Averaging methods and splitting of separatrices

Let $f : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ integrable.

1. We define the spatial average as:

$$\bar{f} = \frac{1}{2\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\varphi)\, d\varphi$$

2. We define the time average as

$$f^*(\varphi_0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi_0 + \omega t)\, dt$$

{ Thm: time average exits and coincides with the sp. av. if f is Riemann integrable and w is rationally independent. }

The (heuristic) averaging principle

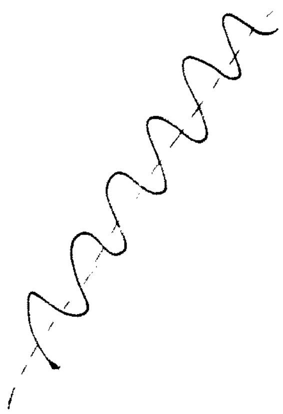
Consider $\dot{z} = \varepsilon f(t, z)$, $z(t_0) = z_0$ (1)

with f T-periodic in t.

The principle states that if

"The solutions of $\dot{z} = \varepsilon \bar{f}(z)$ are close to those of (1)."

The idea behind this ppe is clear. If we have an oscillating system, ~~with a unit~~ the real solution will not be too far from what the average yields.



The averaging ppl can be justified, under suitable conditions:

One-frequency averaging

with flow $\varphi_t(I, \varphi, I_0, \varphi$

Consider the system (1)
$$\dot{I} = \varepsilon f(I, \varphi, \varepsilon)$$
$$\dot{\varphi} = \omega(I) + \varepsilon g(I, \varphi, \varepsilon)$$

and assume $I \in K \subset \mathbb{R}^n$, $0 < m < \omega(I) < M$ in K

f, ω and g 2π-periodic in φ, C^1 in all arguments.

Assume f, $\nabla_I f$, g, ω, $\nabla\omega$ are all cf. bdd on $K \times S^1$ for ε small enough. Let

(2) ~~dd~~ $\dot{J} = \varepsilon \bar{f}(J)$, $\bar{f}(J) := \frac{1}{2\pi}\int_0^{2\pi} f(J, \varphi,$

with flow $\tilde{\varphi}$

Then for all $\varepsilon < \varepsilon_0$ $\exists$ a solution for (1)
~~and~~ for $t \in [0, \tau^*/\varepsilon]$, and if we define K' as ~~K~~

$\{ x \in K \,/\, \tilde{\varphi}_t(x) \in "K - y", t \in [0, \tau^*/\varepsilon]$

then for every initial condition in $K^1 \times S^1$

$$\sup_{t \in [0, \tau^*/\varepsilon]} \| I(t) - J(t) \| = \mathcal{O}(\varepsilon)$$

~~Moreover~~

Sketch of the proof

Consider a change of v. of the form

$$I \longrightarrow P = I + \varepsilon S(I, \varphi)$$

Determine S:

$$\frac{dP}{dt} = \varepsilon \left(f(I, \varphi, 0) + \omega(I) \frac{\partial S}{\partial \varphi} \right) + \underbrace{\mathcal{O}(\varepsilon^2)}_{R(I, \varphi, }$$

Set $$S(I, \varphi) = -\frac{1}{\omega(I)} \int_0^{\varphi} [f(I, \varphi, 0) - \bar{f}(I)] \, d\varphi$$

Now:

$$R(I, \varphi, \varepsilon) = \varepsilon^2 \frac{\partial S}{\partial I} \cdot f(I, \varphi, \varepsilon) + \varepsilon [f(I, \varphi, \varepsilon) - f(I, \varphi, 0)] + \varepsilon^2 g(I, \varphi, \varepsilon) \frac{\partial S}{\partial I}$$

which is (with S as defined!) $\mathcal{O}(\varepsilon^2)$. Then

$$\frac{dP}{dt} = \varepsilon \bar{f}(P) + \mathcal{O}(\varepsilon^2). \quad (1)$$

Consider $x(t) = P(t) - J(t)$ where $(P(t), \varphi(t))$ is the solution in the new variables with initial cond $(P(I_0, \varphi_0), \varphi(0))$.

Then

$$\| \dot{x}(t) \| \leq \varepsilon \cdot \| \nabla \bar{f} \| \cdot \| x \| + \mathcal{O}(\varepsilon^2) \quad \text{by } (1)$$

then by Gronwall's lemma:

$$\| P(t) - J(t) \| \leq e^{\varepsilon c t} \Big(\underbrace{\| P(0) - J(0) \|}_{= P(0) - I(0) = \mathcal{O}(\varepsilon)} + c \cdot \varepsilon^2 t \Big)$$

$$< c \cdot \varepsilon \cdot e^{\varepsilon c t}$$

~~for $t = \mathcal{O}(1/\varepsilon)$~~

We want to show this is valid for $\mathcal{O}(1$

We know this holds as long as $P(t) \in K$.

Let τ be the "moment of leave".

Let $\varepsilon < \varepsilon_0$ such that $\|P(\tau) - J(\tau)\| < \eta/2$.

~~This means $P(t)$ lies in $K - \eta/2$ for $t \in [\tau, \tau^*/\varepsilon]$ which implies~~

Thus $P(\tau) \in K - \eta/2$ $\left(J(\tau) \in K - \eta \right)$
$\tau < \tau^*/\varepsilon$

then I also keep in the same domain for times $O(1/\varepsilon)$. Hence

$$\|I(t) - J(t)\| \leq \|I(t) - P(t)\| + \| P(t) - J(t) \|$$

$$\leq C_1 \varepsilon \left(1 + C_2 e^{C_3 \varepsilon t} \right)$$

∎

Series formulation

Consider a before the system

$$\dot{I} = \varepsilon f(I, \varphi, \varepsilon)$$
$$\dot{\varphi} = \omega(I) + \varepsilon g(I, \varphi, \varepsilon) \qquad (1)$$

We look for a change of variables of the form

$$I, \varphi \longrightarrow J, \psi$$

$$I = J + \varepsilon u_1(J, \psi) + \varepsilon^2 u_2(\quad) + \dots$$

$$\varphi = \psi + \varepsilon v_1(J, \psi) + \varepsilon^2 v_2(\quad) + \dots$$

where u_j, v_j are 2π-periodic in ψ. We want this c. of v. s.t.

$$\dot{J} = \varepsilon F_0(J) + \varepsilon^2 F_1(J) + \dots$$

$$\dot{\psi} = \omega(J) + \varepsilon G_0(J) + \dots$$

i.e. independent of the (fast) phases.

Assume from now on (1) is analytic in all v.

Then :

$$F_0(J) = f(J, \psi, 0) - \frac{\partial U_1}{\partial \psi}\,\omega$$

$$G_0(J) = g(J, \psi, 0) + \frac{\partial \omega}{\partial J} U_1 - \frac{\partial V_1}{\partial \psi}\,\omega$$

$$F_i(J) = X_i(J, \psi) - \frac{\partial U_{i+1}}{\partial \psi}\,\omega \qquad i \geq 1$$

$$G_i(J) = Y_i(J, \psi) + \frac{\partial \omega}{\partial J} U_{i+1} - \frac{\partial V_{i+1}}{\partial \psi}\,\omega$$

where X_i, Y_i are uniquely determined by U_j, V_j $j \leq$

To solve this system, let $h(J, \psi)$ be analytic and 2π-per in ψ,

$$h = h_0(J) + \sum_{k \neq 0} h_k(J)\, e^{i(k, \psi)}$$

i.e. $h_k \neq 0$

and denote

$$\langle h \rangle^{\psi} = h_0(J) \qquad , \qquad \{h\}^{\psi} = \sum_{k \neq 0}' \frac{h_k}{i(k, \omega)}\, e^{i(k, \psi)}$$

Averaging Integration

With these two operators we can solve,

arbitrary

$$F_0(J) = \langle f(J,\psi,0)\rangle_\psi$$

$$u_1(J,\psi) = \{ f(J,\psi,0)\}^\psi + u_1^0(J)$$

⋮

etc.

If we truncate these series at order $r \geq 1$,

$$\dot J = \varepsilon F_\Sigma(J,\varepsilon) + \varepsilon^{r+1}\alpha(J,\psi,\varepsilon)$$

$F_0 \quad F_r$

$$\dot\psi = \omega(J) + \varepsilon G_\Sigma(J,\varepsilon) + \varepsilon^{r+1}\beta(J,\psi,\varepsilon)$$

Observe we assumed $\cancel{(k,\omega)\neq 0}$

$$(k,\omega(J))\neq 0$$

which for 1-fr. system is "easy" to assume (just $\omega(J)>0$). For m.f. systems resonances play an important role.

Exponential estimates (sketch)

Neishtadt's theorem

$$\dot{I} = \varepsilon f(I ; \varphi, \varepsilon) \qquad (1)$$
$$\dot{\varphi} = \omega(I) + \varepsilon g(I, \varphi, \varepsilon)$$

assume $\omega(I) > 0$, all functions real analytic in $D + \delta$, $D = G\{I\} \times S^1\{\varphi\}$ (G region)

and $|f| < C$, $|g| < C$.

Thm: When $y, \psi \in D + 1/2\,\delta$, $0 < \varepsilon < \varepsilon_1$, there exist an analytic c. of v. 2π-per. in ψ of the form

$$I = J + \varepsilon U(J ; \psi, \varepsilon)$$
$$\varphi = \psi + \varepsilon V(J, \psi, \varepsilon)$$

$U, V = \mathcal{O}(1)$ that reduces (1) to

$$\dot{J} = \varepsilon \left(F(J, \varepsilon) + \alpha(J, \psi, \varepsilon)\right)$$
$$\dot{\psi} = \Omega(J, \varepsilon) + \varepsilon \beta(J, \psi, \varepsilon)$$

with

$$F = \bar{f} + O(\varepsilon)$$

$$\Omega = \omega + O(\varepsilon).$$

$$\dot{I} = \varepsilon \left(F_i(I) + \alpha_i \right)$$

Sketch of the proof:

$$\dot{\varphi} = \Omega_i(I) + \varepsilon \beta_i$$

$$\bar{\alpha}_i = \bar{\beta}_i = 0 \quad , \quad D_i \subset D_{i-1}$$

Let $I = J + \varepsilon u(J;\psi)$ $\quad \varphi = \psi + \varepsilon v(J;$

then

$$\dot{J} = \varepsilon \Big(\underbrace{E}_{Id_{n\times n}} + \varepsilon \frac{\partial u}{\partial J} \Big)^{-1} \Big(F_i (J + \varepsilon u)$$

$$+ \alpha_i (J + \varepsilon u, \varphi) - \quad \dots$$

$$\dot{\psi} = \dots$$

Choose u, v s.t. $\quad u(J;\varphi) = \frac{1}{\Omega_i(J)} \int_0^\varphi \alpha_i$

$$v = \dots$$

If we make r steps

$$|\alpha_i| + |\beta_i| < 2^{-i+1} K \varepsilon$$

and after $r = [1/4 \, \delta K/\varepsilon] > K'/\varepsilon$,

$$|\alpha_r| + |\beta_r| < 2^{-r+1} K \varepsilon < c_2 e^{-c^{-1}/\varepsilon}$$

And inductive step & bounds.

Additional remarks

Simple example

$$\dot{I} = \varepsilon (a + b \cos\varphi)$$

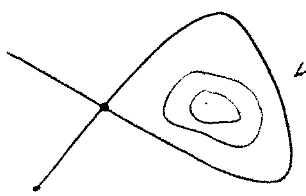
$$\dot{\varphi} = \omega$$

$\hookrightarrow$ $\dot{J} = \varepsilon a$ and

$$I(t) = I_0 + \varepsilon a t + \varepsilon b[\sin(\omega t + \varphi_0) - \sin\varphi_0]/\omega$$

$$J(t) = I_0 + \varepsilon a t$$

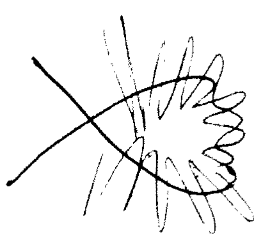
Exponentially small splitting of separatrices.



separatrix (homoclinic/heteroclinic connection)

Integrable system

$\downarrow$ perturbation

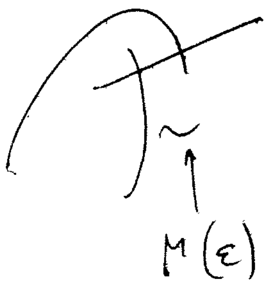


Homo/heteroclinic tangle.

In conservative systems this splitting is (generically) EXPONENTIALLY SMALL.

S. of s. means the system is non-integrable, and there are chaotic zones of very small measure.

S. of s. can be measured, sometimes use the Melnikov method / integral / function.



But it is first order in the parameter: $M(\varepsilon) = M_0 + \varepsilon M_1 + \dots$

Melnikov function

But if an exp. small averaging method can be used, one may be able to prove exp. small estimates.

Neishtadt → "easy" example
Simó → Harder
Treschev → Hardest

Historical use:

Lagrange, Laplace, Poincaré. Celestial mechanics. Simplification of problems to make analytical estimates

Early 20th cent. Numerical computation
Late 20th cent. Adiabatic invariants, splitting.

References:

P. Lochak, C. Meunier: Multiphase Av. for classical system (not really good, but lots of things).

J. Sanders, F. Verhulst: Av. methods in nonlinear d.s. (quite good).

V. I. Arnol'd: M.M. C.M.

Arnold, Kozlov, Neishtadt: Dyn. Sys. 3: Math. aspects of classical & celestial mechanics (must read!)

Neishtadt: The separation of motions in systems with rapidly rotating phase (ask me for a copy)

Simó: Averaging under fast q.p. forcing