Model Equation, Stability and Dynamics for Wavepacket Solitary Waves

Paul Milewski
Mathematics, UW-Madison

Collaborator: Ben Akers, PhD student

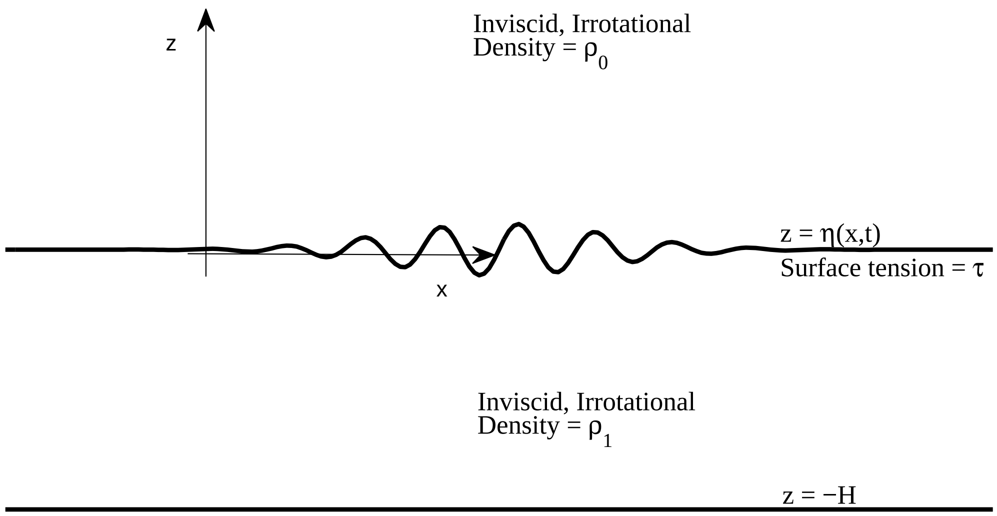
Summary

- Localized solitary waves exist in the water wave problem in the capillary-gravity regime ($\lambda \sim 1cm.$) both for one- and two-dimensional free-surfaces in fluids of *any* depth. This contrasts with gravity only waves where solitary waves exist *only* for a one dimensional free-surface and *only* in the shallow water regime.
- Why do we care about these waves?
  - The initial generation of waves by wind is predominantly in the capillary-gravity regime.
  - Gravity capillary waves are the main scatterers in microwave radar remote sensing.
- What is a good reduced model for these waves?
- What are the dynamics of these waves (stability, collisions, etc...)?

References

- Dias, F., Kharif, C. (1999), “Nonlinear Gravity and Capillary-Gravity Waves”, *Ann. Rev. Fluid Mech.*
- Vanden-Broeck, J.-M., Dias, F. (2002) “Gravity-capillary solitary waves in water of infinite depth and related free surface flows." *J. Fluid Mech.*, **240**, 549Ð557.
- Calvo, David C., Yang, T.-S., Akylas T. R. (2002), “Stability of steep gravity-capillary waves in deep water". *J. Fluid Mech.*,**452**, 123–143.
- Zhang, Xin, (1995) “Capillary-gravity and capillary waves generated in a wind wave tank: observations and theories," *J. Fluid Mech.* **289**, 51–82.
- Falcon, E., Laroche, C., Fauve, S. (2002) “Observation of depression solitary surface waves on a thin fluid layer," *Phys. Rev. Let.* **89(20)**.
- Parau, E. I., Vanden-Broeck, J.-M., Cooker, M. J. (2005) “Nonlinear three-dimensional gravity-capillary solitary waves", *J. Fluid Mech.* **536**, 99–105.
- Kim, B., Akylas, T. R. (2005) “On gravity-capillary lumps", *J. Fluid Mech.* **540**, 337–351.
- Also, Milewski and Vanden-Broeck (1999), Berger and Milewski (2000), Milewski (2005), Akers and Milewski (2008).

Linear Water Waves



$$\eta_{tt} + \frac{|k| sinh(|k|H)(g(\rho_1 - \rho_0) + \tau |k|^2)}{\rho_1 cosh(|k|H) + \rho_0 sinh(|k|H)} \eta = 0, \qquad (1)$$

Here k is the Fourier dual variable to x. The wavelength $L = 2\pi/|k|$. There are 2 dimensionless parameters $\lambda = \rho_0/\rho_1$ and $B = \tau/g\rho_1 H^2$. Renaming kH as k, the frequency of a wave is

$$\Omega^2(k) = \frac{|k| sinh(|k|)(1 - \lambda + B|k|^2)}{cosh(|k|) + \lambda sinh(|k|)}. \qquad (2)$$

Deep Water Formulation

Consider an inviscid, irrotational flow of a fluid of infinite depth with a free-surface. Nondimensionalizing the problem with the surface tension coefficient and gravity we have:

$$\Phi_{zz} + \Delta\Phi = 0, \qquad -\infty < z < \eta(\mathbf{x}, t), \tag{3}$$

$$\eta_t + \nabla\Phi \cdot \nabla\eta = w, \qquad z = \eta(\mathbf{x}, t), \tag{4}$$

$$\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}\Phi_z^2 + \eta - \kappa = 0, \qquad z = \eta(\mathbf{x}, t). \tag{5}$$

Here $\Phi(\mathbf{x}, z, t)$ is the velocity potential such that $\mathbf{u} = (\nabla\Phi, \Phi_z)$, the function η is the departure from a flat free surface, κ is the curvature of the surface. We reduce the problem in the interior by writing

$$\Phi(\mathbf{x}, z, t) = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{\phi(\mathbf{x}, t)\right\} e^{|\mathbf{k}|z}\right\}.$$

Where $\mathcal{F}$ is the Fourier transform in $\mathbf{x}$ with dual variable $\mathbf{k}$. Inserting into the boundary conditions gives a time evolution system of equations for η and ϕ.

Dispersion relation

$\Omega(k)$ called the dispersion relation. There are two important speeds associated to the waves

- Phase speed (propagation speed of wavefronts or crests).

$$c_p = \frac{\Omega(k)}{|k|}\frac{k}{|k|} = C_p \hat{e}_k$$

- Group speed (propagation of energy)

$$c_g = \nabla_k \Omega$$

A necessary condition for solitary waves to bifurcate from linear waves at a given value of k in a weakly nonlinear equation with linear dispersion relation $\Omega(k)$ is that

$$c_g = c_p$$

[Solitary waves are solutions $\eta(x - ct)$ which decay to zero as $|x - ct| \to \infty$]

One-dimensional, One-way, Weakly-nonlinear

In one dimension the evolution for η can be factored into left and right propagating waves. The equation for the right-propagating ones is given approximately by

$$\eta_t + \mathcal{L}\eta = \epsilon\mathcal{N}(\eta), \qquad (6)$$

where $\mathcal{N}$ is a quadratic nonlinear term and $\mathcal{L}$ is a linear operator, diagonal in Fourier space, with the odd Fourier symbol given by the dispersion relation $i\Omega(k)$:

$$\mathcal{L}e^{ikx} = i\Omega(k)e^{ikx}. \qquad (7)$$

Computing the group velocity in terms of the phase velocity:

$$c_g = (kc_p)' = kc_p' + c_p.$$

The condition that phase and group velocity are equal is generically satisfied in the limit as $k \to 0$, and is satisfied at finite k wherever $c_p' = 0$. (Note that at $k = 0$, $c_p' = 0$ also.)

> *Generically, solitary waves bifurcate from linear waves when there is an extremum of $c_p(k)$ in one-dimension, and a minimum of $C_p(|k|)$ in higher dimensions.* [Local extrema usually lead to generalized solitary waves.]

Shallow water ($k \approx 0$) examples

$$\Omega(k) = sign(k)\left[\frac{|k|sinh(|k|)(1-\lambda+B|k|^2)}{cosh(|k|)+\lambda sinh(|k|)}\right]^{1/2}. \tag{8}$$

- If $\rho_0 = 0$, $\Omega \approx k - \frac{1}{2}\left(\frac{1}{3}-B\right)k^3$, $c_p = 1 - \frac{1}{2}\left(\frac{1}{3}-B\right)k^2$ giving the *Korteweg-de Vries* equation:

$$\eta_t + \eta_x + \epsilon\frac{1}{2}\left(\frac{1}{3}-B\right)\eta_{xxx} + \epsilon\frac{3}{2}\eta\eta_x = 0.$$

  In higher dimensions only $B > \frac{1}{3}$ have solitary waves in this limit.

- If $\rho_0 \neq 0$, approximation leads to $\Omega(1-\lambda)^{-1/2} \approx k - \frac{1}{2}\lambda k|k|$, $c_p = (1-\lambda) - \lambda|k|$, giving the *Benjamin-Ono* equation:

$$\eta_t + \eta_x - \frac{1}{2}\epsilon\lambda\mathcal{H}\eta_{xx} + \epsilon\frac{3}{2}\eta\eta_x = 0.$$

  Here, $\mathcal{H}$ is the Hilbert transform in x, with Fourier symbol $-isign(k)$.

Deep Water I

In deep water the full dispersion relation simplifies considerably

$$\Omega(\mathbf{k}) = sign(k)\left[|\mathbf{k}|(1+|\mathbf{k}|^2)\right]^{1/2}, \qquad \mathbf{k}=(k,l),$$

and we expand and rescale about $(k,l)=(1,0)$ for left-travelling waves

$$\tilde{\Omega}(\mathbf{k}) \approx sign(k)\left[-1+2|k|-k^2-2l^2\right].$$

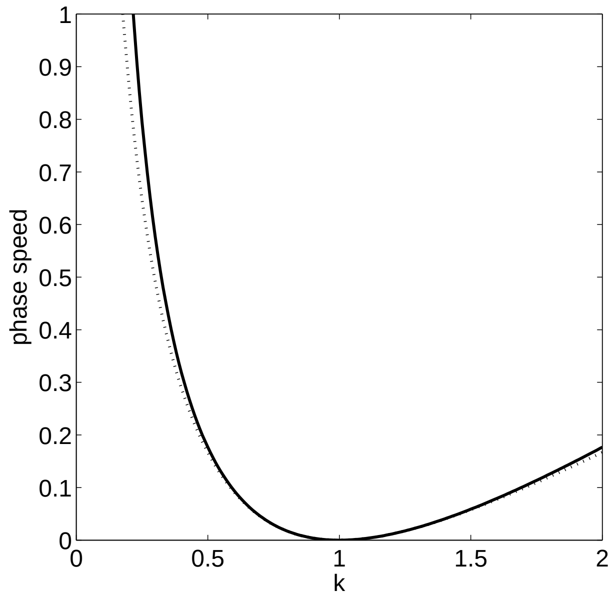
Leading to an approximate equation (the “finite-k KP equation")

$$\eta_t + \mathcal{H}\eta + 2\eta_x - \mathcal{H}\eta_{xx} - \mathcal{H}\eta_{yy} - \epsilon\frac{3}{2}\eta\eta_x = 0. \tag{9}$$

$$\eta_t \qquad - \frac{1}{6}\eta_{xxx} \qquad - \int \eta_{yy} - \epsilon\frac{3}{2}\eta\eta_x = 0. \tag{10}$$

[KPI equation]

Deep Water II



$$M = \int u dx,$$

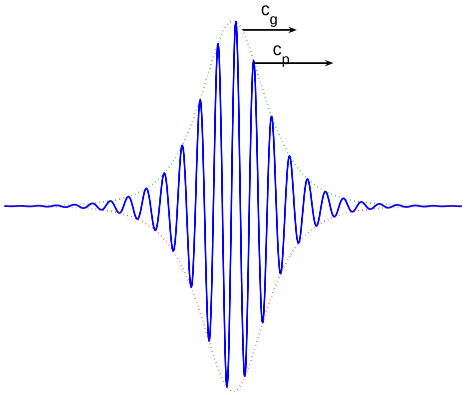
$$E = \int u^2 dx,$$

$$H = \int \frac{1}{2} u \mathcal{C} u + \frac{1}{3} u^3 dx,$$

$$\hat{\mathcal{C}}(k,l) = \frac{1}{|k|} + |k| + 2\frac{l^2}{|k|}.$$

The capillary gravity regime, and corresponds to $\lambda = 1.7cm$ and $c_p = 23cm/sec$ in water.

Role of NLS



Away from $k = 0$ narrow band solutions take the form of wavepackets: consider solutions of the form (where we have chosen the appropriate scalings)

$$\eta = \epsilon A(\epsilon(x - c_g t), \epsilon^2 t)e^{ik(x-c_p t)} + * + O(\epsilon^2), \tag{11}$$

[From this it is clear that approximate traveling waves satisfy $c_g = c_p$.] Define $X = x - c_g t$, $\tau = \epsilon^2 t$. Then $A(X, \tau)$ satisfies the NLS equation:

$$iA_\tau + \lambda A_{XX} = \chi|A|^2 A \tag{12}$$

Here $\lambda = \frac{1}{2}c'_g = \frac{\sqrt{2}}{2}$. The product $\lambda\chi < 0$ corresponds to a focussing NLS.

Limitations of NLS

Solitary wave type solutions

$$A(X,\tau) = d|2\chi/\lambda|^{1/2}\text{sech}(d(X-2k\lambda\tau))e^{ikX-i\lambda(k^2-d^2)\tau}. \tag{13}$$

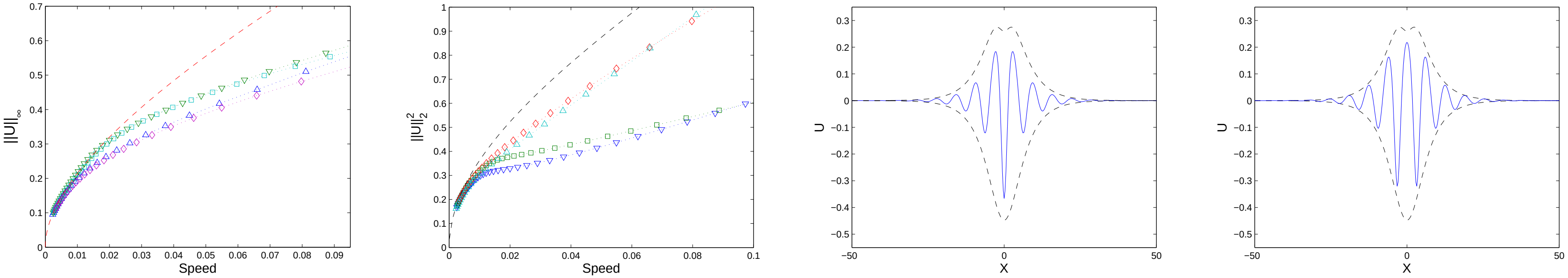
For traveling waves one must have $\epsilon k = -\lambda(\epsilon d)^2/2$, giving a traveling speed

$$c = c_g - \frac{\lambda}{k_0}(\epsilon d)^2$$

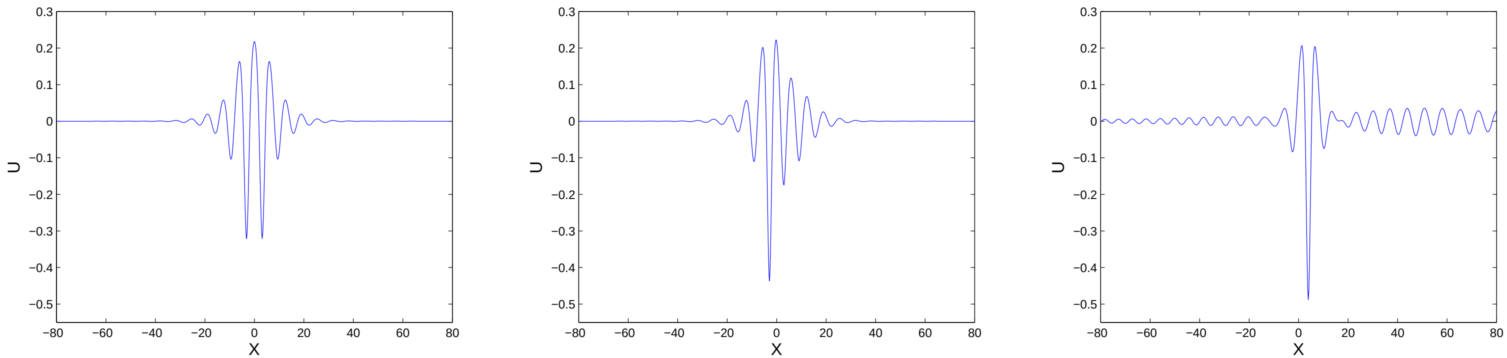
This solution serves to motivate the existence of solitary wave and as an initial guess in numerical computations. However NLS fails at:

- Symmetry of waves. Only symmetric waves are found in models (although there is indication that exotic asymmetric solutions are possible). In NLS the relative phase between carrier and envelope is undetermined.
- Stability of waves. Solitons are stable (in the appropriate sense) in NLS, however in the fluid some solitary waves are stable and others unstable.
- Wave dynamics and interaction. NLS is integrable and collisions are elastic. In all other model equations they are not. (This is being checked numerically in Euler.)

1-D Solutions



$$\text{Energy} \sim \frac{32}{9}(\Delta c)^{1/2} + \frac{64}{9}(\Delta c)^{3/2}$$



Elevation waves are unstable, and the evolution transforms them into depression waves.

This can be confirmed with eigenvalue calculations in models as well as Euler. (Instability of translational mode)

2-D NLS

The “relevant" NLS model in two-dimensions is the focussing case

$$iA_t - \Delta A - |A|^2 A = 0. \tag{14}$$

with conserved quantities

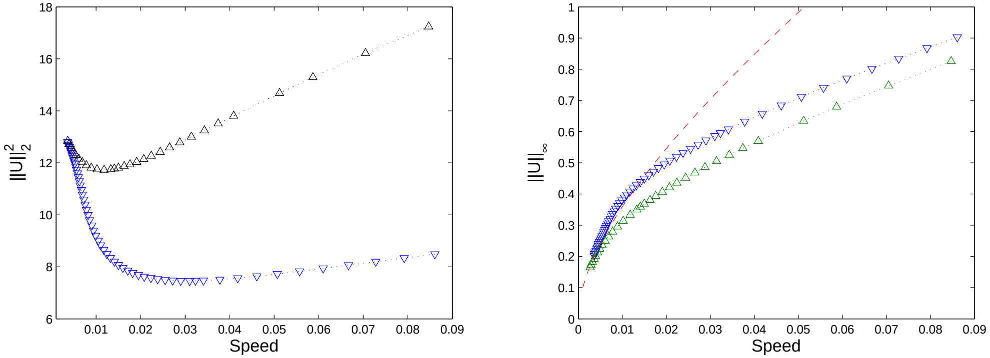
$$M = \int |A|^2, \qquad E = \int |\nabla A|^2 - \frac{1}{2}|A|^4$$

The equation has finite time singularity, depending on the sign of E, according to an elegant argument of Zakharov. Consider

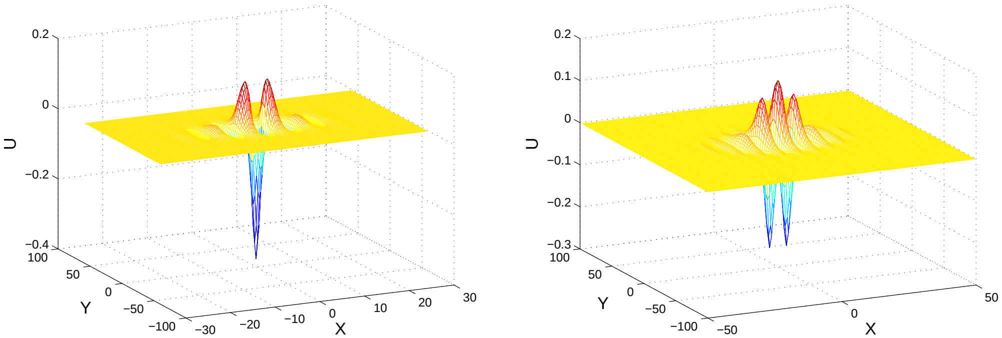
$$G(t) = \int (x^2 + y^2)|A|^2, \qquad \text{then} \qquad G''(t) = 8E.$$

If $E < 0$ then $G \to 0$ and the wave collapses. Ground state solitary waves (which approximate the localized solutions of the water wave problem) have $E = 0$.

2-D Solutions

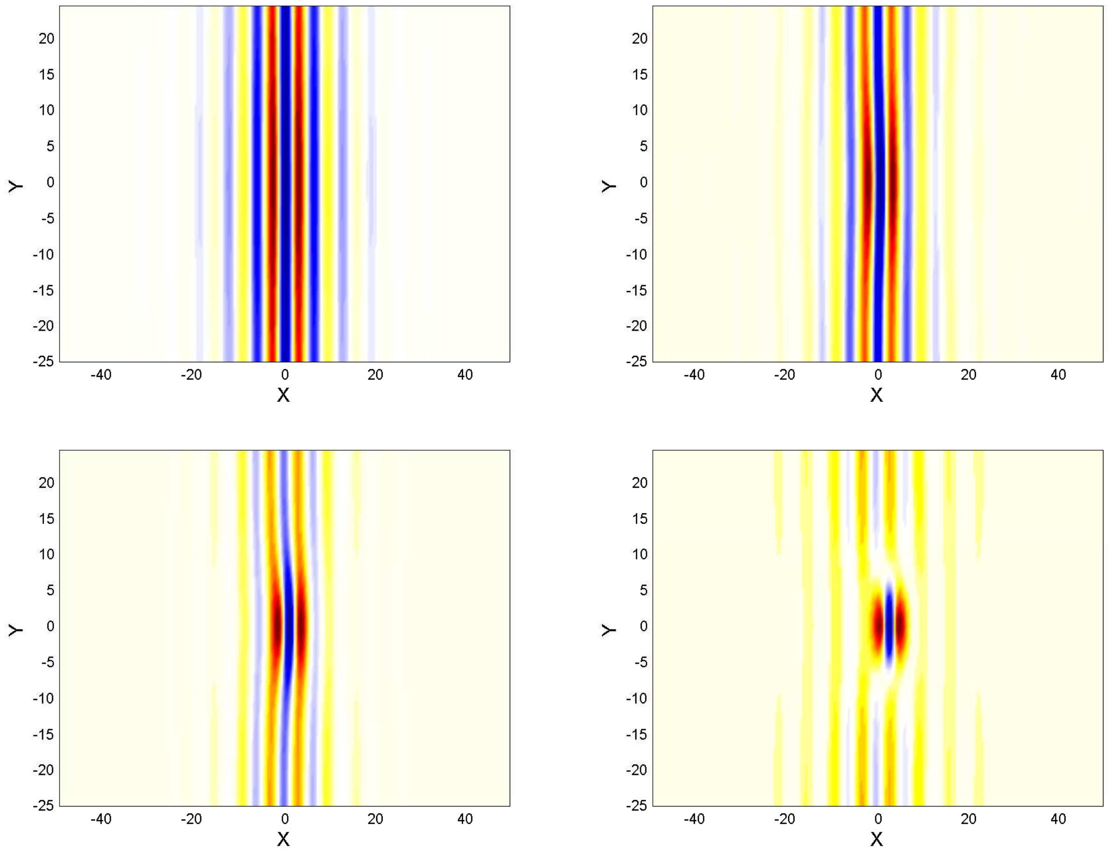


Energy $\approx 14.7 + 29.3\Delta c.$



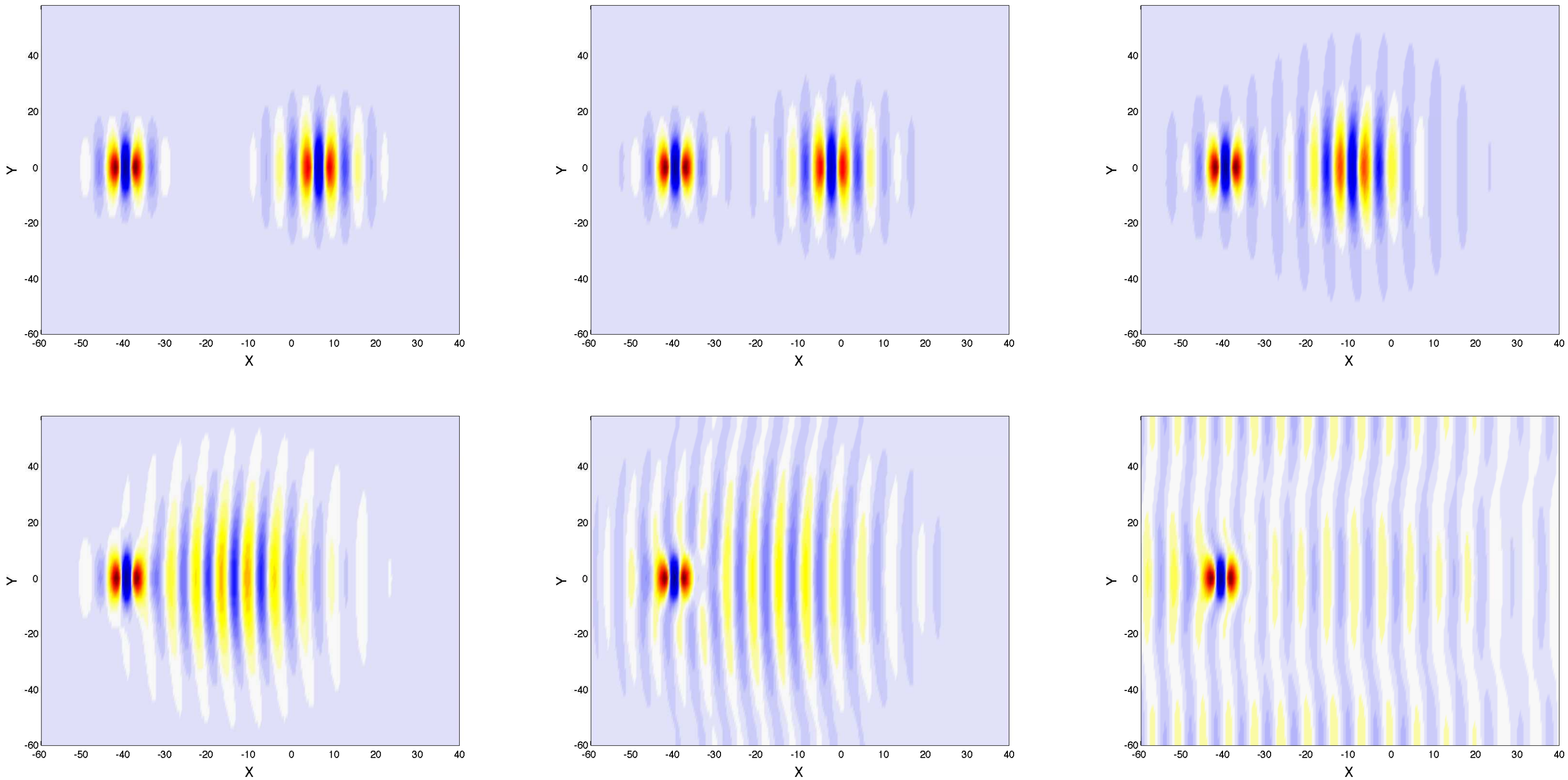
We shall see through computations that elevation waves are unstable and depression waves are sometimes stable. Change of stability at extrema of $E(c)$.

2-d Line Solitary Wave stability



Line solitary waves unstable and focus. Predictable through 2D NLS linear stability analysis or through initial stages of focussing instability.

Collisions (Inelastic)



There are inelastic collisions in the model. Movies.

Conclusions

- *Searching for a correct model in deep water.* In shallow water we have the depression KdV and KP equations (for B>1/3). The NLS type envelope equations are inadequate in this deep water regime.
- Are the 2-d waves observable [see Fauve et al for 1d observations]?
- Does nonlinearity enhance efficiency of energy transfer from wind?
- General principles and approach to 2-d solitary wavepackets. Other examples?
- Little is known mathematically about the finite-k KP: robustness, finite time singularity, etc...?.