\|_\eta + \|\mathrm{D}\mathcal{E}_s\|_\eta + \|\mathrm{D}\mathcal{E}_u\|_\eta$, and the constant $\mathfrak{c}$ depends on j, a, b, ε, x_0, f, h, P.

Alternatively, when $E_\eta \leqslant 1$, we have that

$$\begin{aligned} \|\mathrm{D}^j(X - X_{(0)})|_{(0,+\infty)}\|_\eta &\leqslant \mathfrak{c} E_\eta^{\frac{1}{j+1}}, && \textit{for } 0 \leqslant j \leqslant \ell \\ \|\mathrm{D}^j(\widehat{x}^\sigma - \widehat{x}^\sigma_{(0)})|_{(0,+\infty)}\|_\eta &\leqslant \mathfrak{c} E_\eta^{\frac{1}{j+1}}, && \textit{for } \sigma = s,u, \textit{ and } 0 \leqslant j \leqslant \ell+1, \end{aligned}$$

Remark 4.9. Note that from the assumptions ($\mathrm{H}_0$1), ($\mathrm{H}_0$2), and the theory of normal hyperbolicity, we have that for $\sigma \in \{c,s,u\}$, the maps $\rho \mapsto \Pi^\sigma_\rho$ are $C^{\ell+1+\mathrm{Lip}}$. We will use this fact in the proof of our main result.

Remark 4.10 (on ϕ regularity)*.* The map ϕ in Theorem 4.8 does not belong to $C^{\ell+1+\mathrm{Lip}}$ because ϕ is not bounded in C^0, see Lemma 5.4. However, the composition $\widehat{x} \circ \phi$ is $C^{\ell+1+\mathrm{Lip}}$ when $\widehat{x}$ is $C^{\ell+1+\mathrm{Lip}}$ and $\mathrm{D}\phi$ is $C^{\ell+\mathrm{Lip}}$.

Remark 4.11 (on the perturbative parameter)*.* We allow the perturbative map P to depend on the perturbative parameter ε. In some applications treated in Section 7, P is obtained by power expansion in ε which implies that P may have higher order terms in ε. In other applications such as the small delay case, the equation will be reformulated such that P will explicitly appear.

Moreover, notice that $\varepsilon \in (0,1)$ is not necessarily a restriction since one can always scale the map P or change its sign to admit other ranges of ε.

Remark 4.12 (on the choice of regularity space)*.* The smallness of ε depends on ℓ, therefore, the method claims results on finite regularity only. The results on analytic regularity are false without extra hypothesis, see [MPNP94, MPN11].

Note that if the equation (1) is smooth enough, once obtaining that there are C^1 solutions in time, one can use the equation (1) to bootstrap the regularity of the solution as the particular problem allows. In some cases, one may obtain C^∞ solutions.

Remark 4.13 (on the a-posteriori formulation)*.* The proofs we are going to present are constructive, hence they can be implemented numerically. The operator concatenates several elementary operations, some of these operations for a 2D model have been addressed in a numerical toolkit in [GYdlL21].

The formulation we adopted in Theorem 4.8 admits an a-posteriori format which states that if there is an approximate solution, then close to it there is a true solution. A-posteriori results can be the basis of computer-assisted proofs (CAP's) because if one is able to estimate rigorously non-degeneracy conditions and errors, then one concludes existence of the solution. The error verification in the approximation is a long finite calculation taking care of round-off and truncation errors. Some cases of CAP's have already been used in delay equations, e.g. [GMJ17, GLMY23, SZ18].

4.3.1 Parameter dependence result

Theorem 4.8 is on the case where the parameter $\mu = \mu_0$ is fixed, while we could modify hypotheses $(\mathrm{H}_\varepsilon 1)$–$(\mathrm{H}_\varepsilon 2)$ easily to obtain results on smooth dependence on parameters of the solution in (35).

Indeed, with $\mu \in (0,1)$, we view ϕ and $\widehat{x}$ as maps $\phi\colon \mathbb{R}\times(0,1)\to\mathbb{R}$ and $\widehat{x}\colon \mathbb{R}\times(0,1)\to\mathbb{R}^n$. Therefore, the solution is of the form

$$x(t,\mu) = x_0\circ\phi(t,\mu) + \widehat{x}\big(\phi(t,\mu),\mu\big).$$

For smooth dependence on parameters, in the first hypothesis on the perturbation $(\mathrm{H}_\varepsilon 1)$, we need bounds on the partial derivatives with respect to t and μ by functions of $\|u\|_{C^{j+1}}$, where the norms are understood as the norms for $u\colon \mathbb{R}\times(0,1)\to\mathbb{R}^n$. Meanwhile, the second hypothesis $(\mathrm{H}_\varepsilon 2)$ should be changed to include the parameter as follows

$$|\mathscr{P}[u^2,\varepsilon,\mu](s) - \mathscr{P}[u^1,\varepsilon,\mu](t)| \leqslant \mathfrak{L}_1|s-t| + \mathfrak{L}_2\|u^2_s - u^1_t\|_{C^1([-h,h])},$$

for all $\mu\in(0,1)$. With the changes in the hypotheses, we obtain that for small ε the solution x is jointly $C^{\ell+1+\mathrm{Lip}}$ in t and μ.

Notice that $\mu\in(0,1)$ is not necessarily a restriction since we can apply an affine transformation to it. Also, we can generalize our result to consider higher dimensional parameter μ with similar argument.

Proving naturally the smooth dependence on parameters is one of the advantages of our framework. In general, the smooth parameter dependence is not trivial for solutions of SDDEs, see [HKWW06, Wal03], and it involves extra assumptions. Nevertheless, we admit that we only search for solutions of a certain form, as in [YGdlL21, YGdlL22].

Theorem 4.8 also applies when the parameter appears in the unperturbed system. Indeed, suppose the unperturbed equation takes the form

$$\dot{x}(t) = g(x(t),\mu_0 + h).$$

If there is μ_0 such that $(\mathrm{H}_0 1)$–$(\mathrm{H}_0 2)$ are satisfied for $f(x) \stackrel{\text{def}}{=} g(x,\mu_0)$, then we define Q as

$$Q(x,h) = \int_0^1 \mathrm{D}_\mu g(x,\mu_0+\sigma h)d\sigma,$$

and consider $\dot{x}(t) = g(x(t),\mu_0) + hQ(x(t),h)$. Since g is smooth, we can incorporate Q in the perturbative map P of a model like (1) to satisfy the assumptions. Note that here h also becomes a perturbative parameter. We could treat the two smallness parameters ε (for P) and h (for Q) jointly by ε or separately in the fixed point proof of an operator Γ^ε.

5. Main ingredients of the proofs

The proof involves several steps; some of them are standard bounds but others are strongly related to the type of perturbations we consider. We will start providing a general overview of the tools and steps of the proposed proof. In particular, we will provide a sequence of lemmas, which build up the whole proof.

5.1 Overview of fixed point arguments

Here we give some ideas on the fixed point theorems used and their variants. Following the proof strategy of center manifold theorem in [Lan73], to obtain the conclusions it suffices to show that the operator Γ^ε satisfies two types of bounds:

B1. Propagated bounds, see Section 5.3;

B2. Low regularity contraction, see Section 5.4.

The propagated bounds establish that a ball B in a space of smooth functions is mapped to itself by Γ^ε. If moreover the operator is a contraction in a low regularity norm on B, then we conclude that there exists a unique fixed point in the low regularity closure of the smooth ball (we will denote this by $\overline{B}$).

The desired result of existence and uniqueness of fixed points can be established by two different arguments.

The first argument is to appeal to a version of Schauder (see [Bre11, p. 179]).

Theorem 5.1. *Let X be a Banach space, $C \subset X$ nonempty, closed convex, $K \subset C$ compact, $\Gamma\colon C \to C$ continuous, $\Gamma(C) \subset K$. Then, Γ has a fixed point in K.*

In our applications, $C = K$ is a ball in a space of highly differentiable functions with domain $\mathbb{R}$ and Lipschitz modulus of continuity in the highest derivative. The space X is a Banach space for functions with domain $\mathbb{R}$ equipped with a low regularity norm. The fact that K is compact is a consequence of an easy version of Arzela-Ascoli theorem since $\mathbb{R}$ is separable. Since $C = K$ is a ball, convexity is obvious.

The propagated bounds in Section 5.3 show that that $\Gamma^\varepsilon(K) \subset K$.

A further simplification is that, since K is compact in the low regularity topology, to prove continuity of $\Gamma^\varepsilon : K \to K$ it suffices to show that the graph of Γ^ε is closed in the low regularity topology. This is very easy to verify.

Application of the Schauder theorem obtains the existence of fixed points using only the propagated bounds.

The low regularity contraction shows that the fixed point x^* is unique and provides – as we show below – with a-posteriori bounds using interpolation inequalities.

To obtain uniqueness, we could consider using other arguments (e.g. using that two fixed points satisfy the invariance equation or other geometric properties).

The contraction in low regularity norm has other consequences besides the uniqueness of the fixed point.

Given a point $x_0 \in K$ we obtain that $\Gamma^n(x_0)$ converges exponentially fast to x^* in the low regularity norm. Furthermore, because of the propagated bounds, the smooth distance between $\Gamma^n(x_0)$ and x^* remains bounded. Using interpolation inequalities [Had98, Kol49, dlLO99], we also obtain exponential convergence of $\Gamma^n(x_0)$ to x^* in spaces of regularity in between. This leads to an a-posteriori result estimating the distance between x_0 and x^* based on estimates of $\Gamma(x_0) - x_0$ in spaces of low regularity. One source of interest is that such estimates for a numerical approximation x_0 can be obtained using a computer assisted proof. In our case, there are some extra complications since some of the norms we use are weighted norms. See Section 5.5.

A second method of proof used very often in the theory of center manifolds is to use the theorem in [Lan73]. The method in [Lan73] uses at the same time the propagated bounds and some other argument to produce uniqueness of fixed points, It can be applied even when we are interested in functions whose domain is a non-separable space (so Arzela-Ascoli requires adaptation).

There are many possible variants. For the low regularity contraction we have several choices. We can use weighted norms (we have used the Razumikhin norms (5.4) in some cases) or contractions in any bounded interval. The only role is to get uniqueness.

Note that we need to verify the contraction property in functions which we already know that are smooth. A notable case which appears a lot in state dependent delays is the composition operator. Note that we can use $\|u_1 \circ u_2 - u_1 \circ \tilde{u}_2\|_{C^0} \leqslant \|\mathrm{D}u_1\|_{C^0} \|u_2 - \tilde{u}_2\|_{C^0}$ if the functions are defined in a convex set, or more generally in a balanced domain – i.e. a domain in which a multiple of the distance among two points bounds from above the length of the shortest path joining them.

Remark 5.2 (Improving the $C^{r+\mathrm{Lip}}$ regularity in the conclusions to C^r). In this paper we formulate existence results in $C^{r+\mathrm{Lip}}$ spaces to obtain solutions in the same regularity space.

This has the minor inconvenience that one has to present extra arguments for the last Lipschitz regularity. The Lipschitz constants, in general do not satisfy formulas such as Faà di Bruno. When we have an Euclidean domain, the Lipschitz constant can be approximated as limit. Therefore, we could prove the result for C^r and add a limit argument for the Lipschitz constant in the last regularity level.

Remark 5.3 (Weaker alternative uniqueness result). The approach we adopted for Theorem 4.8 also admits a weaker version. The propagated bounds (B1) tells us that there is $(X^*, \widehat{x}^*)$ in $\mathcal{X}$ such that $\Gamma^\varepsilon[X^*, \widehat{x}^*] = (X^*, \widehat{x}^*)$.

The low regularity contraction step, (B2), provides the uniqueness in such a ball. It needs to prove the contraction for all pair of elements in $\mathcal{X}$. However, known already the existence we can proceed by contradiction and only check the set of possible fixed points in $\mathcal{X}$. That is, if $(X^*, \widehat{x}^*)$ and $(Y^*, \widehat{y}^*)$ were two different solutions, if we prove there is $\kappa \in (0, 1)$ such that

$$d\big((X^*, \widehat{x}^*), (Y^*, \widehat{y}^*)\big) \leqslant \kappa\, d\big((X^*, \widehat{x}^*), (Y^*, \widehat{y}^*)\big), \tag{36}$$

for a suitable distance $d(\cdot, \cdot)$, then the solution is unique. Notice that κ can depend on ε and also that the inequality in (36) does not need to be strict. This argument is indeed weaker since it is does not say anything about other elements in the ball and thus it does not allow an a-posteriori formulation.

5.2 Estimates on evolution

The evolution of a one-dimensional vector field can be estimated in a completely elementary manner. Even if they are elementary, we collect the estimates in Lemma 5.4 for the ease of reference. Note that we cannot claim that $\phi \in C^\ell$ because our definition of C^ℓ spaces involves uniform boundedness (in particular even the identity map is not C^ℓ in our definition). Even if $Id \notin C^\ell$ and $\phi \notin C^\ell$, we can "summarize" the lemma saying that

$$X - 1 \in C^\ell \Rightarrow \phi - Id \in C^{\ell+1}.$$

Moreover, we have that $\widehat{x} \in C^{\ell+1} \Rightarrow \widehat{x} \circ \phi \in C^{\ell+1}$. For higher dimensional vector fields, the estimates are not so strong and, in fact, even for bounded vector fields the flows can have exponential growth. This is a reason why in this work we can only deal with hyperbolic orbits and not with Normally Hyperbolic Invariant Manifolds.

Lemma 5.4. *Let X be a $C^{\ell+\mathrm{Lip}}$ vector field in $\mathbb{R}$ and ϕ be its associated evolution given by $\dot{\phi}(t) = X \circ \phi(t)$ with initial condition $\phi(0) = 0$. Define $\hat{X} \stackrel{\text{def}}{=} X - 1$ and assume that $\hat{X} \in \mathscr{B}^{\ell+\mathrm{Lip}}_{\mathsf{t}}(0)$ with $\mathsf{t} \stackrel{\text{def}}{=} (\mathsf{t}_0, \mathsf{t}_1, \ldots, \mathsf{t}_\ell, \mathsf{t}^{\mathrm{Lip}}_\ell) \in \mathbb{R}^{\ell+2}_+$. If $\mathsf{t}_0 < 1$, then*

1. *ϕ and ϕ^{-1} are strictly increasing functions.*

2. *For all* t *and* s *in* $\mathbb{R}$,

$$\begin{aligned} (1-\mathfrak{t}_0)|t-s| \leqslant |\phi(t)-\phi(s)| \leqslant (1+\mathfrak{t}_0)|t-s|, \\ \frac{1}{1+\mathfrak{t}_0}|t-s| \leqslant |\phi^{-1}(t)-\phi^{-1}(s)| \leqslant \frac{1}{1-\mathfrak{t}_0}|t-s|. \end{aligned} \tag{37}$$

   In particular, $(1-\mathfrak{t}_0)|t| \leqslant |\phi(t)| \leqslant (1+\mathfrak{t}_0)|t|$ *and* $\frac{1}{1+\mathfrak{t}_0}|t| \leqslant |\phi^{-1}(t)| \leqslant \frac{1}{1-\mathfrak{t}_0}|t|$.

3. $|\mathrm{D}^{j+1}\phi(t)| \leqslant \tilde{\mathfrak{t}}_j$ *and* $|\mathrm{D}^{j+1}(\phi^{-1})(t)| \leqslant \hat{\mathfrak{t}}_j$ *for all* $j = 0,\dots,\ell$, *where* $\tilde{\mathfrak{t}}_j$ *and* $\hat{\mathfrak{t}}_j$ *only depend on* $\mathfrak{t}_0,\mathfrak{t}_1,\dots,\mathfrak{t}_j$.

4. $\mathrm{Lip}(\mathrm{D}^{\ell+1}\phi) \leqslant \tilde{\mathfrak{t}}_\ell^{\mathrm{Lip}}$ *and* $\mathrm{Lip}(\mathrm{D}^{\ell+1}(\phi^{-1})) \leqslant \hat{\mathfrak{t}}_\ell^{\mathrm{Lip}}$, *where* $\tilde{\mathfrak{t}}_\ell^{\mathrm{Lip}}$ *and* $\hat{\mathfrak{t}}_\ell^{\mathrm{Lip}}$ *only depend on* $\mathfrak{t}$.

In particular, $\mathrm{D}\phi \in \mathscr{B}_{\tilde{\mathfrak{t}}}^{\ell+\mathrm{Lip}}$ *and* $\mathrm{D}(\phi^{-1}) \in \mathscr{B}_{\hat{\mathfrak{t}}}^{\ell+\mathrm{Lip}}$.

Proof. Let us first observe that $\mathrm{D}(\phi^{-1})(t) = \frac{1}{X(t)}$, hence $\phi^{-1}(t) = \int_0^t \frac{d\sigma}{X(\sigma)}$.

1. Since $\mathfrak{t}_0 < 1$, we have $X > 0$, which implies the monotonicity.

2. Note that

$$\phi(t) - \phi(s) = \int_s^t X \circ \phi(u)\, du.$$

   By the assumption on $\hat{X}$, we obtain $1-\mathfrak{t}_0 \leqslant X \leqslant 1+\mathfrak{t}_0$, then the first inequality in (37) is proved. We can prove the second inequality for ϕ^{-1} similarly. The last argument is true since $\phi(0) = 0$.

3. Clearly, $|\mathrm{D}\phi(t)| \leqslant 1+\mathfrak{t}_0$. We now prove that $|\mathrm{D}^j(\mathrm{D}\phi(t))| \leqslant \tilde{\mathfrak{t}}_j$ for $j = 1,\dots,\ell$.

   i) $\mathrm{D}^2\phi(t) = \mathrm{D}\hat{X} \circ \phi(t)(1+\hat{X}\circ\phi(t))$ which is bounded by $\mathfrak{t}_1(1+\mathfrak{t}_0)$.

   ii) By the Faà di Bruno Formula, we have an expression of the form

$$\mathrm{D}^{r+1}\phi = \mathrm{D}^r(\hat{X}\circ\phi) = \sum_{\substack{(m_1,\dots,m_r)\in\mathbb{N}^r \\ \sum_{j=1}^r jm_j = r}} C_{m_1,\dots,m_r} \mathrm{D}^{m_1+\cdots+m_r}\hat{X}\circ\phi \prod_{j=1}^r (\mathrm{D}^j\phi)^{m_j},$$

   where $C_{m_1,\dots,m_r}$'s are combinatorial numbers. By the induction hypothesis $|\mathrm{D}^{j+1}\phi(t)| \leqslant \tilde{\mathfrak{t}}_j$ for $j = 0,\dots,r-1$ and triangle inequality, we prove that $\mathrm{D}^{r+1}\phi$ is bounded by $\tilde{\mathfrak{t}}_r$ depending on $\mathfrak{t}_0,\mathfrak{t}_1,\dots,\mathfrak{t}_r$.

   Since

$$\mathrm{D}^{r+1}(\phi^{-1}) = \mathrm{D}^r\left(\frac{1}{X}\right), \tag{38}$$

   we derive that $\mathrm{D}^{r+1}(\phi^{-1})(t)$ is bounded by a $\tilde{\mathfrak{t}}_r$ only depending on $\mathfrak{t}_0,\mathfrak{t}_1,\dots,\mathfrak{t}_r$.

4. Straightforward by using Lemma 4.2. □

5.3 Propagated bounds

In this section, we show that for small enough ε, we can find the parameters $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$ of the space $\mathcal{X}$, see (33) and (34), so that if the inputs $(X, \widehat{x}^s, \widehat{x}^u)$ are in the space $\mathcal{X}$, then its image under Γ^ε also lies in $\mathcal{X}$.

To reach this goal, we bound $\Gamma^\varepsilon[X, \widehat{x}^s, \widehat{x}^u]$ and its derivatives by algebraic expressions of $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$ in the following Lemmas. We use basic tools like triangle inequalities and rules of differentiation, including the Leibnitz product formula, Faà di Bruno formula, etc.

At the end of the section, we discuss the choices for the parameters $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$ for small enough ε.

Lemma 5.5. *Let $\ell \geqslant 0$ be a fixed integer. There are constants $a_0, \dots, a_\ell, a_\ell^{\mathrm{Lip}}$, $b_0, \dots, b_\ell, b_\ell^{\mathrm{Lip}}$ such that $T[x_0, \widehat{x}]$ and $\mathcal{B}[X, \widehat{x}]$ defined in (19) and (22) have the following bounds for all $j = 0, \dots, \ell$.*

1. $|\mathrm{D}^j T[x_0, \widehat{x}](\rho)| \leqslant a_j$ *for a_j's depending on $\mathfrak{s}_0, \dots, \mathfrak{s}_j$, $\mathfrak{u}_0, \dots, \mathfrak{u}_j$, $\|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$;*
2. $|\mathrm{D}^j \mathcal{B}[X, \widehat{x}](\rho)| \leqslant b_j$ *for b_j's depending on $\mathfrak{t}_0, \dots, \mathfrak{t}_j$, $\mathfrak{s}_0, \dots, \mathfrak{s}_j$, $\mathfrak{u}_0, \dots, \mathfrak{u}_j$, $\|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$;*
3. $\mathrm{Lip}(\mathrm{D}^\ell T[x_0, \widehat{x}]) \leqslant a_\ell^{\mathrm{Lip}}$ *and* $\mathrm{Lip}(\mathrm{D}^\ell \mathcal{B}[X, \widehat{x}]) \leqslant b_\ell^{\mathrm{Lip}}$ *for a_ℓ^{Lip} and b_ℓ^{Lip} depending on $\mathfrak{t}$, $\mathfrak{s}$, $\mathfrak{u}$, $\|x_0\|_{C^{\ell+\mathrm{Lip}}}$, and $\|f\|_{C^{\ell+2+\mathrm{Lip}}}$.*

Where the norms of f are evaluated on a neighborhood of $\{x_0(t)\}_{t \in \mathbb{R}}$.

Proof. This lemma is proved by using Leibnitz product formula and Faà di Bruno's formula. By the classical Taylor error bound,

$$T[x_0, \widehat{x}](\rho) = \int_0^1 \int_0^\sigma \mathrm{D}^2 f \circ (x_0 + s\widehat{x})(\rho)\widehat{x}(\rho)^2 \, ds \, d\sigma, \tag{39}$$

where we use $\mathrm{D}^2 f \circ (x_0 + s\widehat{x})(\rho)\widehat{x}(\rho)^2$ to denote the result of the bilinear operator $\mathrm{D}^2 f \circ (x_0 + s\widehat{x})(\rho)$ acting on $\widehat{x}(\rho)$ and $\widehat{x}(\rho)$. Then, we can bound $\mathrm{D}^j T[x_0, \widehat{x}]$ in terms of $\mathfrak{s}_0 + \mathfrak{u}_0, \dots, \mathfrak{s}_j + \mathfrak{u}_j$, $\|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$. Similarly, $\mathrm{Lip}(\mathrm{D}^\ell T[x_0, \widehat{x}])$ can be bounded by $\mathfrak{s}$, $\mathfrak{u}$, $\|x_0\|_{C^{\ell+\mathrm{Lip}}}$, and $\|f\|_{C^{\ell+2+\mathrm{Lip}}}$.

Since

$$\mathrm{D}^j \mathcal{B}[X, \widehat{x}] = \mathrm{D}^j[(1 - X)\mathrm{D}f \circ x_0 \widehat{x}] + \mathrm{D}^j T[x_0, \widehat{x}],$$

$\mathrm{D}^j \mathcal{B}[X, \widehat{x}]$ is bounded by an algebraic expression of $\mathfrak{t}_0, \dots, \mathfrak{t}_j$, $\mathfrak{s}_0, \dots, \mathfrak{s}_j$, $\mathfrak{u}_0, \dots, \mathfrak{u}_j$, $\|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$. □

The operator Γ^ε defined in Section 4.1 involves quotients. To bound the center component of Γ^ε, we need to assume that the vector field f is bounded away from zero on the unperturbed hyperbolic solution $\{x_0(t)\}$, i.e. there is $b > 0$ such that $b \leqslant \inf\{|f \circ x_0|\}$. For the stable and unstable components, we use the fact that $1 - \mathfrak{t}_0 < X$ for $X \in \mathscr{B}_{\mathfrak{t}}^{\ell+\mathrm{Lip}}(1)$.

We use rules of differentiation, Cauchy-Schwartz inequality, and Lemma 5.5 to prove:

Proposition 5.6 (center correction)**.** *There are constants $\mathfrak{b}_{c,j}$ and $\mathfrak{d}_{c,j}$ such that for all $\rho \in \mathbb{R}$ and $(X, \widehat{x}^s, \widehat{x}^u) \in \mathcal{X}$, the operator Γ_c^ε defined in (27) satisfies*

1. $|\Gamma_c^\varepsilon[X, \widehat{x}^s, \widehat{x}^u](\rho) - 1| \leqslant \mathfrak{b}_{c,0} + \varepsilon\mathfrak{d}_{c,0}$;
2. $|\mathrm{D}^j \Gamma_c^\varepsilon[X, \widehat{x}^s, \widehat{x}^u](\rho)| \leqslant \mathfrak{b}_{c,j} + \varepsilon\mathfrak{d}_{c,j}$ *and* $j = 1, \dots, \ell$;
3. $\mathrm{Lip}\left(\mathrm{D}^\ell \Gamma_c^\varepsilon[X, \widehat{x}^s, \widehat{x}^u]\right) \leqslant \mathfrak{b}_{c,\ell}^{\mathrm{Lip}} + \varepsilon\mathfrak{d}_{c,\ell}^{\mathrm{Lip}}$,

where $\mathfrak{b}_{c,j}$'s and $\mathfrak{d}_{c,j}$'s depend on $\|x_0\|_{C^j}$, $\|\Pi_t^c\|_{C^j}$, $\|f\|_{C^{j+2}}$, $\mathfrak{t}_0, \dots, \mathfrak{t}_j$, $\mathfrak{s}_0, \dots, \mathfrak{s}_j$, and $\mathfrak{u}_0, \dots, \mathfrak{u}_j$. The constants $b_{c,\ell}^{\mathrm{Lip}}$ and $\mathfrak{d}_{c,\ell}^{\mathrm{Lip}}$ depend on $\|x_0\|_{C^{\ell+\mathrm{Lip}}}$, $\|\Pi_t^c\|_{C^{\ell+\mathrm{Lip}}}$, $\|f\|_{C^{\ell+2+\mathrm{Lip}}}$, $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$.

Proof. For the first bound, we use (H_ε1) and Lemma 5.5, and note that the terms involving $\mathcal{B}[X, \widehat{x}]$ and $\varphi[X, \widehat{x}]$ are bounded by $\mathfrak{b}_{c,0}$ and $\varepsilon\mathfrak{d}_{c,0}$ respectively, where

$$\mathfrak{b}_{c,0} = \frac{C_\Pi \|f\|_{C^0}}{b^2}\left[\mathfrak{t}_0\|f\|_{C^1}(\mathfrak{s}_0 + \mathfrak{u}_0) + \frac{1}{2}\|f\|_{C^2}(\mathfrak{s}_0 + \mathfrak{u}_0)^2\right] \qquad \mathfrak{d}_{c,0} = \frac{C_\Pi \|f\|_{C^0}}{b^2}\|\varphi[X, \widehat{x}]\|_{C^0}.$$

Then we use (H_ε1), Faà di Bruno formula, Leibnitz product formula, the quotient rule, Lemma 5.4, and Lemma 5.5 to obtain the bounds for the derivatives of Γ_c^ε. □

The operators Γ_s^ε and Γ_u^ε in (29) gain one derivative thanks to the integration, which is the reason why we define the space $\mathcal{X}$ in (33) with different regularities in the components. Here we bound the derivatives of Γ_s^ε and Γ_u^ε up to order $\ell + 1$.

Proposition 5.7 (stable and unstable corrections)**.** *There are constants $\mathfrak{b}_{\sigma,k}$ and $\mathfrak{d}_{\sigma,k}$, $\sigma \in \{s,u\}$, such that for all $\rho \in \mathbb{R}$ and $(X, \widehat{x}^s, \widehat{x}^u) \in \mathcal{X}$, the operators Γ^ε_s and Γ^ε_u defined in (29) satisfy*

1. $|\Gamma^\varepsilon_\sigma[X, \widehat{x}^s, \widehat{x}^u](\rho)| \leqslant \mathfrak{b}_{\sigma,0} + \varepsilon \mathfrak{d}_{\sigma,0}$.
2. $|\mathrm{D}^j \Gamma^\varepsilon_\sigma[X, \widehat{x}^s, \widehat{x}^u](\rho)| \leqslant \mathfrak{b}_{\sigma,j} + \varepsilon \mathfrak{d}_{\sigma,j}$ *for* $j = 1, \ldots, \ell+1$;
3. $\mathrm{Lip}\left(\mathrm{D}^{\ell+1}\Gamma^\varepsilon_\sigma[X, \widehat{x}^s, \widehat{x}^u]\right) \leqslant \mathfrak{b}^{\mathrm{Lip}}_{\sigma,\ell+1} + \varepsilon \mathfrak{d}^{\mathrm{Lip}}_{\sigma,\ell+1}$,

where for $j > 0$, $\mathfrak{b}_{\sigma,j}$'s and $\mathfrak{d}_{\sigma,j}$'s depend on $\|x_0\|_{C^{j-1}}$, λ_σ, $\|\Pi^\sigma_t\|_{C^{j-1}}$, $\|f\|_{C^{j+1}}$, $\mathfrak{t}_0, \ldots, \mathfrak{t}_{j-1}$, $\mathfrak{s}_0, \ldots, \mathfrak{s}_{j-1}$, and $\mathfrak{u}_0, \ldots, \mathfrak{u}_{j-1}$. Moreover, $\mathfrak{d}_{\sigma,j}$'s also depend on $\mathfrak{s}_j$ and $\mathfrak{u}_j$. Similarly, the constants $b^{\mathrm{Lip}}_{\sigma,\ell+1}$ and $\mathfrak{d}^{\mathrm{Lip}}_{\sigma,\ell+1}$ depend on $\|x_0\|_{C^{\ell+\mathrm{Lip}}}$, λ_σ, $\|\Pi^\sigma_t\|_{C^{\ell+\mathrm{Lip}}}$, $\|f\|_{C^{\ell+2+\mathrm{Lip}}}$, $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$, where the dependence on $\mathfrak{s}^{\mathrm{Lip}}_{\ell+1}$ and $\mathfrak{u}^{\mathrm{Lip}}_{\ell+1}$ is only for $\mathfrak{d}^{\mathrm{Lip}}_{\sigma,\ell+1}$.

Proof. Note that

$$\mathfrak{b}_{\sigma,0} = \frac{C_\Pi C_U}{\lambda_\sigma(1-\mathfrak{t}_0)}\left[\mathfrak{t}_0\|f\|_{C^1}(\mathfrak{s}_0+\mathfrak{u}_0) + \frac{1}{2}\|f\|_{C^2}(\mathfrak{s}_0+\mathfrak{u}_0)^2\right] \qquad \mathfrak{d}_{\sigma,0} = \frac{C_\Pi C_U}{\lambda_\sigma(1-\mathfrak{t}_0)}\|\varphi[X,\widehat{x}]\|_{C^0}.$$

In order to bound the derivatives of $\Gamma^\varepsilon_\sigma$, we use the fact that $\Gamma^\varepsilon_\sigma$ solves the differential equation

$$\mathrm{D}\Gamma^\varepsilon_\sigma[X, \widehat{x}^s, \widehat{x}^u](\rho) = \mathrm{D}f \circ x_0(\rho)\Gamma^\varepsilon_\sigma[X, \widehat{x}^s, \widehat{x}^u](\rho) + \Pi^\sigma_\rho \frac{1}{X(\rho)}\Big[\mathcal{B}[X,\widehat{x}](\rho) + \varepsilon\varphi[X,\widehat{x}](\rho)\Big], \quad \sigma \in \{s,u\}. \tag{40}$$

Then we can obtain the bounds with Faà di Bruno formula, Leibnitz product formula, the quotient rule, Lemma 5.4, and Lemma 5.5. □

It remains to show that it is possible to choose the components of $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$ so that Γ^ε maps $\mathcal{X}$ into itself as long as ε is small enough. As we will see, we need to choose a small $\mathfrak{t}_0$. Without loss of generality, we assume that $\mathfrak{t}_0 \leqslant \frac{1}{2}$ so that $\frac{1}{1-\mathfrak{t}_0} \leqslant 2$ and we do not need to worry about $1-\mathfrak{t}_0$ in the denominator.

Indeed, the zero order constants should satisfy

$$\begin{aligned} \mathfrak{b}_{c,0} + \varepsilon\mathfrak{d}_{c,0} &\leqslant \mathfrak{t}_0, \\ \mathfrak{b}_{s,0} + \varepsilon\mathfrak{d}_{s,0} &\leqslant \mathfrak{s}_0, \\ \mathfrak{b}_{u,0} + \varepsilon\mathfrak{d}_{u,0} &\leqslant \mathfrak{u}_0. \end{aligned} \tag{41}$$

where the left sides of the inequalities come from Proposition 5.6 and Proposition 5.7. As $\mathfrak{b}_{c,0}$, $\mathfrak{b}_{s,0}$, and $\mathfrak{b}_{u,0}$ being quadratic in $\mathfrak{t}_0$, $\mathfrak{s}_0$, and $\mathfrak{u}_0$, we can choose small enough $\mathfrak{t}_0$, $\mathfrak{s}_0$, and $\mathfrak{u}_0$ so that when ε is small enough, the set of inequalities (41) are satisfied.

For the i-th order, the following inequalities should be satisfied.

$$\begin{aligned} G^i_c(\varepsilon, \mathfrak{t}_0, \ldots, \mathfrak{t}_i, \mathfrak{s}_0, \ldots, \mathfrak{s}_i, \mathfrak{u}_0, \ldots, \mathfrak{u}_i) &\leqslant \mathfrak{t}_i, \\ G^i_s(\varepsilon, \mathfrak{t}_0, \ldots, \mathfrak{t}_{i-1}, \mathfrak{s}_0, \ldots, \mathfrak{s}_{i-1}, \mathfrak{u}_0, \ldots, \mathfrak{u}_{i-1}) &\leqslant \mathfrak{s}_i, \\ G^i_u(\varepsilon, \mathfrak{t}_0, \ldots, \mathfrak{t}_{i-1}, \mathfrak{s}_0, \ldots, \mathfrak{s}_{i-1}, \mathfrak{u}_0, \ldots, \mathfrak{u}_{i-1}) &\leqslant \mathfrak{u}_i, \end{aligned} \tag{42}$$

where G^i_σ, $\sigma \in \{c,s,u\}$ are polynomials of ε and the components of $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{u}$. Moreover, one factor in the coefficient of $\mathfrak{t}_i$ in G^i_c is $(\mathfrak{s}_0+\mathfrak{u}_0)$. In order to guarantee (42), we first fix $\mathfrak{s}_i$ and $\mathfrak{u}_i$, and then choose $\mathfrak{t}_i$. Similar arguments hold for $\mathfrak{t}^{\mathrm{Lip}}_\ell$, $\mathfrak{s}^{\mathrm{Lip}}_{\ell+1}$, and $\mathfrak{u}^{\mathrm{Lip}}_{\ell+1}$. Indeed, in this process, we may have to ask for smaller ε, $\mathfrak{s}_0$, and $\mathfrak{u}_0$ at each step, so we will not be able to obtain C^∞ result with our method in general.

5.4 Low regularity contraction

The operator Γ^ε defined in Section 4.1 is a contraction on $\mathcal{X}^\ell_{\mathfrak{t},\mathfrak{s},\mathfrak{u}}$ in (33) if there is $\kappa \in (0,1)$ such that

$$d\Big(\big(\Gamma^\varepsilon_c[X,\widehat{x}^s,\widehat{x}^u],\Gamma^\varepsilon_s[X,\widehat{x}^s,\widehat{x}^u],\Gamma^\varepsilon_u[X,\widehat{x}^s,\widehat{x}^u]\big),\big(\Gamma^\varepsilon_c[Y,\widehat{y}^s,\widehat{y}^u],\Gamma^\varepsilon_s[Y,\widehat{y}^s,\widehat{y}^u],\Gamma^\varepsilon_u[Y,\widehat{y}^s,\widehat{y}^u]\big)\Big) < \kappa\, d\Big(\big(X,\widehat{x}^s,\widehat{x}^u\big),\big(Y,\widehat{y}^s,\widehat{y}^u\big)\Big), \quad (43)$$

for all $(X,\widehat{x}^s,\widehat{x}^u)$ and $(Y,\widehat{y}^s,\widehat{y}^u)$ in $\mathcal{X}^\ell_{\mathfrak{t},\mathfrak{s},\mathfrak{u}}$ and $d(\cdot,\cdot)$ a distance.

We consider the distance in a low regularity space, $\mathcal{X}^\ell_{\mathfrak{t},\mathfrak{s},\mathfrak{u}}$ where $\ell = 0$. The space $\mathcal{X}^0_{\mathfrak{t},\mathfrak{s},\mathfrak{u}}$ has information of the vector field X, $\widehat{x}$, $\mathrm{D}\widehat{x}$. Because we accept time-dependence in the perturbative map P, our construction requires to bound the difference of backward flows associated to the center correction. More precisely, if ϕ and ψ are flows of X and Y respectively, we need to bound $\phi^{-1}(\rho) - \psi^{-1}(\rho)$. Neverthelss, this may fail to be bounded in C^0 (e.g. if the vector fields differ by a constant) but, however, it can be bounded in C_η space for $\eta > 0$, see (32). Therefore, we consider the distance:

$$d\Big(\big(X,\widehat{x}^s,\widehat{x}^u\big),\big(Y,\widehat{y}^s,\widehat{y}^u\big)\Big) \stackrel{\text{def}}{=} \|X-Y\|_\eta + \|\widehat{x}^s-\widehat{y}^s\|_\eta + \|\widehat{x}^u-\widehat{y}^u\|_\eta + \|\mathrm{D}\widehat{x}^s-\mathrm{D}\widehat{y}^s\|_\eta + \|\mathrm{D}\widehat{x}^u-\mathrm{D}\widehat{y}^u\|_\eta,$$

where the norm $\|\cdot\|_\eta$ on $\mathbb{R}$ is defined by

$$\|x-y\|_\eta \stackrel{\text{def}}{=} \sup_{\rho\in\mathbb{R}} |x(\rho)-y(\rho)|e^{-\eta|\rho|}.$$

In what follows and for typographical reasons, we may just write $\widehat{x}$ for $(\widehat{x}^s,\widehat{x}^u)$ to express, for instance, $\|\widehat{x}-\widehat{y}\|$ instead of $\|\widehat{x}^s-\widehat{y}^s\| + \|\widehat{x}^u-\widehat{y}^u\|$. Similarly for $\mathrm{D}\widehat{x}$ and $\mathrm{D}\widehat{y}$.

Assuming that for $(X,\widehat{x})$ and $(Y,\widehat{y})$ in $\mathcal{X}$, we have bounds in the differences involving $\mathcal{B}$ defined in (22) and φ defined in (23) as in inequalities (47) and (48). Then if $0 < b \leqslant \inf\{|f\circ x_0|\}$, we can obtain

$$\|\Gamma^\varepsilon_c[X,\widehat{x}] - \Gamma^\varepsilon_c[Y,\widehat{y}]\|_\eta \leqslant \frac{C_\Pi\|f\|_{C^0}}{b^2}\Big[(\mathfrak{d}_\mathcal{B}+\varepsilon\mathfrak{d}_\varphi)\|X-Y\|_\eta + (\mathfrak{c}_\mathcal{B}+\varepsilon\mathfrak{c}_\varphi)\|\widehat{x}-\widehat{y}\|_\eta + \varepsilon\mathfrak{e}_\varphi\|\mathrm{D}\widehat{x}-\mathrm{D}\widehat{y}\|_\eta\Big], \quad (44)$$

and for $\sigma \in \{s,u\}$,

$$\|\Gamma^\varepsilon_\sigma[X,\widehat{x}] - \Gamma^\varepsilon_\sigma[Y,\widehat{y}]\|_\eta \leqslant \frac{2C_\Pi C_U}{(\lambda_\sigma-\eta)(1-\mathfrak{t}_0)}\Big[(\mathfrak{d}_\mathcal{B}+\varepsilon\mathfrak{d}_\varphi)\|X-Y\|_\eta + (\mathfrak{c}_\mathcal{B}+\varepsilon\mathfrak{c}_\varphi)\|\widehat{x}-\widehat{y}\|_\eta + \varepsilon\mathfrak{e}_\varphi\|\mathrm{D}\widehat{x}-\mathrm{D}\widehat{y}\|_\eta\Big], \quad (45)$$

where $\mathfrak{c}_\mathcal{B}$, $\mathfrak{d}_\mathcal{B}$, $\mathfrak{c}_\varphi$, $\mathfrak{d}_\varphi$, and $\mathfrak{e}_\varphi$ are constants specified in Propositions 5.8 and 5.12. Notably, we can make the constants $\mathfrak{c}_\mathcal{B}$ and $\mathfrak{d}_\mathcal{B}$ small. For the stable and unstable components of Γ^ε, we have used the bounds in (9). We discuss the idea for the stable one. With the Razumikhin norm, we have to bound an integral of the form:

$$I_1 = \int_{-\infty}^{\rho} e^{-\lambda_s(\rho-v)}e^{-\eta|\rho|}e^{\eta|v|}dv.$$

We consider the cases when $\rho \leqslant 0$ and $\rho > 0$, and obtain that $I_1 \leqslant \frac{2}{\lambda_s-\eta}$ for both cases, provided $\eta < \lambda_s$. The unstable direction could be estimated similarly when $\eta < \lambda_u$. Therefore, we derive the estimates in (45) when $\eta < \min\{\lambda_s,\lambda_u\}$.

Using the expression of $\mathrm{D}\Gamma^\varepsilon_\sigma$ for $\sigma \in \{s,u\}$ in (40), the differences of $\mathrm{D}\Gamma^\varepsilon_\sigma$ can be estimated by

$$\|\mathrm{D}\Gamma^\varepsilon_\sigma[X,\widehat{x}] - \mathrm{D}\Gamma^\varepsilon_\sigma[Y,\widehat{y}]\|_\eta \leqslant \|f\|_{C^1}\|\Gamma^\varepsilon_\sigma[X,\widehat{x}] - \Gamma^\varepsilon_\sigma[Y,\widehat{y}]\|_\eta + \frac{C_\Pi}{1-\mathfrak{t}_0}\Big[(\mathfrak{d}_\mathcal{B}+\varepsilon\mathfrak{d}_\varphi)\|X-Y\|_\eta + (\mathfrak{c}_\mathcal{B}+\varepsilon\mathfrak{c}_\varphi)\|\widehat{x}-\widehat{y}\|_\eta + \varepsilon\mathfrak{e}_\varphi\|\mathrm{D}\widehat{x}-\mathrm{D}\widehat{y}\|_\eta\Big]. \quad (46)$$

With the bounds in (44)–(46), we prove that the operator is a contraction if ε is small enough.

We now bound the differences in $\mathcal{B}$ and φ for $(X,\widehat{x})$ and $(Y,\widehat{y})$ in Propositions 5.8 and 5.12.

Proposition 5.8. *Let $\mathcal{B}$ be the map defined in* (22) *and let X, Y be in $\mathscr{B}^0_{\mathfrak{t}_0}(1)$ and let $\widehat{x}$, $\widehat{y}$ be in $\mathscr{B}^0_{\mathfrak{s}_0+\mathfrak{u}_0}(0)$. Then there are constants $\mathfrak{c}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{B}}$ only depending on $\|f\|_{C^{2+\mathrm{Lip}}}$, $\mathfrak{t}_0$, $\mathfrak{s}_0$, and $\mathfrak{u}_0$ such that*

$$|\mathcal{B}[X,\widehat{x}](\rho) - \mathcal{B}[Y,\widehat{y}](\rho)|e^{-\eta|\rho|} \leqslant \mathfrak{c}_{\mathcal{B}}\|\widehat{x}-\widehat{y}\|_\eta + \mathfrak{d}_{\mathcal{B}}\|X-Y\|_\eta. \tag{47}$$

Proof. $\mathcal{B}$ consists of two terms. The one coming from the Taylor error is bounded using the integral formulation,

$$T[x_0,\widehat{x}](\rho) = \int_0^1\int_0^\sigma \mathrm{D}^2 f\circ(x_0+s\widehat{x})(\rho)\widehat{x}(\rho)^2\,ds\,d\sigma.$$

Hence by adding and subtracting

$$|T[x_0,\widehat{x}](\rho) - T[x_0,\widehat{y}](\rho)|e^{-\eta|\rho|} \leqslant (\mathfrak{s}_0+\mathfrak{u}_0)\left(\mathrm{Lip}(\mathrm{D}^2 f)(\mathfrak{s}_0+\mathfrak{u}_0) + \|\mathrm{D}^2 f\|\right)\|\widehat{x}-\widehat{y}\|_\eta.$$

The other term in $\mathcal{B}$ is also bounded similarly, by adding and subtracting, which ends up to the final bound

$$\begin{aligned}|\mathcal{B}[X,\widehat{x}](\rho) - \mathcal{B}[Y,\widehat{y}](\rho)|e^{-\eta|\rho|} \leqslant \Big(\|f\|_{C^1}\mathfrak{t}_0 + (\mathfrak{s}_0+\mathfrak{u}_0)\left(\mathrm{Lip}(\mathrm{D}^2 f)(\mathfrak{s}_0+\mathfrak{u}_0) + \|\mathrm{D}^2 f\|\right)\Big)\|\widehat{x}-\widehat{y}\|_\eta \\ + \|f\|_{C^1}(\mathfrak{s}_0+\mathfrak{u}_0)\|X-Y\|_\eta.\end{aligned}$$

Defining

$$\begin{aligned}\mathfrak{c}_{\mathcal{B}} &\stackrel{\text{def}}{=} \|f\|_{C^1}\mathfrak{t}_0 + (\mathfrak{s}_0+\mathfrak{u}_0)\left(\mathrm{Lip}(\mathrm{D}^2 f)(\mathfrak{s}_0+\mathfrak{u}_0) + \|\mathrm{D}^2 f\|\right),\\ \mathfrak{d}_{\mathcal{B}} &\stackrel{\text{def}}{=} \|f\|_{C^1}(\mathfrak{s}_0+\mathfrak{u}_0),\end{aligned}$$

we have the desired inequality. Moreover, the constants $\mathfrak{c}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{B}}$ are small if $\mathfrak{t}_0$, $\mathfrak{s}_0$, and $\mathfrak{u}_0$ are small. □

Remark 5.9. Notice that the smallness of $\mathfrak{c}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{B}}$ is ensured by choosing small enough $\mathfrak{t}_0$, $\mathfrak{s}_0$, and $\mathfrak{u}_0$.

To bound the difference in φ, we prove two preliminary lemmas. Lemma 5.10 shows how to bound the difference of two backward flows, which motivates our choice of Razumikhin norm. Lemma 5.11 bounds difference of two forward flows composed with backward ones. The last result is essential for the type of functional perturbations we are interested in.

Lemma 5.10. *Let X and Y be vector fields in $\mathbb{R}$ in a ball $\mathscr{B}^0_{\mathfrak{t}_0}(1)$ with $\mathfrak{t}_0 \in (0,1)$ and let $\eta > 0$. If $\dot{\phi} = X\circ\phi$ and $\dot{\psi} = Y\circ\psi$ with zero initial conditions at zero, then*

$$|\phi^{-1}(\rho) - \psi^{-1}(\rho)|e^{-\eta|\rho|} \leqslant \frac{\|X-Y\|_\eta}{\eta(1-\mathfrak{t}_0)^2}.$$

In particular, $\|\phi^{-1}-\psi^{-1}\|_\eta \leqslant \frac{1}{\eta(1-\mathfrak{t}_0)^2}\|X-Y\|_{C^0}$.

Proof. Since $\phi(0)=\psi(0)=0$,

$$\phi^{-1}(\rho) = \int_0^\rho \frac{d\sigma}{X(\sigma)} \quad\text{and}\quad \psi^{-1}(\rho) = \int_0^\rho \frac{d\sigma}{Y(\sigma)}.$$

Therefore

$$|\phi^{-1}(\rho) - \psi^{-1}(\rho)|e^{-\eta|\rho|} \leqslant \frac{\|X-Y\|_\eta}{(1-\mathfrak{t}_0)^2}\int_0^1 |\rho|e^{(\sigma-1)\eta|\rho|}\,d\sigma = \frac{\|X-Y\|_\eta}{\eta(1-\mathfrak{t}_0)^2}(1-e^{-\eta|\rho|}) \leqslant \frac{\|X-Y\|_\eta}{\eta(1-\mathfrak{t}_0)^2}. \qquad \square$$

Lemma 5.11. *Let ϕ, ψ be flows of vector fields X, $Y \in \mathscr{B}^{\mathrm{Lip}}_{(\mathfrak{t}_0,\mathfrak{t}_1)}(1)$ respectively with zero initial conditions at zero. For all $s \in [-h,h]$, define*

$$\alpha(\rho,s) \stackrel{\mathrm{def}}{=} \phi(\phi^{-1}(\rho)+s) \quad \textit{and} \quad \beta(\rho,s) \stackrel{\mathrm{def}}{=} \psi(\psi^{-1}(\rho)+s).$$

Then there is a constant $\mathfrak{z}$ depending on $\mathfrak{t}_0, \mathfrak{t}_1$, η, h such that

$$\sup_{s\in[-h,h]} |\alpha(\rho,s)-\beta(\rho,s)|e^{-\eta|\rho|} \leqslant \mathfrak{z}\|X-Y\|_\eta.$$

Proof. In order to consider different signs of $s \in [-h,h]$, we define $\alpha_\pm(\rho,s) \stackrel{\mathrm{def}}{=} \alpha(\rho,\pm s)$ and $\beta_\pm(\rho,s) \stackrel{\mathrm{def}}{=} \beta(\rho,\pm s)$ for $s \in [0,h]$. Then

$$\sup_{s\in[-h,h]} |\alpha(\rho,s)-\beta(\rho,s)|e^{-\eta|\rho|} = \max\left\{ \sup_{s\in[0,h]} |\alpha_+(\rho,s)-\beta_+(\rho,s)|e^{-\eta|\rho|}, \sup_{s\in[0,h]} |\alpha_-(\rho,s)-\beta_-(\rho,s)|e^{-\eta|\rho|} \right\}.$$

By expanding in s,

$$\alpha_\pm(\rho,s) = \rho \pm \int_0^s X\circ\alpha_\pm(\rho,\sigma)\,d\sigma \qquad \text{and} \qquad \beta_\pm(\rho,s) = \rho \pm \int_0^s Y\circ\beta_\pm(\rho,\sigma)\,d\sigma.$$

Adding and subtracting,

$$|\alpha_\pm(\rho,s)-\beta_\pm(\rho,s)|e^{-\eta|\rho|} \leqslant \int_0^s |X\circ\alpha_\pm(\rho,\sigma)-Y\circ\alpha_\pm(\rho,\sigma)|e^{-\eta|\rho|} + \mathrm{Lip}(Y)|\alpha_\pm(\rho,\sigma)-\beta_\pm(\rho,\sigma)|e^{-\eta|\rho|}\,d\sigma.$$

Notice that

$$\begin{aligned}\int_0^s |X\circ\alpha_\pm(\rho,\sigma)-Y\circ\alpha_\pm(\rho,\sigma)|e^{-\eta|\rho|}d\sigma &\leqslant \|X-Y\|_\eta \int_0^s e^{\eta\left[|\alpha_\pm(\rho,\sigma)|-|\rho|\right]}\,d\sigma \\ &\leqslant \|X-Y\|_\eta \int_0^s e^{\eta(1+\mathfrak{t}_0)\sigma}\,d\sigma = \|X-Y\|_\eta \frac{e^{\eta(1+\mathfrak{t}_0)h}-1}{\eta(1+\mathfrak{t}_0)}.\end{aligned}$$

By Grönwall's inequality,

$$|\alpha_\pm(\rho,s)-\beta_\pm(\rho,s)|e^{-\eta|\rho|} \leqslant e^{\mathfrak{t}_1 h}\frac{e^{\eta(1+\mathfrak{t}_0)h}-1}{\eta(1+\mathfrak{t}_0)}\|X-Y\|_\eta. \qquad \square$$

Proposition 5.12. *There are constants $\mathfrak{c}_\varphi$, $\mathfrak{d}_\varphi$, and $\mathfrak{e}_\varphi$ such that for all X, $Y \in \mathscr{B}^{\mathrm{Lip}}_{(\mathfrak{t}_0,\mathfrak{t}_1)}(1)$ and $\widehat{x}$, $\widehat{y} \in \mathscr{B}^{1+\mathrm{Lip}}_{(\mathfrak{s}_0+\mathfrak{u}_0,\mathfrak{s}_1+\mathfrak{u}_1,\mathfrak{s}_2+\mathfrak{u}_2)}(0)$, the following inequality holds for the map φ defined in (23).*

$$|\varphi[X,\widehat{x}](\rho)-\varphi[Y,\widehat{y}](\rho)|e^{-\eta|\rho|} \leqslant \mathfrak{c}_\varphi\|\widehat{x}-\widehat{y}\|_\eta + \mathfrak{d}_\varphi\|X-Y\|_\eta + \mathfrak{e}_\varphi\|\mathrm{D}\widehat{x}-\mathrm{D}\widehat{y}\|_\eta. \tag{48}$$

Proof. By the definition of φ and the assumption ($\mathrm{H}_\varepsilon 2$), we have that

$$|\varphi[X,\widehat{x}](\rho)-\varphi[Y,\widehat{y}](\rho)| \leqslant \mathfrak{L}_1|\phi^{-1}(\rho)-\psi^{-1}(\rho)| + \mathfrak{L}_2\left\|\big((x_0+\widehat{x})\circ\phi\big)_{\phi^{-1}(\rho)} - \big((x_0+\widehat{y})\circ\psi\big)_{\psi^{-1}(\rho)}\right\|_{C^1([-h,h])}.$$

Using Lemma 5.10, $|\phi^{-1}(\rho)-\psi^{-1}(\rho)|e^{-\eta|\rho|}$ is bounded by a constant multiple of $\|X-Y\|_\eta$. In order to bound the second part of the above inequality, we first consider

$$\sup_{s\in[-h,h]} |(x_0+\widehat{x})\circ\phi(\phi^{-1}(\rho)+s)-(x_0+\widehat{y})\circ\psi(\psi^{-1}(\rho)+s)|e^{-\eta|\rho|}. \tag{49}$$

Using the α, β notation from Lemma 5.11 and adding/subtracting, (49) is equivalent to

$$\sup_{s\in[-h,h]} \Big|(x_0 \circ \alpha - x_0 \circ \beta) + (\widehat{x} \circ \alpha - \widehat{y} \circ \alpha) + (\widehat{y} \circ \alpha - \widehat{y} \circ \beta)\Big|_{(\rho,s)} \big| e^{-\eta|\rho|}.$$

The first and third terms are bounded using Lemma 5.11 and Lipschitz property of x_0 and $\widehat{y}$. The second term is controlled as follows

$$|\widehat{x} \circ \alpha(\rho, s) - \widehat{y} \circ \alpha(\rho, s)| e^{-\eta|\rho|} \leqslant e^{\eta(|\alpha(\rho,s)|-|\rho|)} \|\widehat{x} - \widehat{y}\|_\eta \leqslant e^{\eta(1+\mathfrak{t}_0)h} \|\widehat{x} - \widehat{y}\|_\eta.$$

Now we consider the derivative

$$\begin{aligned} \frac{d}{ds}\Big[\big((x_0 + \widehat{x}) \circ \phi\big)_{\phi^{-1}(\rho)}(s) - \big((x_0 + \widehat{y}) \circ \psi\big)_{\psi^{-1}(\rho)}(s) \Big] & \\ = (x_0 + \widehat{x})' \circ \alpha(\rho, s) X \circ \alpha(\rho, s) - (x_0 + \widehat{y})' \circ \beta(\rho, s) Y \circ \beta(\rho, s), & \end{aligned}$$

which equals to the following sum evaluated at (ρ, s) by adding/subtracting

$$\begin{aligned} & (x_0' \circ \alpha - x_0' \circ \beta) X \circ \alpha \tag{L1}\\ & + x_0' \circ \beta (X \circ \alpha - Y \circ \alpha) \tag{L2}\\ & + x_0' \circ \beta (Y \circ \alpha - Y \circ \beta) \tag{L3}\\ & + (\widehat{x}' \circ \alpha - \widehat{x}' \circ \beta) X \circ \alpha \tag{L4}\\ & + (\widehat{x}' \circ \beta - \widehat{y}' \circ \beta) X \circ \alpha \tag{L5}\\ & + \widehat{y}' \circ \beta (X \circ \alpha - X \circ \beta) \tag{L6}\\ & + \widehat{y}' \circ \beta (X \circ \beta - Y \circ \beta). \tag{L7} \end{aligned}$$

Each line can be bounded directly or by using Lemma 5.11. Indeed,

$$\begin{aligned} |(\mathtt{L1})(\rho, s)| e^{-\eta|\rho|} &\leqslant (1 + \mathfrak{t}_0) \operatorname{Lip}(x_0') \mathfrak{z} \|X - Y\|_\eta, \\ |(\mathtt{L2})(\rho, s)| e^{-\eta|\rho|} &\leqslant \|x_0\|_{C^1} e^{\eta(1+\mathfrak{t}_0)h} \|X - Y\|_\eta, \\ |(\mathtt{L3})(\rho, s)| e^{-\eta|\rho|} &\leqslant \|x_0\|_{C^1} \mathfrak{t}_1 \mathfrak{z} \|X - Y\|_\eta, \\ |(\mathtt{L4})(\rho, s)| e^{-\eta|\rho|} &\leqslant (1 + \mathfrak{t}_0)(\mathfrak{s}_2 + \mathfrak{u}_2) \mathfrak{z} \|X - Y\|_\eta, \\ |(\mathtt{L5})(\rho, s)| e^{-\eta|\rho|} &\leqslant (1 + \mathfrak{t}_0) e^{\eta(1+\mathfrak{t}_0)h} \|\widehat{x}' - \widehat{y}'\|_\eta, \\ |(\mathtt{L6})(\rho, s)| e^{-\eta|\rho|} &\leqslant (\mathfrak{s}_1 + \mathfrak{u}_1) \mathfrak{t}_1 \mathfrak{z} \|X - Y\|_\eta, \\ |(\mathtt{L7})(\rho, s)| e^{-\eta|\rho|} &\leqslant (\mathfrak{s}_1 + \mathfrak{u}_1) e^{\eta(1+\mathfrak{t}_0)h} \|X - Y\|_\eta. \end{aligned}$$

Collecting all the intermediate bounds we have explicit $\mathfrak{c}_\varphi$, $\mathfrak{d}_\varphi$, $\mathfrak{e}_\varphi$ depending on $\|x_0\|_{C^{1+\mathrm{Lip}}}$, $\mathfrak{L}_1$, $\mathfrak{L}_2$, h, η, $\mathfrak{t}_0$, $\mathfrak{t}_1$, $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_1$, and $\mathfrak{u}_2$. □

5.5 A-posteriori results

By the propagated bounds (B1), there is a fixed point v^* of the operator Γ^ε. For an initial guess $v \stackrel{\text{def}}{=} (X_{(0)}, \widehat{x}^s_{(0)}, \widehat{x}^u_{(0)})$ of the fixed point method, we have

$$\begin{aligned} \|X_{(0)} - \Pi^c[v^*]\|_{C^{\ell+\mathrm{Lip}}} &\leqslant M_c < +\infty, \\ \|\widehat{x}^s_{(0)} - \Pi^s[v^*]\|_{C^{\ell+1+\mathrm{Lip}}} &\leqslant M_s < +\infty, \\ \|\widehat{x}^u_{(0)} - \Pi^u[v^*]\|_{C^{\ell+1+\mathrm{Lip}}} &\leqslant M_u < +\infty. \end{aligned}$$

On the other hand, from the low regularity contraction (43) and the Banach fixed point Theorem,

$$d(v, v^*) \leqslant (1 - \kappa)^{-1} d(v, \Gamma^\varepsilon[v]), \tag{50}$$

where κ is the contraction rate.

The a-posteriori formulation consists in controlling derivatives of $v-v^*$ by the low regularity norm of the initial error. If the initial error is small, this formulation assures that there is a true solution close to such initial guess in the sense of C^j. The estimation is done using interpolation inequalities.

5.5.1 A-posteriori argument on a bounded interval

On an interval $[a,b]$, the inequality (50) implies

$$\|(v-v^*)|_{[a,b]}\|_{C^0} \leqslant e^{\delta\eta}(1-\kappa)^{-1} d(v,\Gamma^\varepsilon[v]),$$

where $\delta = \max\{|a|,|b|\}$. Thus, by using the interpolation inequalities in (31), we deduce that there are constants $\mathfrak{c}_c$, $\mathfrak{c}_s$, and $\mathfrak{c}_u$ such that

$$\begin{aligned}
\|(X_{(0)}-\Pi^c[v^*])|_{[a,b]}\|_{C^j} &\leqslant \mathfrak{c}_c e^{\delta\eta\frac{\ell+1-j}{\ell+1}}(1-\kappa)^{-\frac{\ell+1-j}{\ell+1}} d(v,\Gamma^\varepsilon[v])^{\frac{\ell+1-j}{\ell+1}} && 0\leqslant j\leqslant \ell,\\
\|(\widehat{x}^s_{(0)}-\Pi^s[v^*])|_{[a,b]}\|_{C^j} &\leqslant \mathfrak{c}_s e^{\delta\eta\frac{\ell+2-j}{\ell+2}}(1-\kappa)^{-\frac{\ell+2-j}{\ell+2}} d(v,\Gamma^\varepsilon[v])^{\frac{\ell+2-j}{\ell+2}} && 0\leqslant j\leqslant \ell+1,\\
\|(\widehat{x}^u_{(0)}-\Pi^u[v^*])|_{[a,b]}\|_{C^j} &\leqslant \mathfrak{c}_u e^{\delta\eta\frac{\ell+2-j}{\ell+2}}(1-\kappa)^{-\frac{\ell+2-j}{\ell+2}} d(v,\Gamma^\varepsilon[v])^{\frac{\ell+2-j}{\ell+2}} && 0\leqslant j\leqslant \ell+1,
\end{aligned}$$

where $\Pi^\sigma[w](t) \stackrel{\text{def}}{=} \Pi^\sigma_t w$ for $\sigma\in\{c,s,u\}$. Note that for the stable and unstable directions, we could use the interpolation with C^1 and $C^{\ell+1+\text{Lip}}$ spaces as well.

5.5.2 A-posteriori argument on semi lines

Let $g\colon \mathbb{R}\to\mathbb{R}^n$ be a smooth function and define $g_\eta\colon \mathbb{R}\to\mathbb{R}^n$ as

$$g_\eta(t) \stackrel{\text{def}}{=} e^{-\eta|t|}g(t) = \begin{cases} e^{-\eta t}g(t) & t>0\\ e^{\eta t}g(t) & t\leqslant 0.\end{cases}$$

In general, the function g_η is not differentiable at $t=0$ for $\eta>0$. Therefore, we provide interpolation inequalities for $t>0$ in C_η space (recall Section 4.2.2) in the following Lemma 5.13. Similar results hold for $t<0$.

Lemma 5.13. *Let $g\colon (0,+\infty)\to\mathbb{R}^n$ be a $C^{\ell+\text{Lip}}$ function ($\ell\geqslant 0$). Then*

$$\|g\|_{C_\eta}\leqslant 1 \quad \textit{implies} \quad \|\mathrm{D}^j g\|_{C_\eta}\leqslant \mathfrak{c}_j\|g\|_{C_\eta}^{\frac{1}{j+1}},$$

for all $j=0,\dots,\ell$ and some constants $\mathfrak{c}_j$'s depending on η, j, and $\|g\|_{C^{j+1}}$ ($\|g\|_{C^{\ell+\text{Lip}}}$ when $j=\ell$).

Proof. Let us prove the result by induction:

i.) For $j=1$, we use the interpolation inequality (31). Noticing that $\|g_\eta\|_{C^0}=\|g\|_{C_\eta}$ and $\|g\|_{C_\eta}\leqslant\|g\|_{C_\eta}^{1/2}$ as $\|g\|_{C_\eta}\leqslant 1$, we have

$$\begin{aligned}
|e^{-\eta t}\mathrm{D}g(t)| &= |\mathrm{D}g_\eta(t)+\eta e^{-\eta t}g(t)| \leqslant \|g_\eta\|_{C^1}+\eta\|g\|_{C_\eta}\leqslant M_{0,2}\|g_\eta\|_{C^0}^{1/2}\|g_\eta\|_{C^2}^{1/2}+\eta\|g_\eta\|_{C^0}\\
&= \big(M_{0,2}\|g_\eta\|_{C^2}^{1/2}+\eta\big)\|g_\eta\|_{C^0}^{1/2}.
\end{aligned}$$

We take $\mathfrak{c}_1 = M_{0,2}\|g_\eta\|_{C^2}^{1/2}+\eta$, so that the proof of the case $j=1$ is done.

ii.) Assume that the result is true up to $j-1$. By Leibnitz product formula, we have

$$\mathrm{D}^j g_\eta(t) = \sum_{k=0}^{j}\binom{j}{k}\mathrm{D}^k(e^{-\eta t})\mathrm{D}^{j-k}g(t) = e^{-\eta t}\Big(\mathrm{D}^j g(t)+\sum_{k=1}^{j}\binom{j}{k}(-\eta)^k\mathrm{D}^{j-k}g(t)\Big).$$

Then by induction hypotheses and interpolation inequality,

$$|e^{-\eta t}\mathrm{D}^j g(t)| = \left|\mathrm{D}^j g_\eta + \sum_{k=1}^{j}\binom{j}{k}(-\eta)^{k+1}\mathrm{D}^{j-k}g(t)\right| \leqslant \|g_\eta\|_{C^j} + \sum_{k=1}^{j}\mathfrak{a}_k\|g_\eta\|_{C^0}^{\frac{1}{k}}$$
$$\leqslant \left(M_{0,j+1}\|g_\eta\|_{C^{j+1}}^{\frac{j}{j+1}} + \sum_{k=1}^{j}\mathfrak{a}_k\|g_\eta\|_{C^0}^{\frac{1}{k}-\frac{1}{j+1}}\right)\|g_\eta\|_{C^0}^{\frac{1}{j+1}},$$

for some $\mathfrak{a}_k$ involving combinatorial numbers. We let $c_j = M_{0,j+1}\|g_\eta\|_{C^{j+1}}^{\frac{j}{j+1}} + \sum_{k=1}^{j}\mathfrak{a}_k$ so that the result for j is proved. □

Then from (50) and Lemma 5.13, we conclude that for a good enough initial guess,

$$\|\mathrm{D}^j(X_{(0)} - \Pi^c[v^*])|_{(0,+\infty)}\|_{C_\eta} \leqslant \mathfrak{c}_c(1-\kappa)^{-\frac{1}{j+1}}d(v,\Gamma^\varepsilon[v])^{\frac{1}{j+1}} \qquad 0 \leqslant j \leqslant \ell,$$
$$\|\mathrm{D}^j(\widehat{x}^s_{(0)} - \Pi^s[v^*])|_{(0,+\infty)}\|_{C_\eta} \leqslant \mathfrak{c}_s(1-\kappa)^{-\frac{1}{j+1}}d(v,\Gamma^\varepsilon[v])^{\frac{1}{j+1}} \qquad 0 \leqslant j \leqslant \ell+1,$$
$$\|\mathrm{D}^j(\widehat{x}^u_{(0)} - \Pi^u[v^*])|_{(0,+\infty)}\|_{C_\eta} \leqslant \mathfrak{c}_u(1-\kappa)^{-\frac{1}{j+1}}d(v,\Gamma^\varepsilon[v])^{\frac{1}{j+1}} \qquad 0 \leqslant j \leqslant \ell+1,$$

where $\Pi^\sigma[w](t) \stackrel{\text{def}}{=} \Pi^\sigma_t w$ for $\sigma \in \{c,s,u\}$. Similar results hold on $(-\infty,0)$.

6. Further results

In this section, we discuss bootstrap of the regularity and non-autonomous unperturbed systems.

6.1 Estimates on the growth of higher derivatives

For ODEs, one could bootstraps the regularity of the solution: An initial value problem of an ODE, say $\dot{y}(t) = g\circ y(t)$ with $y(0) = y_0$, has the property that if one is able to find a C^1 solution and g is $C^{\ell+1}$, then automatically such a solution will be $C^{\ell+2}$ for $\ell \geqslant 0$. If we considered $\mathscr{P}\colon C^{\ell+1} \to C^{\ell+1}$ in Theorem 4.8, we would have the same bootstrap property as in ODE's and we could first find solution in C^1 space. However, this setting would not cover applications with neutral or small delays.

Instead, we consider $\mathscr{P}\colon C^{\ell+1} \to C^{\ell}$. Therefore, we are not able to bootstrap regularity directly in this case. Hence, the fixed point method should be performed on a suitable space up to the right regularity level (beyond C^1), see Section 4.2. Nevertheless, once the Theorem 4.8 is proved, we can bootstrap other type of solution behaviors; solutions with exponential derivative growth.

We stress that we are looking for C^1 solutions such that higher derivatives can arbitrarily grow. A simple example is the function $z(t) = \int_0^t \sin e^s\,ds$, which is C^1 but from the second derivative on grows exponentially. Because we will have C^1 solutions, then the perturbation P will at least be C^0 (otherwise it would not be possible to be controlled by the perturbative parameter ε).

Let us now deduce how adding some slightly different assumptions to P we can include new type of solutions. Indeed, given a C^1 solution $x(t)$ of (1), if we consider the second and third derivatives, then we have

$$\ddot{x}(t) = \mathrm{D}f\circ x(t)\dot{x}(t) + \varepsilon\frac{d}{dt}P(t,x_t,\varepsilon,\mu),$$
$$\dddot{x}(t) = \mathrm{D}^2 f\circ x(t)\dot{x}(t)^{\otimes 2} + \mathrm{D}f\circ x(t)\ddot{x}(t) + \varepsilon\frac{d^2}{dt^2}P(t,x_t,\varepsilon,\mu).$$

If $\ddot{x}$ has an exponential growth, then it necessarily comes from the perturbation P since $\mathrm{D}f$ and $\dot{x}$ are bounded. More precisely, if there are $\gamma \geqslant 0$ and $C_j > 0$ such that

$$\left|\frac{\partial^j}{\partial t^j}P(t,x_t,\varepsilon,\mu)\right| \leqslant C_j e^{\gamma j|t|},$$

then for $j = 1, \dots, \ell$, the solution $\ddot{x}(t)$ will also be bounded exponentially and, in general, what we have is that $|\mathrm{D}^{j+1}x(t)|e^{-\gamma j|t|} < +\infty$ for $j \geqslant 0$.

To provide a formal statement, let us define the exponential derivative growth space:

Definition 6.1 (Finitely differentiable space with exponential growth)**.** Let $\gamma \geqslant 0$ and let $C^\ell_\gamma(I, \mathbb{R}^n)$ be the space of ℓ times differentiable functions on the interior of interval $I \subset \mathbb{R}$ and with finite norm:

$$\|g\|_{C^\ell_\gamma} \stackrel{\text{def}}{=} \max_{j=0,\dots,\ell} \sup_{t\in I} |\mathrm{D}^j g(t)| e^{-\gamma j|t|} \qquad \text{for all } g \in C^\ell_\gamma.$$

Notice that when $\gamma > 0$, an element in C^ℓ_γ does not have Lipschitz boundedness in all its derivatives. Corollary 6.2 is a bootstrap result in the space with exponential growth defined above.

Corollary 6.2. *Let $\gamma \geqslant 0$, let $x(t)$ be a solution from Theorem 4.8 for $\ell = 1$, and let $\ell' \geqslant \ell$.*

Assume that the unperturbed system satisfies:

$H_{0,\gamma}1$) The hypothesis ($\mathrm{H}_0 2$) *in Theorem 4.8 holds for ℓ',*

and that the perturbation $\mathcal{P}$ in Theorem 4.8 satisfies:

$H_{\varepsilon,\gamma}1$) For all $\varepsilon \in (0, \varepsilon_0)$, $t \in \mathbb{R}$ and for $j = 1, \dots, \ell'$, $u \in C^1(\mathbb{R}, \mathbb{R}^n)$, $\mathrm{D}u \in C^{j-1}_\gamma(\mathbb{R}, \mathbb{R}^n)$,

$$\left| \frac{d^j}{dt^j} \mathcal{P}[u, \varepsilon, \mu](t) \right| \leqslant C_j e^{\gamma j|t|} F(\|u\|_{C^1}, \|\mathrm{D}u\|_{C^{j-1}_\gamma}),$$

where $C_j > 0$ and $F : \mathbb{R} \to \mathbb{R}$ is increasing and continuous.

Then the C^1-solution $x(t)$ of (1) *is such that* $\mathrm{D}x$ *is in $C^{\ell'}_\gamma$.*

Notice that Corollary 6.2 does not change the range of $\varepsilon \leqslant \varepsilon_0$ and it can incorporate smooth parameter dependence with the approach introduced in Section 4.3.1 using C^ℓ_γ spaces.

6.2 Non-autonomous unperturbed system

Our set up can incorporate non-autonomous systems using the standard method of adding an extra variable. Consider a non-autonomous system $\dot{x}(t) = g(x(t), t)$, where g is ℓ-times differentiable and Lipschitz. We introduce a new variable s, and let $y \stackrel{\text{def}}{=} (x, t)$, then

$$y'(s) = \frac{d}{ds}\begin{pmatrix} x(s) \\ t(s) \end{pmatrix} = G \circ y(s) = \begin{pmatrix} g(x(s), t(s)) \\ 1 \end{pmatrix}.$$

Let us define an affine differentiable space $\widetilde{C}^\ell \stackrel{\text{def}}{=} Id + C^\ell$. This space has a well-defined Lipschitz constant. Thus, a solution y belongs to the product space $C^{\ell+1} \times \widetilde{C}^{\ell+1}$.

Remark 6.3. Note that our setting for hyperbolic orbits does not involve the orbit to lie on a bounded set, it only requires that the vector field are bounded in a neighborhood of uniform size of the orbit. In the non-autonomous case, this amounts to uniform for all the derivatives of small enough order of g – including derivatives with respect to time – in a neighborhood of uniform size of the orbit. Hence, we can remake all the unperturbed hypothesis admitting these affine differentiable spaces and derive a similar result like in Theorem 4.8 that explicitly includes non-autonomous unperturbed systems.

In the applications to delay equations, we will include, for technical reasons that the delays are bounded.

Even if this very direct approach gives results for many applications, it can be improved. Indeed, It is well known [MNnO17] that one can obtain a theory of evolutions of the equation $\dot{x}(t) = g(x(t), t)$ by assuming only that g is measurable with respect to t (several mild integrability assumptions are needed). This is usually called *Caratheodory theory*. Under rather mild assumptions, the Caratheodory theory allows to write variational equations and the remainder. The operator Γ^ε in this paper can then be formulated just as well. At the moment, we are not aware of any significant applications.

7. Some models covered by the general results

This section is devoted to providing examples of perturbations P which satisfy the assumptions of our main theorem. We show how to verify the hypotheses in Theorem 4.8 and we add some important remarks.

7.1 ODE Perturbation

A very particular case of Theorem 4.8 is when $P(t, x_t) = g(t, x(t))$, where the history segment x_t is evaluated at zero to obtain $x(t)$. This case corresponds to ODE perturbations.

When there is a hyperbolic orbit in the unperturbed system satisfying Definition 2.1, we obtain that there is a solution close to the unperturbed hyperbolic orbit applying Theorem 4.8. The hyperbolicity of the perturbed solution can be seen from [Mos69]. This is a version of Anosov shadowing theorem [Ano69]. The precise version is close to the version in [Mos69] as modified in [dlLMM86].

We show that hyperbolic orbits have a counterpart in the perturbed system. As a corollary of our formalism (as in [dlLMM86]), we obtain smooth dependence on parameters, see Section 4.3.1.

Note that the range of perturbation parameters for which the orbit persists depends on the hyperbolicity parameters of the orbit. Also the size affected by perturbations on an orbit depends on the hyperbolicity parameters. For Anosov systems for which all the orbits have uniform hyperbolicity constants, the validity range of perturbations is uniform and the size of the perturbation effects is uniform. In non-uniformly hyperbolic sets, the allowed values of the perturbation and the size of the responses will depend a lot on the orbits.

7.2 State and time dependent delay equations

Let us consider the model

$$\dot{x}(t) = f \circ x(t) + \varepsilon Q\big(t, x(t + r(t, x(t)))\big). \tag{51}$$

where the perturbative map P is

$$P(t, \vartheta, \varepsilon) \stackrel{\text{def}}{=} Q\big(t, \vartheta \circ r(t, \vartheta(0))\big).$$

The perturbative hypotheses in Theorem 4.8 are satisfied by considering $r\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a $C^{\ell+\text{Lip}}$ map and $Q\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ a $C^{\ell+\text{Lip}}$ map. In this case, the history segment can be taken as $h \stackrel{\text{def}}{=} \|r\|_{C^0}$.

The hypothesis (H$_\varepsilon$1) can be verified using the chain rule and Faá di Bruno formula. To check hypothesis (H$_\varepsilon$2), we analyze $\mathscr{P}$ in Theorem 4.8:

$$\mathscr{P}[u, \varepsilon](t) = \mathscr{P}[u](t) = Q\big(t, u(t + r(t, u(t)))\big),$$

where we have omitted the ε and μ in $\mathscr{P}$ since in this example Q does not depend on them. Hence,

$$|\mathscr{P}[u^2](s) - \mathscr{P}[u^1](t)| \leqslant \text{Lip}(Q)|s-t| + \text{Lip}(Q)|u^2(s + r(s, u^2(s)) - u^1(t + r(t, u^1(t))|. \tag{52}$$

The second term is bounded by adding/subtracting and triangle inequality, that is,

$$\begin{aligned} |u^2(s + r(s, u^2(s)) - u^1(t + r(t, u^1(t))| &\leqslant |u^2(s + r(s, u^2(s)) - u^2(s + r(t, u^1(t))| \\ &\quad + |u^2(s + r(t, u^1(t)) - u^1(t + r(t, u^1(t))| \\ &\leqslant \|u^2\|_{C^1}\|r\|_{C^1}\big[|s-t| + \|u^2_s - u^1_t\|_{C^0[-h,h]}\big] \\ &\quad + \|u^2_s - u^1_t\|_{C^0[-h,h]}. \end{aligned}$$

Then we can take constants $\mathfrak{L}_1$ and $\mathfrak{L}_2$ so that (H$_\varepsilon$2) is true for all u^1, u^2 in a ball of $C^{\ell+1+\text{Lip}}(\mathbb{R}, \mathbb{R}^n)$.

7.3 Nested delay equations

Let us consider a differential equation with nested delay/advance terms

$$\dot{x}(t) = f \circ x(t) + \varepsilon Q\big(t, x(t + r(t, x(t + r_1 \circ x(t))))\big).$$

In this case, the perturbative map for (1) is

$$P(t, \vartheta, \varepsilon) \stackrel{\text{def}}{=} Q(t, \vartheta \circ r(t, r_1 \circ \vartheta(0))),$$

and the "history segment" is $h \stackrel{\text{def}}{=} \max\{\|r\|_{C^0}, \|r_1\|_{C^0}\}$. If $r\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{\ell+\mathrm{Lip}}$ map, $r_1\colon \mathbb{R}^n \to \mathbb{R}^n$ a $C^{\ell+\mathrm{Lip}}$ map, and $Q\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ a $C^{\ell+\mathrm{Lip}}$ map, then the perturbative hypotheses in Theorem 4.8 are satisfied.

The idea is similar to Section 7.2. Now we need to bound

$$|u^2(s + r(s, u^2(s + r_1 \circ u^2(s))) - u^1(t + r(t, u^1(t + r_1 \circ u^1(t)))|. \tag{53}$$

We obtain that

$$\begin{aligned}
(53) &\leqslant |u^2(s + r(s, u^2(s + r_1 \circ u^2(s))) - u^2(s + r(t, u^1(t + r_1 \circ u^1(t)))| \\
&\quad + |u^2(s + r(t, u^1(t + r_1 \circ u^1(t)) - u^1(t + r(t, u^1(t + r_1 \circ u^1(t))| \\
&\leqslant \|u^2\|_{C^1} \|r\|_{C^1} \big[|s - t| + \|u^2\|_{C^1} \|r_1\|_{C^1} \|u^2_s - u^1_t\|_{C^0[-h,h]} + \|u^2_s - u^1_t\|_{C^0[-h,h]}\big] \\
&\quad + \|u^2_s - u^1_t\|_{C^0[-h,h]}.
\end{aligned}$$

Therefore, for all u^1, u^2 in a ball of $C^{\ell+1+\mathrm{Lip}}(\mathbb{R}, \mathbb{R}^n)$, there are constants $\mathfrak{L}_1$ and $\mathfrak{L}_2$ such that ($\mathrm{H}_\varepsilon 2$) is satisfied.

7.4 Neutral delay equations

As an example of neutral delay/advance equation, we consider

$$\dot{x}(t) = f \circ x(t) + \varepsilon Q\big(t, x(t + r(t, \tfrac{d}{dt} x(t)))\big), \tag{54}$$

Where $Q : \mathbb{R} \times \mathbb{R}^n$ is a smooth function.

This can be made into the form (1) taking.

$$P(t, \vartheta, \varepsilon, \mu) \stackrel{\text{def}}{=} Q\big(t, \vartheta \circ r(t, \tfrac{d}{ds} \vartheta(0))\big),$$

that depends on time and on the derivative of the state. Note that we used the fact that $\frac{dx_t}{ds}(0) = \frac{dx}{dt}(t)$. The history segment in this case is $h \stackrel{\text{def}}{=} \|r\|_{C^0}$. Note that we are not assuming that that the sign of r is negative, so that we can just as well have advanced equations.

Using the standard adding of extra variables, the unperturbed equation could be an equation of order $n+1$, but the R.H.S cannot introduce derivatives of order higher than $n+1$,

To apply Theorem 4.8, we assume regularities on Q and r such that the perturbation in (54) satisfies ($\mathrm{H}_\varepsilon 1$). In particular, if $r\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{\ell+\mathrm{Lip}}$ map and $Q\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{\ell+\mathrm{Lip}}$ map, ($\mathrm{H}_\varepsilon 1$) is verified.

To check ($\mathrm{H}_\varepsilon 2$), we bound the term

$$\Big|u^2\big(s + r(s, \tfrac{d}{ds} u^2(s)\big) - u^1\big(t + r(t, \tfrac{d}{dt} u^1(t))\big)\Big|, \tag{55}$$

and obtain

$$\begin{aligned}
(55) &\leqslant \Big|u^2\big(s + r(s, \tfrac{d}{ds} u^2(s)\big) - u^2\big(s + r(t, \tfrac{d}{dt} u^1(t))\big)\Big| \\
&\quad + \Big|u^2\big(s + r(t, \tfrac{d}{dt} u^1(t))\big) - u^1\big(t + r(t, \tfrac{d}{dt} u^1(t))\big)\Big| \\
&\leqslant \|u^2\|_{C^1} \|r\|_{C^1} \big[|s - t| + \|u^2_s - u^1_t\|_{C^1[-h,h]}\big] \\
&\quad + \|u^2_s - u^1_t\|_{C^0[-h,h]}.
\end{aligned}$$

Therefore, $(\mathrm{H}_\varepsilon 2)$ is satisfied.

The modification of the verification for several delays/advances is left to the reader. Note that we can let some of the r's be delays and others be advances.

Similar to Section 7.3, one can also consider nested delays involving first derivative in the state, or more generally $\frac{d}{ds}\vartheta(s)$ for $s \in [-h, h]$. Of course, particular cases such as constant delays satisfy the assumptions of our result.

7.5 Small delays and small advances

There are problems in the literature in which the time at which the solution needs to be evaluated contains very small time changes. An important case, which involves special challenges, is the motion of charged particles, studied in more detail in Section 7.6.

In this section we will show that the terms with small delay or small advances can be included in the formalism of Theorem 4.8. The delays allowed are very general and could be functionals on the history segment. Only some mild regularity assumptions will be imposed. In particular, we do not need to assume that the delays are positive, so we can also consider advanced perturbations (or perturbations that include both advanced and retarded terms). This generality becomes useful in the treatment of motion of point charges where the delay can depend on the whole trajectory. Some of the physical theories proposed in [WF49, WF45] involve both advanced and retarded term and, hence, they could be included in our framework.

A small delay is a singular perturbation because the nature of the problem changes completely. Heuristically, the expansions on the perturbative parameter, involve derivatives of the function. If the delay is not zero – even if small – the phase space may be an infinite dimensional solution manifold or something more complicated.

The simplest non-trivial case is

$$\dot{x}(t) = f \circ x(t - \varepsilon\tau(x_t)), \tag{56}$$

where τ is a functional of the trajectory segment. This case (56) fits into our framework by rewriting it as

$$\dot{x}(t) = f \circ x(t) + \varepsilon\left[\frac{1}{\varepsilon} f \circ x(t - \varepsilon\tau(x_t)) - \frac{1}{\varepsilon} f \circ x(t)\right].$$

To estimate the perturbation and verify $(\mathrm{H}_\varepsilon 1)$–$(\mathrm{H}_\varepsilon 2)$ we use the heuristic idea that

$$\frac{1}{\varepsilon} f \circ x(t - \varepsilon\tau(x_t)) - \frac{1}{\varepsilon} f \circ x(t) \approx -\mathrm{D}f \circ x(t) x'(t)\tau(x_t).$$

Therefore, the effect of the small delay is similar to including a functional losing one derivative, which is incorporated in Theorem 4.8 fortunately.

More precisely:

$$\frac{1}{\varepsilon} f \circ x(t - \varepsilon\tau(x_t)) - \frac{1}{\varepsilon} f \circ x(t) = -\int_0^1 \mathrm{D}f \circ x(t - \sigma\varepsilon\tau(x_t)) x'(t - \sigma\varepsilon\tau(x_t))\tau(x_t)\, d\sigma.$$

Hence, equation (56) with small delays or advances fits our setting where the functional perturbative map is

$$\mathcal{Q}[\vartheta, \varepsilon] \stackrel{\text{def}}{=} -\int_0^1 \mathrm{D}f \circ \vartheta(-\sigma\varepsilon\tau \circ \vartheta)\vartheta'(-\sigma\varepsilon\tau \circ \vartheta)\tau \circ \vartheta\, d\sigma, \tag{57}$$

where $\tau\colon C^{\ell+1+\mathrm{Lip}}([-h, h], \mathbb{R}^n) \to \mathbb{R}$. Note that, we can also consider several delays τ_i in (56). Applying Theorem 4.8, we obtain that

Theorem 7.1. *Consider the equation*

$$\dot{x}(t) = f \circ \Big(x\big(t - \varepsilon\tau_1(t, x_t)\big), \dots, x\big(t - \varepsilon\tau_L(t, x_t)\big) \Big) + \varepsilon P(t, x_t, \varepsilon, \mu). \tag{58}$$

Assume $(\mathrm{H}_0 1)$–$(\mathrm{H}_0 2)$ *in Theorem 4.8 and that for all* $i = 1, \dots, L$*, if* x_t *is in a ball in* $C^{\ell+1+\mathrm{Lip}}$ *space, then* $\tau_i(t, x_t)$ *ranges in a ball in* $C^{\ell+\mathrm{Lip}}$ *space, and that* τ_i*'s have Lipschitz properties.*

Then, hypotheses $(\mathrm{H}_\varepsilon 1)$ *and* $(\mathrm{H}_\varepsilon 2)$ *hold for the* $\mathcal{Q}_i$ *given as*

$$\mathcal{Q}_i(t, \vartheta, \varepsilon) \stackrel{\text{def}}{=} \int_0^1 \mathrm{D}_i f\Big(\vartheta\big(-\sigma\varepsilon\tau_1(t,\vartheta)\big), \dots, \vartheta\big(-\sigma\varepsilon\tau_L(t,\vartheta)\big)\Big)\vartheta'\big(-\sigma\varepsilon\tau_i(t,\vartheta)\big)\tau_i(t,\vartheta)\, d\sigma,$$

Assume in addition that, $\mathscr{P}$ *defined in* (23) *corresponding to* P *in equation* (58) *satisfies* $(\mathrm{H}_\varepsilon 1)$ *and* $(\mathrm{H}_\varepsilon 2)$ *in Theorem 4.8, then we have the same conclusions as Theorem 4.8.*

Maybe the most unclear part of the proof of Theorem 7.1 is that $\mathcal{Q}_i$ satisfies $(\mathrm{H}_\varepsilon 2)$ (since $(\mathrm{H}_\varepsilon 1)$ comes from the fact that τ_i maps a ball in $C^{\ell+1+\mathrm{Lip}}$ to another ball in $C^{\ell+\mathrm{Lip}}$). We illustrate $(\mathrm{H}_\varepsilon 2)$ with $\mathcal{Q}$ in (57). Let ϑ and ϱ in a $C^{\ell+1+\mathrm{Lip}}$ ball and let us bound $\mathcal{Q}[\vartheta,\varepsilon] - \mathcal{Q}[\varrho,\varepsilon]$ whose integrand is (after adding and subtracting)

$$\begin{aligned}
&\mathrm{D}f \circ \vartheta(-\sigma\varepsilon\tau\circ\vartheta)\vartheta'(-\sigma\varepsilon\tau\circ\vartheta)\tau\circ\vartheta - \mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\varrho)\varrho'(-\sigma\varepsilon\tau\circ\varrho)\tau\circ\varrho \\
=&\big[\mathrm{D}f\circ\vartheta(-\sigma\varepsilon\tau\circ\vartheta) - \mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\vartheta)\big]\vartheta'(-\sigma\varepsilon\tau\circ\vartheta)\tau\circ\vartheta && (\texttt{S1})\\
&+\big[\mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\vartheta) - \mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\varrho)\big]\vartheta'(-\sigma\varepsilon\tau\circ\vartheta)\tau\circ\vartheta && (\texttt{S2})\\
&+\mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\varrho)\big[\vartheta'(-\sigma\varepsilon\tau\circ\vartheta) - \varrho'(-\sigma\varepsilon\tau\circ\vartheta)\big]\tau\circ\vartheta && (\texttt{S3})\\
&+\mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\varrho)\big[\varrho'(-\sigma\varepsilon\tau\circ\vartheta) - \varrho'(-\sigma\varepsilon\tau\circ\varrho)\big]\tau\circ\vartheta && (\texttt{S4})\\
&+\mathrm{D}f\circ\varrho(-\sigma\varepsilon\tau\circ\varrho)\varrho'(-\sigma\varepsilon\tau\circ\varrho)\big[\tau\circ\vartheta - \tau\circ\varrho\big]. && (\texttt{S5})
\end{aligned}$$

The individual intermediate lines admit straightforward bounds in terms of the inputs, i.e. for $s \in [-h, h]$,

$$\begin{aligned}
|(\texttt{S1})(s)| &\leqslant \|f\|_{C^2}\|\vartheta\|_{C^1}\|\tau\|_{C^0}\|\vartheta-\varrho\|_{C^0}\\
|(\texttt{S2})(s)| &\leqslant \sigma\varepsilon\,\mathrm{Lip}(\mathrm{D}f\circ\varrho)\,\mathrm{Lip}(\tau)\|\vartheta-\varrho\|_{C^0}\|\vartheta\|_{C^1}\|\tau\|_{C^0}\\
|(\texttt{S3})(s)| &\leqslant \|f\|_{C^1}\|\vartheta-\varrho\|_{C^1}\|\tau\|_{C^0}\\
|(\texttt{S4})(s)| &\leqslant \sigma\varepsilon\|f\|_{C^1}\|\varrho\|_{C^2}\,\mathrm{Lip}(\tau)\|\vartheta-\varrho\|_{C^0}\|\tau\|_{C^0}\\
|(\texttt{S5})(s)| &\leqslant \|f\|_{C^1}\|\varrho\|_{C^1}\,\mathrm{Lip}(\tau)\|\vartheta-\varrho\|_{C^0}.
\end{aligned}$$

Thus, adding up all these bounds, there is a constant, say $\mathfrak{L}_2$, such that $|\mathcal{Q}[\vartheta,\varepsilon] - \mathcal{Q}[\varrho,\varepsilon]| \leqslant \mathfrak{L}_2\|\vartheta-\varrho\|_{C^1}$.

7.6 Delays implicitly defined by the solution. Applications to electrodynamics

In this section, we show that our framework applies to the problem of electrodynamics of point charges and formulate Theorem 7.3 which is obtained by applying Theorem 4.8 to (59), the model of particles moving on the electromagnetic fields generated by others.

From the mathematical point of view, the main new problem is that the delays that appear in the equation – the time the signals emitted by one particle take to reach another – depend on the trajectory. The delays can only be found by solving an implicit equation that involves the whole trajectory, see (60). We refer to this situation as implicitly defined delays.

7.6.1 Formulation of the mathematical problem

The motion of point charges under the electromagnetic field generated by others has several physical problems due to *self-energy* (see [Spo04] and also [Jac99, Chapter 16]) which we will not discuss.

We will follow the formulation of on [WF49] which avoids the self-energy problems. The basic idea of [WF45, WF49] is that each charge moves in the field generated by the others and by external sources (not in the field generated by themselves!). The expression of the electromagnetic fields generated by charges in motion is obtained by solving Maxwell equations. The explicit expression of the solution of Maxwell equations generated by charges (knowing their positions and velocities) is well known since the turn of the XX century and is called Liénard-Weichert potentials ([LL62, Roh07, Jac99, Zan13]). The motion of the particle in this potential is given by Newton's laws taking the relativistic expression of the mass. One can think of the Liénard-Wiechert potentials as the standard Coulomb/Ampere expressions taking into account delays and Fitzgerald contractions. As in the wave equation, the solutions of the Maxwell equations can be advanced or retarded, or convex combinations of both. In Physics literature, it is customary to take only retarded solutions, but this restriction does not follow from Maxwell equations or the boundary conditions and we do not need it for the results in this paper.

Hence, the equation of the ith particle are of the form:

$$\ddot{q}_i(t) = A_{\text{ext}}(t, q_i(t), \dot{q}_i(t)) + \sum_{j \neq i} A_{i,j}\big(q_i(t), \dot{q}_i(t), q_j(t-\tau_{ij}), \dot{q}_j(t-\tau_{ij}), q_j(t+\sigma_{ij}), \dot{q}_j(t+\sigma_{ij})\big), \tag{59}$$

The q_i represents the position of the i point charge, The term A_{ext} denotes the external force, and $A_{i,j}$ is an explicit expression given by the Liénard-Weichert potentials that depend on the time delay/advance defined by solving the implicit equations.

$$\begin{aligned} \tau_{ij}(t) &= \frac{1}{c}\big|q_i(t) - q_j(t-\tau_{ij}(t))\big|, \\ \sigma_{ij}(t) &= \frac{1}{c}\big|q_i(t) - q_j(t+\sigma_{ij}(t))\big|, \end{aligned} \tag{60}$$

with c being the speed of light. We think of c as large so that $\varepsilon = \frac{1}{c}$ is a small parameter.

The most salient mathematical feature of (59) is that it involves delays (or advances) which correspond to the time that the light takes to travel from the source particle to the dynamic particle. This delay depends on the whole trajectory of both particles (one needs to solve implicitly for the trajectory of a light ray to intersect the trajectory of the source particle). Since the Liénard-Wiechert potentials can be retarded or advanced (or convex combinations of both)[1] we get that the resulting equations can be retarded or advanced also.

Some minor complications are that (59) presents some singularities when $q_i(t) = q_j(t); i \neq j$ or when $|\dot{q}_i(t)| = c$.

The explicit form of the expressions of $A_{i,j}$ and A_{ext} in (59) can be found in any advanced textbook and the detailed expression is not relevant for our treatment.

Our treatment is rather general and applies to other models of the same structure. Due to relativity, all models of particles interacting by pairwise interactions are of the form (59). The structure (59) includes models incorporating gravity or more manageable approximations of Liénard-Wiechert potentials (it is common to keep the first order in ε and ignore higher orders in ε such as Fitzgerald contractions).

The treatment presented here extends to modifications of (59) that involve interactions among 3 or more bodies.

$$\begin{aligned} \ddot{q}_i(t) = A_{\text{ext}}(t, q_i(t), \dot{q}_i(t)) + F_i\big(&\{q_j(t-\tau_{jk}(t))\}_{j,k=1}^N, \{\dot{q}_j(t-\tau_{jk}(t))\}_{j,k=1}^N, \\ &\{q_j(t+\sigma_{jk}(t))\}_{j,k=1}^N, \{\dot{q}_j(t+\sigma_{jk}(t))\}_{j,k=1}^N\big), \end{aligned} \tag{61}$$

where F is an expression with the same type of singularities and the delays/advances τ_{jk}, σ_{jk} are defined in (60) or even by more general procedures that involve all the trajectories. The only requirement is that the delays range in a $C^{\ell+\text{Lip}}$ ball if the trajectories range in a $C^{\ell+\text{Lip}}$ ball.

[1] In [WF49], it is suggested that combining the delay and advance with coefficients 1/2 is physically important.

If there are no external electromagnetic forces (or if the external electromagnetic forces are time independent), the classical model conserves energy, so that it does not have any hyperbolic orbits. On the other hand, in time dependent (e.g. periodic) external fields, many interesting models (e.g. ion traps, mirror magnet machines) have many hyperbolic orbits [Gho96, Kaj22].

In this paper, we show that near hyperbolic orbits of the unperturbed model (59), i.e. $\varepsilon = 0$ in (60) [2], and if the perturbed model avoids the singularities and it has bounded delays, Theorem 4.8 applies. Moreover, constructed solutions are similar to those hyperbolic orbits for the unperturbed model (59), as long as $0 < \varepsilon \ll 1$ in (60). See Theorem 7.3 for a precise formulation.

Before proceeding to the detailed analysis, let us make some comments on the equations and their physical properties.

1. The expressions defining the forces are algebraic expressions (arithmetic operations and square roots). They have singularities when there are collisions ($q_i(t) = q_j(t)$ for some $i \neq j$) or when a particle reaches the speed of light ($|\dot{q}_i(t)| = c$ for some i).

   We will assume that the unperturbed solutions we consider are a finite distance away from these singularities so that the expressions given the second derivatives and given the positions and velocities are smooth functions in a neighborhood of the unperturbed solution. See Definition 7.2.

2. The delays τ_{ij} or advances σ_{ij} as in (60) involve solving implicit equations that involve the pairwise trajectories $q_i(t)$ and $q_j(t)$.

   The physical meaning of the delays/advances are the time it takes a light signal to travel from the particle i to the particle j. Since the speed of the particles is bounded away from the speed of light, this time exists and is unique. Nevertheless, finding the delay requires to solve an equation that depends on the pairwise trajectories. See (60).

   Fortunately, for the method used in this paper, the main property needed is that τ_{ij} and σ_{ij} are uniformly smooth assuming that the trajectories are smooth (and that they are away from the singularities).

   Note that in general $\tau_{ij} \neq \tau_{ji}$ even if $\tau_{ij} - \tau_{ji} = O(\varepsilon^2)$. Indeed, after expanding the solution of (60) up to first order, i.e.

   $$\tau_{ij}(t) = \varepsilon|q_i(t) - q_j(t)| + \varepsilon^2\big(q_i(t) - q_j(t)\big) \cdot \dot{q}_j(t) + O(\varepsilon^3). \tag{62}$$

   (hint: It is easy to consider the expansions for τ_{ij}^2 obtained by squaring both sides of (60) and express the square of length as an inner product). The above derivation (62) of an approximate form of the delay is purely formal. Note that it is only valid for differentiable q and that the error incurred in the approximation depends on higher derivatives of q. Of course in a delay equation, modifying the form of the delay is a very singular perturbation so, substituting the approximation above may lead to equations with different solutions.

   In Theorem 4.8 the main objects are sets of uniformly differentiable q. In these sets of uniformly differentiable functions, the approximations in (62) are uniform (in a norm involving one less derivative than the uniform).

3. Theorem 4.8 also applies to many modifications of the model and produces solutions. For example, ignoring The Figtgerald contractions [Ver16] or changing the delay by its state dependent approximation (62) [Dri84] (they are formally second order in ε). The exact solutions of these models obtained applying Theorem 4.8 will be approximate solutions of (59).

   In theoretical Physics, there are also other methods to produce approximate solutions using formal power series [CCdlL20, GdlLY25, MS78] or using numerical approximations.

[2] For $\varepsilon = 0$, the delays and advances vanish so that (59) is an ODE.

The a-posteriori formulation of Theorem 4.8 shows that these approximate solutions are close to solutions of (59).

4. The delays/advances τ_{ij}, σ_{ij} may be unbounded if the particles i, j get far apart. This is not covered by our Theorem 4.8. We assume in this paper that the unperturbed orbits remain in a bounded region.

   One can hope that this assumption can be weakened because when the delays are large, the interactions are weak. The problem of charges scattering is, of course, of great physical importance and has been considered many times. Starting with the pioneering work of [Dri63]. We refer to the recent [BDDH17] for an account of progress in this line of research and several other approximate models of electrodynamics.

5. Other recent advances in the theory of constructing solutions for electrodynamics exploit that periodic solutions of these equations have a variational structure. This allows to use deep variational tools such as Floer theory. We refer to [AFS20, Fra21].

   The approach for periodic orbits in [YGdlL22] does not require a variational structure, but on the other hand, requires proximity to an ODE.

Denoting $y(t) \stackrel{\text{def}}{=} (q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t))$, we can write the equation (59) in the form of (56) with the delays being implicitly defined. Note that there are $N(N-1)$ delays and $N(N-1)$ advances. Of course, we can consider cases where the expression of the forces does not depend on the advances.

Note that formally, the effect of delays is $O(1/c)$ whereas the effect of Lorentz-Fitzgerald contractions is $O(1/c^2)$. Therefore, it is common to consider models in which only the delays are considered and the Lorentz-Fitzgerald contraction effects are ignored [Ver16, CM07].

These models are of the form considered in (59). So, the results of this paper on persistence of hyperbolic solutions of the non-relativistic models apply. Furthermore, thanks to the a-posteriori formulation of Theorem 4.8 we obtain that the exact solutions in these models are at a distance $O(1/c^2)$ from the solutions to the full model.

Similarly, the hyperbolic solutions produced by other models that solve the relativistic equations to order n (e.g. [MS78] to order 4) will be $O(\varepsilon^{n+1})$ close to the solutions produced here.

However, it is important to note that the justification of the results here is only for hyperbolic solutions and that the quantitative aspects of the corrections needed depend on the hyperbolicity constants. In particular, in non-uniformly hyperbolic sets, where the hyperbolic constants deteriorate, the range of validity of the approximations will become smaller and the errors will be affected by larger constants. This is consistent with the impossibility of formulating effective equations valid everywhere [CJS63], which still allows formulating approximations in some sets.

7.6.2 The result

Since the equation (59) has singularities, we have to assume that the unperturbed solution is away from the singularities.

Definition 7.2. We say that a solution of the classical equations of motion is non-singular when there exist $0 < \xi_1 < 1$ and $\xi_2 > 0$, such that for all t:

$$\begin{aligned} |\dot{q}_j(t)| &\leqslant \xi_1 c, \qquad \text{for all } j, \\ |q_i(t) - q_j(t)| &\geqslant \xi_2, \qquad \text{for all } i \neq j. \end{aligned}$$

The internal forces and the masses are analytic around non-singular solutions, by Definition 7.2. Therefore, the regularity assumptions of Theorem 4.8 on the unperturbed equation concern only the external fields.

Assuming that the solutions remain in a bounded set and that the trajectories are uniformly away from collisions, we obtain that the vector fields giving the evolution are uniformly differentiable.

Theorem 7.3. *Consider the model* (59) *with the delays or advances defined in* (60). *Denote* $1/c$ *as* ε, *and treat it as a parameter.*

Assume that the external fields are A_{ext} *are* $C^{\ell+2+\text{Lip}}$.

Assume that for $\varepsilon = 0$, *the ODE* (59) *has a solution such that:*

1. *It is hyperbolic in the sense of Definition 2.1,*
2. *It is non singular in the sense of Definition 7.2.*
3. *It lies in a bounded set.*

Then, Theorem 4.8 applies to the problem given by (59) *and for small enough values of* ε, *we can find solutions of* (59) *of the form in* (35).

The proof follows from Theorem 7.1 once we have the estimates on regularity bounds and low regularity contraction of the delays and advances (60). These estimates will be obtained in the following section.

Given the formulation of the fixed point argument, we only need to prove that the delays are differentiable when the history segments are assumed to be differentiable (propagated bounds) and to show that the operator Γ is a contraction in a low regularity space when the history segments are differentiable.

Results on the regularity properties of the delay In this section, we study (60) as an equation for $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ when we prescribe the trajectories q_i and q_j. This makes precise the notion that the delay/advance is a functional of the whole trajectory and it shows that Theorem 7.3 follows from Theorem 7.1.

Proposition 7.4. *Let* q_i *and* q_j *be continuously differentiable trajectories that satisfy Definition 7.2 and have derivatives bounded uniformly away from the speed of light* c.

Then, for all $t \in \mathbb{R}$, *we can find unique* $\tau_{ij}(t), \sigma_{ij}(t) > 0$ *solving* (60).

Furthermore, let $\tilde{\tau}_{ij}, \tilde{\sigma}_{ij}$ *be the delay and advance of trajectories* $\tilde{q}_i$ *and* $\tilde{q}_j$, *there exists a constant* C *such that*

$$\|\tau_{ij} - \tilde{\tau}_{ij}\|_{C^0}, \|\sigma_{ij} - \tilde{\sigma}_{ij}\|_{C^0} \leqslant C\big(\|q_i - \tilde{q}_i\|_{C^0} + \|q_j - \tilde{q}_j\|_{C^0}\big).$$

Moreover, if the trajectories q_i *and* q_j *are* $C^{\ell+\text{Lip}}$ *and satisfy Definition 7.2. Then the* τ_{ij}, σ_{ij} *are* $C^{\ell+\text{Lip}}$, *and there is an explicit algebraic expression* g *such that*

$$\|\tau_{ij}\|_{C^{\ell+\text{Lip}}}, \|\sigma_{ij}\|_{C^{\ell+\text{Lip}}} \leqslant g(\|q_i\|_{C^{\ell+\text{Lip}}}, \|q_j\|_{C^{\ell+\text{Lip}}}, \xi_1, \xi_2).$$

Proof. The first part of Proposition 7.4 follows from the standard contraction mapping theorem applied to (60). The second part also follows from the contraction principle, remembering that we are assuming uniform differentiability of the q_i.

Let us define the operator $\mathcal{N}[\tau; q_i, q_j](t) \stackrel{\text{def}}{=} \varepsilon|q_i(t) - q_j(t - \tau(t))|$ and let us first prove that $\mathcal{N}$ is a contraction for small enough ε. That is,

$$\begin{aligned} |\mathcal{N}[\tau](t) - \mathcal{N}[\tilde{\tau}](t)| &= \varepsilon\big||q_i(t) - q_j(t-\tau(t))| - |q_i(t) - q_j(t - \tilde{\tau}(t))|\big| \\ &\leqslant \varepsilon|q_j(t-\tau(t)) - q_j(t-\tilde{\tau}(t))| \leqslant \varepsilon \operatorname{Lip}(q_j)\|\tau - \tilde{\tau}\|_{C^0}. \end{aligned}$$

Let $\varepsilon < 1/\|q_j\|_{C^1}$, and define $\kappa \stackrel{\text{def}}{=} \varepsilon\|q_j\|_{C^1}$, then $\mathcal{N}$ is a contraction with rate κ.

We first bound $\|\mathcal{N}[\tilde{\tau}; q_i, q_j] - \tilde{\tau}\|_{C^0} \leqslant B$ for $\tilde{\tau}$ a fixed point of $\mathcal{N}[\cdot; \tilde{q}_i, \tilde{q}_j]$ of some particles $\tilde{q}_i$ and $\tilde{q}_j$, and B related to the difference between q and $\tilde{q}$. This implies $\|\tau - \tilde{\tau}\|_{C^0} \leqslant \frac{B}{1-\kappa}$.

$$\begin{aligned} |\mathcal{N}[\tilde{\tau}; q_i, q_j](t) - \tilde{\tau}(t)| &= \big|\varepsilon|q_i(t) - q_j(t - \tilde{\tau}(t))| - \varepsilon|\tilde{q}_i(t) - \tilde{q}_j(t - \tilde{\tau}(t))|\big| \\ &\leqslant \varepsilon|q_i(t) - \tilde{q}_i(t)| + \varepsilon|q_j(t - \tilde{\tau}(t)) - \tilde{q}_j(t - \tilde{\tau}(t))| \\ &\leqslant \varepsilon\big(\|q_i - \tilde{q}_i\|_{C^0} + \|q_j - \tilde{q}_j\|_{C^0}\big). \end{aligned}$$

The estimates on the derivatives of τ_{ij}, σ_{ij} follow from applying the implicit function theorem for the solutions of (60). □

By Proposition 7.4, for τ_{ij}, σ_{ij}, we have Lipschitz estimates and that they are $C^{\ell+\mathrm{Lip}}$ if q_i, q_j are $C^{\ell+\mathrm{Lip}}$. Therefore, we can use the procedure in Section 7.5 and apply Theorem 7.1. Note that the τ's in Theorem 7.1 correspond to τ_{ij}/ε or σ_{ij}/ε since both τ_{ij}, σ_{ij} are of order $O(\varepsilon)$.

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Statements and Declarations

The authors declare that they have no conflict of interest.

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