Numerical Fourier analysis of quasi–periodic functions

G. Gómez,[1] J.M. Mondelo[2] C. Simó[1]

[1]Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona

[2]Departament de Matemàtiques, Universitat Autònoma de Barcelona

WSIMS08
IMUB dec 1–5, 2008

Outline

Outline

Setting

We are given an **analytic, quasi–periodic** function

$$f(t) = \sum_{\boldsymbol{k} \in \mathbb{Z}^m} a_{\boldsymbol{k}} e^{i2\pi \langle \boldsymbol{k}, \boldsymbol{\omega} \rangle t},$$

satisfying the Cauchy estimates

$$|a_{\boldsymbol{k}}| \leq C e^{-\delta |\boldsymbol{k}|} \quad \big(\exists C > 0, \delta > 0, \quad |\boldsymbol{k}| = |(k_1, \ldots, k_m)| = |k_1| + \cdots + |k_m| \big)$$

and with a vector of basic frequencies $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ satisfying a Diophantine condition

$$|\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle| > \frac{D}{|k|^{\tau}}, \quad \big(\exists D, \tau > 0 \big).$$

We want to **numerically compute** the frequencies $\{\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle\}_{|\boldsymbol{k}|=0}^{\text{maxor}}$ and amplitudes $a_{\boldsymbol{k}}$ from the values of f.

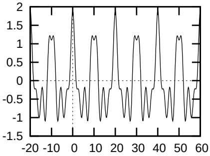
Fourier Transform

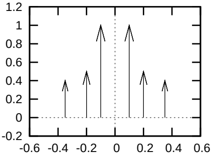
The Fourier Transform will be denoted as

$$f(t) \quad \xrightarrow{\mathcal{F}} \quad \mathcal{F}\big(f(t)\big)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t} dt$$

If $f(t)$ is quasi–periodic, its Fourier transform is a discrete set of impulses based at the frequencies:

$$f(t) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} a_{\boldsymbol{k}} e^{i2\pi\langle \boldsymbol{k},\boldsymbol{\omega}\rangle t} \quad \xrightarrow{\mathcal{F}} \quad \hat{f}(\omega) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} a_{\boldsymbol{k}} \delta_{\langle \boldsymbol{k},\boldsymbol{\omega}\rangle}(\omega)$$



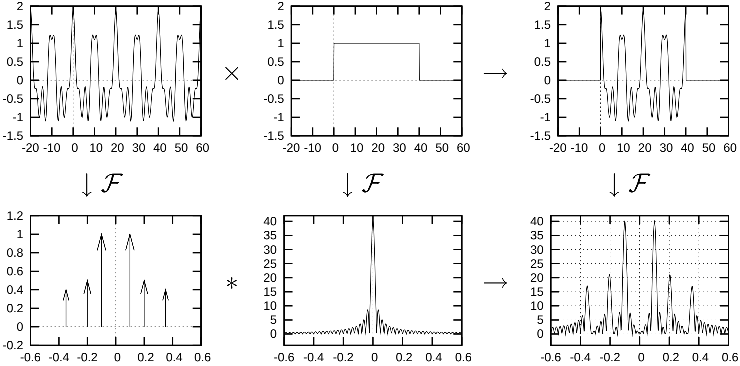


Example: $f(t) = \cos(2\pi 0.1t) + 0.5\cos(2\pi 0.2t) + 0.4\cos(2\pi 0.35t)$

Time truncation $\longrightarrow$ WFT

Graphical development (E.O. Brigham, 1988)

Time truncation gives rise to the phenomenon known as *leakage*.
Example: $T = 40, f(t) = \cos(2\pi 0.1 t) + 0.5\cos(2\pi 0.2 t) + 0.4\cos(2\pi 0.35 t)$.



The maxima of the WFT (bottom right) are displaced from the true frequencies.

Time truncation $\longrightarrow$ WFT

Explicit formulae

- Windowed Fourier Transform:

$$\begin{aligned}\phi_{f,T}(\omega) &:= \frac{1}{T}\mathcal{F}\left(\chi_{[0,T]}f(t)\right)(\omega)\\ &= \frac{1}{T}\int_0^T \chi_{[0,T]}(t)f(t)e^{-i2\pi\omega t}dt.\end{aligned}$$

- Leakage of a complex exponential term.

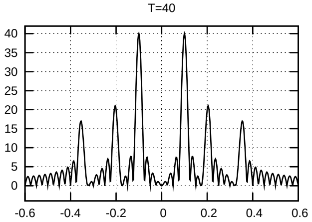
$$\begin{aligned}\left|\phi_{e^{i2\pi\nu t},T}(\omega)\right| &= \left|\frac{e^{i2\pi(\nu-\omega)T}-1}{i2\pi(\nu-\omega)T}\right|\\ &= \left|\frac{\sin\pi(\nu-\omega)T}{\pi(\nu-\omega)T}\right|\\ &= \left|\operatorname{sinc}((\nu-\omega)T)\right|\end{aligned}$$

Reducing leackage

There are two strategies:

- Increase the window length.

$$|\phi_{e^{i2\pi\nu t},T}(\omega)| = |\operatorname{sinc}((\nu-\omega)T)| = \left|\frac{\sin\pi(\nu-\omega)T}{\pi(\nu-\omega)T}\right|$$



Reducing leackage

There are two strategies:

- Use a smoother window.
  We use Hanning's:

$$H_T^{n_h}(t) = q_{n_h}\left(1 - \cos\frac{2\pi t}{T}\right)^{n_h}.$$

being $q_{n_h} = n_h!/\big((2n_h - 1)!!\big)$.

The corresponding WFT is denoted by

$$\phi_{f,T}^{n_h}(\omega) := \mathcal{F}\big(H^{n_h} f\big)(\omega) = \frac{1}{T}\int_0^T H_T^{n_h}(t) f(t) e^{-i2\pi\omega t} dt,$$

Reducing leackage

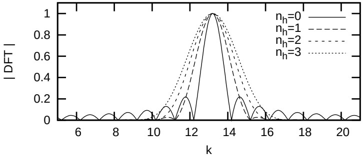
There are two strategies:

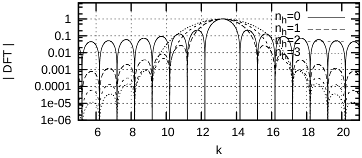
- Use a smoother window.

$$\phi_{e^{i2\pi\nu t},T}(\omega) = \frac{e^{i2\pi(\nu-\omega)T}-1}{i2\pi(\nu-\omega)T} = O\left(\frac{1}{(\nu-\omega)T}\right),$$

vs

$$\phi^{n_h}_{e^{i2\pi\nu t},T}(\omega) = \frac{(-1)^{n_h}(n_h!)^2\left(e^{i2\pi(\nu-\omega)T}-1\right)}{i2\pi\prod_{j=-n_h}^{n_h}\left((\nu-\omega)T+j\right)} = O\left(\frac{1}{\left((\nu-\omega)T\right)^{1+2n_h}}\right)$$

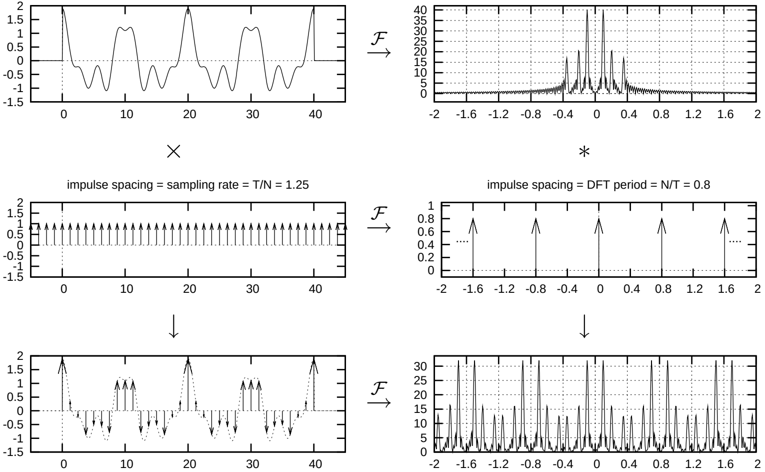




Discretization ⟶ DFT

Graphical development (E.O. Brigham, 1988)

$$T = 40, N = 32, f(t) = \cos(2\pi 0.1t) + 0.5\cos(2\pi 0.2t) + 0.4\cos(2\pi 0.35t)$$



Sampling $\longrightarrow$ DFT

Explicit formulae

- DFT of $\{f(j\frac{T}{N})\}_{j=0}^{N-1}$ defined as $\{F_{f,T,N}(k)\}_{k=0}^{N-1}$, being

$$\begin{aligned} F_{f,T,N}(k) &:= \frac{1}{N}\mathcal{F}\left(\sum_{j\in\mathbb{Z}} \chi_{[0,T]}\left(j\frac{T}{N}\right) f\left(j\frac{T}{N}\right)\delta_{j\frac{T}{N}}\right)\left(\frac{k}{T}\right) \\ &= \frac{1}{N}\sum_{j=0}^{N-1} f\left(j\frac{T}{N}\right) e^{-i2\pi kj/N}. \end{aligned}$$

- With Hanning's window:

$$F_{f,T,N}^{n_h}(k) = \frac{1}{N}\sum_{j=0}^{N-1} H_T^{n_h}\left(j\frac{T}{N}\right) f\left(j\frac{T}{N}\right) e^{-i2\pi kj/N}.$$

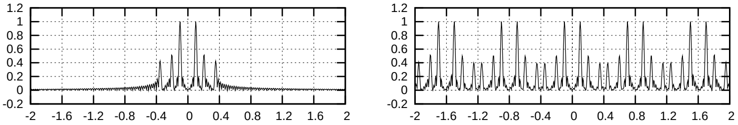
Sampling ⟶ DFT

Explicit formulae

- Relation with the WFT:

$$F_{f,T,N}(k) = \phi_{f,T,N}\left(\frac{k}{T}\right) + \underbrace{\sum_{l \in \mathbb{Z}\setminus\{0\}} \left(\phi_{f,T,N}\left(\frac{k+lN}{T}\right) + \phi_{f,T,N}\left(\frac{k-lN}{T}\right)\right)}_{\text{error term}}$$

- The **fundamental domain** of the DFT for real signals is $[0, T/(2N)]$. $T/(2N)$ is Nyquist's critical frequency.



Sampling $\longrightarrow$ DFT

Explicit formulae

- Relation with the WFT:

$$F_{f,T,N}(k) = \phi_{f,T,N}\left(\frac{k}{T}\right) + \underbrace{\sum_{l\in\mathbb{Z}\setminus\{0\}} \left(\phi_{f,T,N}\left(\frac{k+lN}{T}\right) + \phi_{f,T,N}\left(\frac{k-lN}{T}\right)\right)}_{\text{error term}}$$

- The **fundamental domain** of the DFT for real signals is $[0, T/(2N)]$. $T/(2N)$ is Nyquist's critical frequency.
- The error term above can produce **aliasing**:
  if a frequency of the signal is outside the fundamental domain of the DFT, we will detect an **alias** of it.
- Aliasing is avoided increasing N.

Outline

Algorithm

Parameters: T (time length), N (number of samples), n_h (Hanning index) $b_{\min}$ minimum threshold, several tolerances.

1. Set an starting **threshold** for collecting **peaks of the modulus of the DFT** of $f(t)$.
2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.
3. Find the **amplitudes** of the frequencies found in the previous step, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

Algorithm

Parameters: T (time length), N (number of samples), n_h (Hanning index) $b_{\min}$ minimum threshold, several tolerances.

1. Set an starting **threshold** for collecting **peaks of the modulus of the DFT** of $f(t)$.
2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.
3. Find the **amplitudes** of the frequencies found in the previous step, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

Algorithm

Parameters: T (time length), N (number of samples), n_h (Hanning index)
$b_{\min}$ minimum threshold, several tolerances.

1. Set an starting **threshold** for collecting **peaks of the modulus of the DFT** of $f(t)$.
2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.
3. Find the **amplitudes** of the frequencies found in the previous step, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

Algorithm

Parameters: T (time length), N (number of samples), n_h (Hanning index)
$b_{\min}$ minimum threshold, several tolerances.

1. Set an starting **threshold** for collecting **peaks of the modulus of the DFT** of $f(t)$.
2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.
3. Find the **amplitudes** of the frequencies found in the previous step, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

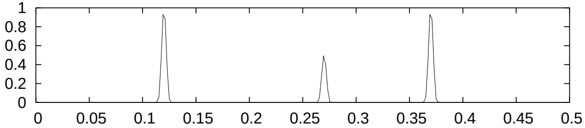
Algorithm

Parameters: T (time length), N (number of samples), n_h (Hanning index)
$b_{\min}$ minimum threshold, several tolerances.

1. Set an starting **threshold** for collecting **peaks of the modulus of the DFT** of $f(t)$.
2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.
3. Find the **amplitudes** of the frequencies found in the previous step, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f, by solving $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$.
5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

An illustration of the algorithm

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
1. Starting thresold: 0.8
modulus of the DFT of the input data:



$\Rightarrow$ peaks $j = 61, j = 189$.

An illustration of the algorithm

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.

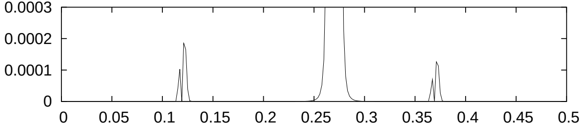
2. Approximation of frequencies:

$$\begin{aligned} \text{peak } 67 &\quad \Rightarrow \text{ frequency } 0.130859375 \\ \text{peak } 189 &\quad \Rightarrow \text{ frequency } 0.369140625 \end{aligned}$$

3. Computation of amplitudes from known frequencies:

Frequency	Cosine amplitude	Sine amplitude
0.369140625	0.702312716711	0.136800713691
0.130859375	0.137731069235	0.699288924190

modulus of the DFT of the residual



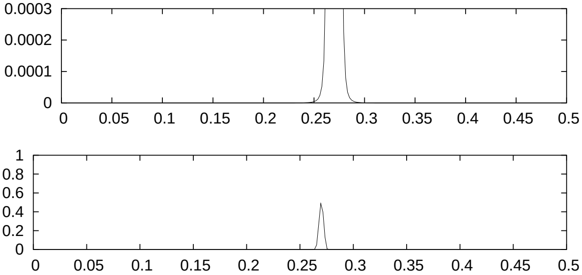
An illustration of the algorithm

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.

4. Iterative refinement:

Frequency	Cosine amplitude	Sine amplitude
0.369995932915	0.005462459021	1.000450861577
0.129998625183	0.999908805689	-0.002241420351

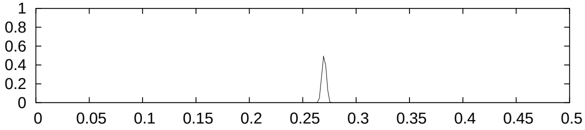
5. modulus of the DFT of input signal minus step 4:



New threshold: 0.2

An illustration of the algorithm

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
5. modulus of the DFT of input signal minus step 4:



New threshold: 0.2
2. Approximation of frequencies:

$$\text{peak } 138 \quad \Rightarrow \text{frequency } 0.26953125$$

3. Amplitudes from known frequencies:

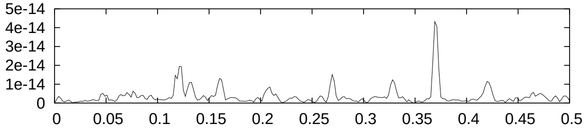
Frequency	Cosine amplitude	Sine amplitude
0.369995932915	0.005462459021	1.000450861577
0.129998625183	0.999908805689	-0.002241420352
0.269531250000	-0.309714556917	-0.330986794067

An illustration of the algorithm

For $f(t) = \cos(2\pi 0.13t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t)$,
$T = N = 512$, $n_h = 0$.
4. Iterative refinement:

Frequency	Cosine amplitude	Sine amplitude
0.3700000000000000	0.0000000000000009	1.0000000000000022
0.1300000000000000	0.9999999999999997	0.0000000000000010
0.2700000000000000	-0.0000000000000028	-0.4999999999999995

modulus of the DFT of the residual:



Computing amplitudes from known frequencies

We ask $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$, being

$$Q_f(t) = A_0^c + \sum_{l=1}^{N_f} \left(A_l^c \cos(2\pi \frac{\nu_l}{T} t) + A_l^s \sin(2\pi \frac{\nu_l}{T} t\right).$$

Since we work with **real** signals, we use the sine and cosine transforms:

$$\begin{aligned} c_{f,T,N}^{n_h}(k) &= \frac{2}{N} \sum_{j=0}^{N-1} f(j\tfrac{T}{N}) H_N^{n_h}(j) \cos\left(2\pi \tfrac{k}{N} j\right), \quad k = 0, ..., \tfrac{N}{2}, \\ s_{f,T,N}^{n_h}(k) &= \frac{2}{N} \sum_{j=0}^{N-1} f(j\tfrac{T}{N}) H_N^{n_h}(j) \sin\left(2\pi \tfrac{k}{N} j\right), \quad k = 1, ..., \tfrac{N}{2} - 1. \end{aligned}$$

They are realted to the DFT in complex form by

$$F_{f,T,N}^{n_h}(k) = \frac{1}{2}\left(c_{f,T,N}^{n_h}(k) - i s_{f,T,N}^{n_h}(k)\right), \qquad k = 0, \ldots, N/2.$$

Computing amplitudes from known frequencies

The system of equations to be solved is **linear** and $(1+2N_f)\times(1+2N_f)$:

$$\begin{aligned}
A_0^c c^{n_h}_{1,T,N}(0) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(0) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(0)\right) &= c^{n_h}_{f,T,N}(0)\\
A_0^c c^{n_h}_{1,T,N}(j) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(j) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(j)\right) &= c^{n_h}_{f,T,N}(j)\\
\sum_{l=1}^{N_f}\left(A_l^c \overline{s}^{n_h}_{\nu_l,T}(j) + A_l^c \widetilde{s}^{n_h}_{\nu_l,T}(j)\right) &= s^{n_h}_{f,T,N}(j)
\end{aligned}$$

where $j=[\nu_l+0.5]$, $l=1\div N_f$ (collocation harmonics), and

$$\begin{aligned}
c_1^{n_h}(j) &= c^{n_h}_{1,T,N}(j), & & \\
\overline{c}^{n_h}_{\nu_l,N}(j) &= c^{n_h}_{\cos(\frac{2\pi\nu_l}{T}),T,N}(j), & \overline{s}^{n_h}_{\nu_l,N}(j) &= s^{n_h}_{\cos(\frac{2\pi\nu_l}{T}),T,N}(j),\\
\widetilde{c}^{n_h}_{\nu_l,N}(j) &= c^{n_h}_{\sin(\frac{2\pi\nu_l}{T}),T,N}(j), & \widetilde{s}^{n_h}_{\nu_l,N}(j) &= s^{n_h}_{\sin(\frac{2\pi\nu_l}{T}),T,N}(j).
\end{aligned}$$

Simultaneous improvement of frequencies and amplitudes

We solve by **Newton's method** the following $(1+3N_f)\times(1+3N_f)$ **non–linear system**:

$$
\begin{aligned}
A_0^c c_{1,T,N}^{n_h}(0) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}_{\nu_l,N}^{n_h}(0) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(0)\right) &= c_{f,T,N}^{n_h}(0)\\
A_0^c c_{1,T,N}^{n_h}(j_i) + \sum_{l=1}^{N_f}\left(A_l^c \overline{c}_{\nu_l,N}^{n_h}(j_i) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(j_i)\right) &= c_{f,T,N}^{n_h}(j_i)\\
\sum_{l=1}^{N_f}\left(A_l^c \overline{s}_{\nu_l,N}^{n_h}(j_i) + A_l^s \widetilde{s}_{\nu_l,N}^{n_h}(j_i)\right) &= s_{f,T,N}^{n_h}(j_i)\\
A_0^c cs_{1,T,N}^{n_h}(j_i^+) + \sum_{l=1}^{N_f}\left(A_l^c \overline{cs}_{\nu_l,N}^{n_h}(j_i^+) + A_l^s \widetilde{cs}_{\nu_l,N}^{n_h}(j_i^+)\right) &= cs_{f,T,N}^{n_h}(j_i^+)
\end{aligned}
$$

being $j_i = [\nu_i + 0.5]$, $j_i^+ = [\nu_i] + 1 - (j_i^+ - [\nu_i])$.

Outline

Strategy

Let us denote

- f_{r_0}: the truncation of f to the frequencies we want to determine:

$$f_{r_0}(t) = A_0^c + \sum_{\substack{|\boldsymbol{k}|\le r_0-1\\ \langle \boldsymbol{k},\boldsymbol{\omega}\rangle>0}} \left(A_k^c \cos(2\pi\langle \boldsymbol{k},\boldsymbol{\omega}\rangle t) + A_k^s \sin(2\pi\langle \boldsymbol{k},\boldsymbol{\omega}\rangle t)\right).$$

- $y = (A_0, \nu_1, A_1^c, A_1^s, \dots, \nu_{N_f}, A_{N_f}^c, A_{N_f}^s)$: the **exact** frequencies and amplitudes.
- $y + \Delta y$: the **computed** frequencies and amplitudes.

The **system we solve** for iterative improvement of frequencies and amplitudes is

$$\underbrace{\mathrm{DFT}(Q_f)}_{g(y+\Delta y)} = \underbrace{\mathrm{DFT}(f_{r_0})}_{b} + \underbrace{\mathrm{DFT}(f - f_{r_0})}_{\Delta b}$$

We would get the **exact** frequencies and amplitudes **if** $\Delta b = 0$.

Strategy

- System for iterative improvement of frequencies and amplitudes:

$$\begin{aligned}
A_0^c + \sum_{l=1}^{N_f} \left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(0) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(0) \right) &= c^{n_h}_{f_{r_0},T,N}(0) + c^{n_h}_{f-f_{r_0},T,N}(0) \\
A_0^c c_1^{n_h}(j_i) + \sum_{l=1}^{N_f} \left(A_l^c \overline{c}^{n_h}_{\nu_l,N}(j_i) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(j_i) \right) &= c^{n_h}_{f_{r_0},T,N}(j_i) + c^{n_h}_{f-f_{r_0},T,N}(j_i) \\
\sum_{l=1}^{N_f} \left(A_l^c \overline{s}^{n_h}_{\nu_l,N}(j_i) + A_l^s \widetilde{s}^{n_h}_{\nu_l,N}(j_i) \right) &= s^{n_h}_{f_{r_0},T,N}(j_i) + s^{n_h}_{f-f_{r_0},T,N}(j_i) \\
A_0^c cs_1^{n_h}(j_i^+) + \sum_{l=1}^{N_f} \left(A_l^c \overline{cs}^{n_h}_{\nu_l,N}(j_i^+) + A_l^s \widetilde{cs}^{n_h}_{\nu_l,N}(j_i^+) \right) &= cs^{n_h}_{f_{r_0},T,N}(j_i^+) + cs^{n_h}_{f-f_{r_0},T,N}(j_i^+).
\end{aligned}$$

  where $f - f_{r_0} = \sum_{|\boldsymbol{k}|\geq r_0} a_{\boldsymbol{k}} e^{i2\pi\langle \boldsymbol{k},\boldsymbol{\omega}\rangle t}$.

- The error term Δb consists of DFT
  - of **periodic terms** with **frequencies not being computed**,
  - evaluated in **harmonics** corresponding to **frequencies being computed**.

  Therefore, the error term Δb can be considered **leakage of the remainder**, $f - f_{r_0}$.

Strategy

- The error term Δb can be considered **leakage of the remainder**

$$\mathrm{DFT}(f - f_{r_0}) = \sum_{|\boldsymbol{k}| \geq r_0} a_{\boldsymbol{k}} \, \mathrm{DFT}(e^{i2\pi \langle \boldsymbol{\omega}, \boldsymbol{k} \rangle t})$$

- The effect of the terms of the remainder on the error Δb is
  - The DFT of terms corresponding to **low–order frequencies**, $\{\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle\}_{|k| \gtrsim r_0}$, evaluated at the harmonics $\{j_i, j_i^+\}$, will be **small** if the harmonics $T\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle$ are far from $\{j_i, j_i^+\}$.
    This can be achieved by increasing T **as long as there is no aliasing**.
  - The DFT of terms corresponding to **high–order frequencies may not be small** ($T\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle$ can be made arbitrarily close to a j_i for large enough $|\boldsymbol{k}|$).
    However, the corresponding **amplitudes** will be small due to the Cauchy estimates

$$|a_{\boldsymbol{k}}| \leq C e^{-\delta |\boldsymbol{k}|} \quad \forall \boldsymbol{k} \in \mathbb{Z}^m,$$

so they will be harmless.

Bounding

- The **system we solve** for iterative improvement of frequencies and amplitudes is

$$\underbrace{\mathrm{DFT}(Q_f)}_{g(y+\Delta y)} = \underbrace{\mathrm{DFT}(f_{r_0})}_{b} + \underbrace{\mathrm{DFT}(f - f_{r_0})}_{\Delta b}$$

  We would get the **exact** frequencies and amplitudes **if** $\Delta b = 0$.
- The **error in frequencies and amplitudes** is given, at first order, by

$$\|\Delta y\|_\infty \leq \|Dg(y)^{-1}\|_\infty \|\Delta b\|_\infty.$$

- **Bounds can be obtained** for $\|Dg(y)^{-1}\|_\infty$ and $\|\Delta b\|$.
- **Main idea**: instead of the DFT,
  - bound the WFT, and
  - the difference WFT − DFT.

Bound for $\|Dg(y)^{-1}\|_\infty$

We can write

$$Dg(y) =: M = \begin{pmatrix} 2 & B_{0,1} & \dots & B_{0,N_f} \\ 0 & B_{1,1} & \dots & B_{1,N_f} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{N_f,1} & \dots & B_{N_f,N_f} \end{pmatrix}.$$

We **split** $M = M_D + M_O$,

$$M = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & B_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{N_f,N_f} \end{pmatrix} + \begin{pmatrix} 0 & B_{0,1} & \dots & B_{0,N_f} \\ 0 & 0 & \dots & B_{1,N_f} \\ 0 & \vdots & \ddots & \vdots \\ 0 & B_{N_f,1} & \dots & 0 \end{pmatrix}.$$

M is **close to block-diagonal**, so the idea is to obtain **bounds for** $\|M_D^{-1}\|$, $\|M_O\|$ and use

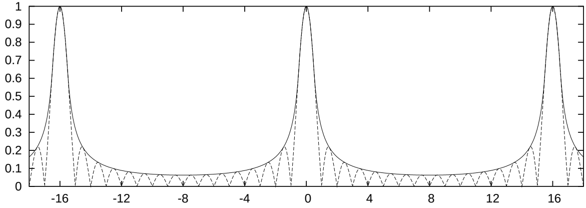
$$\|(M_D + M_O)^{-1}\| \leq \frac{\|M_D^{-1}\|}{1 - \|M_D^{-1}\|\|M_O\|}.$$

Bound for $\|\Delta b\|_\infty$

We have

$$\|\Delta b\| \leq 2C \max_{j \in J} \sum_{|\boldsymbol{k}|=r_0}^{\infty} e^{-\delta|\boldsymbol{k}|} |\widetilde{h}_N^{n_h}(T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle - j)|$$

where $|\widetilde{h}_N^{n_h}|$ is the envelope displayed below ($N = 16$, $n_h = 0$).



Bound for $\|\Delta b\|_\infty$

We have

$$\|\Delta b\| \leq 2C \max_{j\in J} \sum_{|\boldsymbol{k}|=r_0}^{\infty} e^{-\delta|\boldsymbol{k}|} |\widetilde{h}_N^{n_h}(T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle - j)|$$

The **Diophantine condition** gives a **lower bound for** $|T\langle \boldsymbol{k}, \omega\rangle - j|$:

$$|T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle - j| \quad \geq \quad \frac{TD}{(|\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle| + |\boldsymbol{k}_j|)^{\tau}} - 1.$$

For $|\boldsymbol{k}|$ **small**, $|\widetilde{h}_N^{n_h}(T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle - j)| \ll 1$.
After some order r_*, $|\widetilde{h}_N^{n_h}(T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle - j)|$ may approach 1.
Therefore,

$$\|\Delta b\| \leq 2C\left(\max_{j\in J} \sum_{|k|=r_0}^{r_*-1} e^{-\delta|k|} |\widetilde{h}_N^{n_h}(T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle - j)| + \max_{j\in J} \sum_{|k|=r_*}^{\infty} e^{-\delta|k|}\right).$$

Bound for $\|\Delta b\|_\infty$

In

$$\|\Delta b\| \leq 2C\left(\max_{j\in J}\sum_{|\boldsymbol{k}|=r_0}^{r_*-1} e^{-\delta|\boldsymbol{k}|}|\widetilde{h}_N^{n_h}(T\langle \boldsymbol{k},\boldsymbol{\omega}\rangle - j)| + \max_{j\in J}\sum_{|\boldsymbol{k}|=r_*}^{\infty} e^{-\delta|\boldsymbol{k}|}\right),$$

- The first term is bounded by **replacing the DFT by the WFT**. This introduces an additional error term due to this approximation.
- All the sums are reduced to **sums of the form** $\sum_j j^\alpha e^{-\delta j}$, which are bounded by **incomplete Gamma functions**.

Explicit bounds

Hypotheses:

1. Assume $f(t) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} a_{\boldsymbol{k}} e^{i2\pi\langle \boldsymbol{k},\boldsymbol{\omega}\rangle t}$,
   Cauchy estimates: $|a_{\boldsymbol{k}}| \leq Ce^{-\delta|\boldsymbol{k}|}$,
   $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ rac ind.,
   Diophantine condition $|\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle| > D/|k|^\tau$.

2. Apply the numerical Fourier analysis procedure with T, N, n_h with minimum "amplitude barrier" $b_{\min}$.
   $\longrightarrow$ approximations $\widetilde{A}_0, \{(\widetilde{\nu}_k, \widetilde{A}^c_k, \widetilde{A}^s_k)\}_{k=1}^{N_f}$
   (denote by $A_0, \{(\nu_k, A^c_k, A^s_k)\}_{k=1}^{N_f}$ the exact values)

3. Assume $\{T\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle\}_{|\boldsymbol{k}|=1}^{r_0} \subset \{\nu_k\}_{k=1}^{N_f}$, for some order r_0,

4. T, N satisfy some technical (non–demanding) lower bounds.

Explicit bounds

Then the error can be bounded in first–order as:

$$\|\Delta y\| \leq \|M^{-1}\| \|\Delta b\|,$$

with

- $\|M^{-1}\| \leq \dfrac{G_{n_h}}{\min(1, A_{\min})}$ + small terms

n_h	0	1	2	3
G_{n_h}	4.84	8.83	13.3	17.7

- $\|\Delta b\| \leq \underbrace{\dfrac{C_1(n_h, m, C, \delta, D, \tau, r_0, r_*)}{T^{1+2n_h}}}_{\text{leakage from orders } r_0, \ldots, r_*} + \underbrace{\dfrac{C_2(n_h, m, C, \delta, D, \tau, r_0, r_*)}{(D_a^*)^{1+2n_h}}}_{\text{“aliasing” from orders } r_0, \ldots, r_*}$

$$+ \underbrace{\text{tail}(n_h, m, C, \delta, D, \tau, r_*))}_{\text{harmless amplitudes}}$$

where $D_a^* := N - T(r_0 + r_* - 2)\|\omega\|_\infty - 1$
is related to the distance of frequencies up to order r_* to the right end of the fundamental domain of the DFT.

Rules of Thumb for high accuracy

1. Choose T such that the closest frequencies we want to determine are several harmonics away.

2. Choose N such that the largest frequency we want to determine is away from the right end of the fundamental domain of the DFT.

3. Take $n_h = 2$.

Rules of Thumb for high accuracy

1. Choose T such that the closest frequencies we want to determine are several harmonics away.

2. Choose N such that the largest frequency we want to determine is away from the right end of the fundamental domain of the DFT.

3. Take $n_h = 2$.

Rules of Thumb for high accuracy

1. Choose T such that the closest frequencies we want to determine are several harmonics away.

2. Choose N such that the largest frequency we want to determine is away from the right end of the fundamental domain of the DFT.

3. Take $n_h = 2$.

Outline

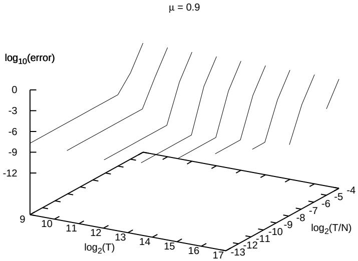
Accuracy test

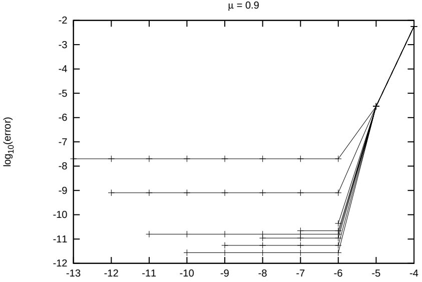
We consider the **quasi–periodic function** ($\omega = (1, \sqrt{2})$, $\varphi = (0.2, 0.3)$)

$$f_\mu(t) = \frac{\sin(2\pi\omega_1 t + \varphi_1)}{1 - \mu\cos(2\pi\omega_1 t + \varphi_1)} \cdot \frac{\sin(2\pi\omega_2 t + \varphi_2)}{1 - \mu\cos(2\pi\omega_2 t + \varphi_2)}, \quad \mu = 0.9.$$

Explicit formulae for frequencies and amplitudes can be obtained, as well as the **Cauchy estimates** and the **Diophantine condition**.

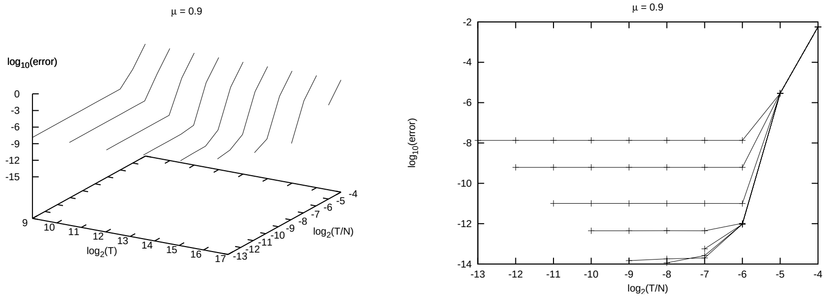
We have performed **Fourier analysis** of this function for **several** T, N, computing the first 20 frequencies ($|k| \le 5$).





Accuracy test

Error in amplitudes only:



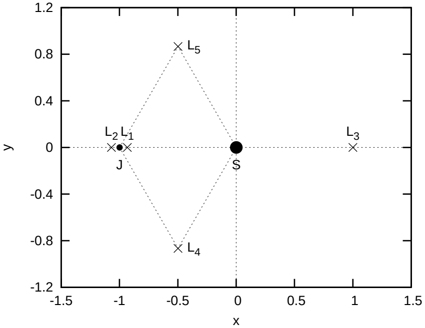
For these functions, the Cauchy estimates are equalitites:

$$f_\mu(t) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} a_{\boldsymbol{k}} e^{i2\pi\langle \boldsymbol{k},\boldsymbol{\omega}\rangle t}, \quad m = 2, \quad |a_{\boldsymbol{k}}| = \frac{1}{\mu^2} c^{|\boldsymbol{k}|} = 1.23 \cdot (0.627)^{|\boldsymbol{k}|}$$

For $|\boldsymbol{k}| = 6$, $|a_{\boldsymbol{k}}| = 6.06 \times 10^{-2}$, but we get nearly full double–precision accuracy in frequencies and amplitudes.

Outline

The circular, planar RTBP



Equation of motion:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \partial_x \Omega(x,y), \\ \ddot{y} + 2\dot{x} &= \partial_y \Omega(x,y), \end{aligned}$$

where

$$\begin{aligned} r_1 &= \sqrt{(x-\mu)^2 + y^2}, \\ r_2 &= \sqrt{(x-\mu+1)^2 + y^2}. \end{aligned}$$

$$\Omega(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu).$$

Mass parameter: $\mu = \dfrac{m_1}{m_1+m_2}$.

Data for the Sun–Jupiter case

- Sun–Jupiter mass parameter:

$$\mu_{SJ} = 1/1048.3486 \quad = \quad 9.5388\,118 \times 10^{-4}$$

- L_5 is center $\times$ center: $\quad \mathrm{Spec}\, \boldsymbol{Df}(L_5) = \{\omega_{\text{long}}^{L_5}, \omega_{\text{short}}^{L_5}\}$,

$$\omega_{\text{long}}^{L_5} \quad = \quad \left(\frac{1-\sqrt{1-27\mu(1-\mu)}}{2}\right)^{1/2} \quad = \quad 0.08046412,$$

$$\omega_{\text{short}}^{L_5} \quad = \quad \left(\frac{1+\sqrt{1-27\mu(1-\mu)}}{2}\right)^{1/2} \quad = \quad 0.99675750.$$

Data for the Sun–Jupiter case

- Sun–Jupiter mass parameter:

$$\mu_{SJ} = 1/1048.3486 \quad = \quad 9.5388\,118 \times 10^{-4}$$

- L_5 is center × center: $\quad \mathrm{Spec}\, Df(L_5) = \{\omega^{L_5}_{\mathrm{long}}, \omega^{L_5}_{\mathrm{short}}\}$,

$$\omega^{L_5}_{\mathrm{long}} = 0.08046412, \qquad \omega^{L_5}_{\mathrm{short}} = 0.99675750.$$

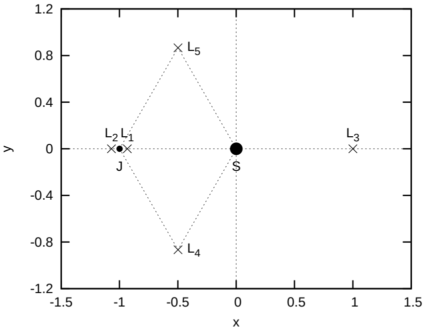
- We'll work with frequencies in cycles per unit of synodic time:

$$\begin{aligned} \nu^{L_5}_{\mathrm{short}} &= \omega^{L_5}_{\mathrm{short}}/(2\pi) = 0.01280626, \\ \nu^{L_5}_{\mathrm{long}} &= \omega^{L_5}_{\mathrm{long}}/(2\pi) = 0.15863888, \end{aligned}$$

- NOTE: $\nu^{L_5}_{\mathrm{short}}/\nu^{L_5}_{\mathrm{long}} = 12.3876.$

The stability domain

Numerical computation (G. Gómez, À. Jorba, J.J. Masdemont, C. Simó, ESA report 1993)



Parametrize the neighborhood of L_5 by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu \\ 0 \end{pmatrix} + (1+\rho) \begin{pmatrix} \cos(2\pi\alpha) \\ \sin(2\pi\alpha) \end{pmatrix}$$

For a grid of values of α, ρ, take i.c.

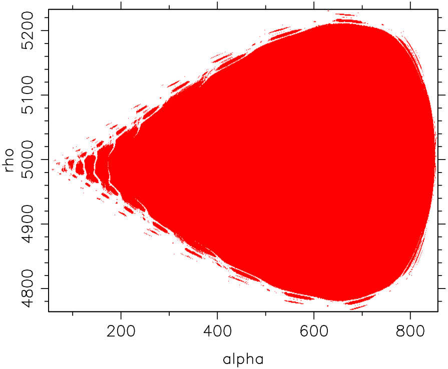
$$\begin{aligned} x_0 &= \mu + (1+\rho)\cos(2\pi\alpha), \\ y_0 &= (1+\rho)\sin(2\pi\alpha), \\ \dot{x}_0 &= \dot{y}_0 = 0. \end{aligned}$$

Try to integrate up to time $T_{\max}$, satisfying:

- Projection on (x, y) not encircling the main primary.
- Not too close aproaches to primaries.
- $y > y_c = -0.5$.

The stability domain

Refinement (C. Simó, 2006, 2008)

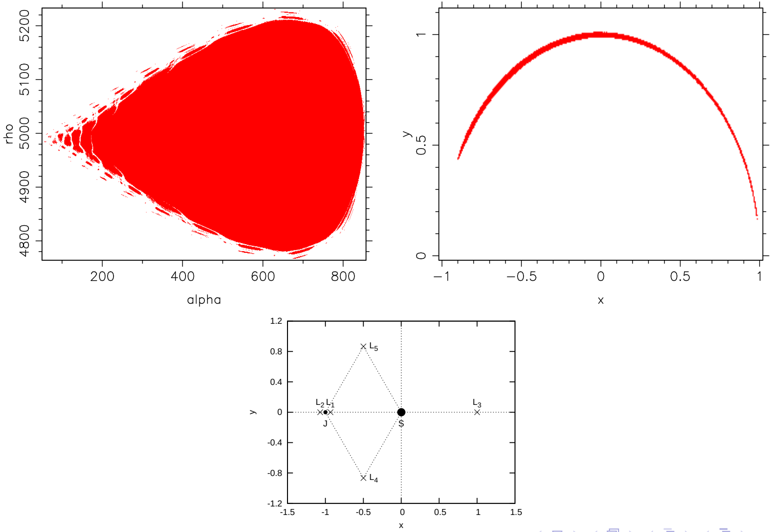


- First run: up to $T_{\max} = 2^{20}(2\pi)$.
  Subsisting points: 215673.
- Second run: try the previous points up to $T_{\max} = 2^{24}(2\pi)$.
  Not all points are tested, but:
  - From the border to the inside.
  - Stop testing when 5 consecutive points stay for 2^{24} Jupiter revolutions.

  Subsisting points: 215115.

Note: This is **not** the phase portrait on an area-preserving map. The initial conditions correspond to different energy levels.
Goal: to relate the frontier of the domain of stability and the island structure to resonances.

The stability domain



Fourier exploration

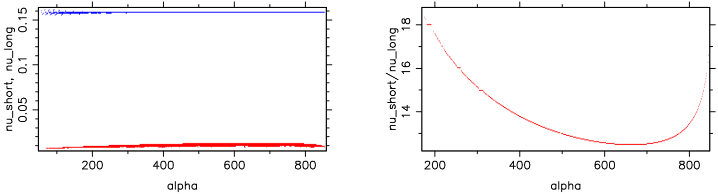
- The Fourier analysis procedure has been applied to each of the subsisting points, with

$$T = 65536,\ N = 262144,\ n_h = 2,\ N_{\max} = 100,\ b_{\min} = 10^{-6}$$

- Total computing time: 352.52 hours (using 28 processors: 12.59 hours)
- Statistics:

status	#analyses	
OK	205 779	95.41%
frequencies too close	8 722	4.04%
refinement did not converge	878	0.41%
the two of the above	294	0.14%
TOTAL	215 673	100%

Basic frequencies



- Left:
  - Blue: freq. of maximum amplitude. It is close to $\nu_{\text{long}}^{L_5}$
    $\longrightarrow \nu_{\text{long}}$
  - Red: frequency of maximum amplitude inside $[0.155, 0.165]$. It is close to $\nu_{\text{short}}^{L_5}$
    $\longrightarrow \nu_{\text{short}}$
- Right: the quotient $\nu_{\text{short}}/\nu_{\text{long}}$ for $\rho = 4950$.

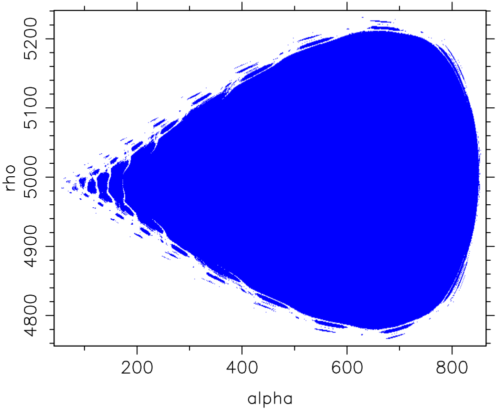
Results

A basic set has been extracted from each set of frequencies, and all frequencies have been written as linear combinations of the basic set. This allows to classify all the points in 4 groups:

1. Analyses ending with an error code.
   9894 (4.54%)
2. Error in determination of linear combinations $\geq 10^{-10}$.
   20416 (9.47%)
3. ν_{short} is not a rational multiple of ν_{long}.
   170389 (79.09%)
4. ν_{short} is a rational multiple of ν_{long}.
   14914 (6.91%)

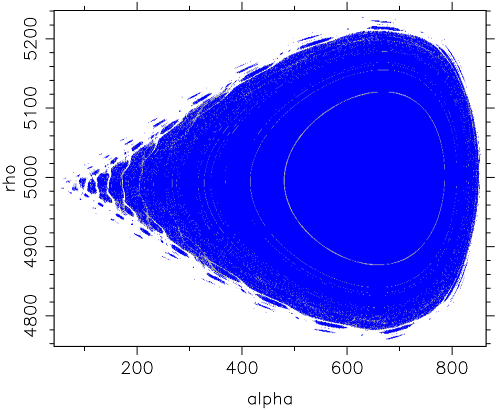
1 + 2 : diffusing (chaotic) orbits.
3 : regular, non–resonant motion.
4 : regular, resonant motion.

Graphical representation



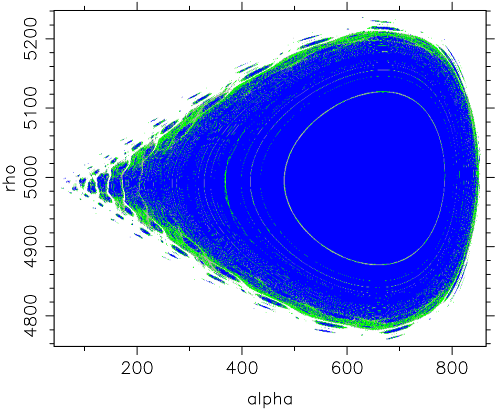
- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Graphical representation



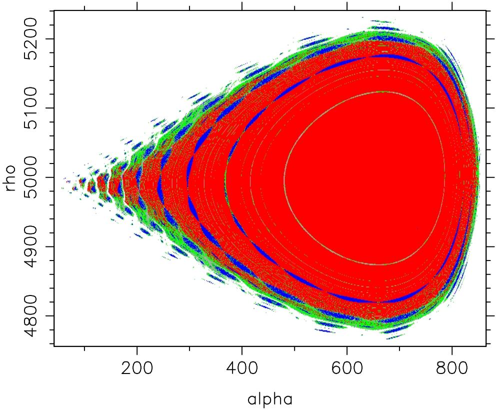
- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Graphical representation



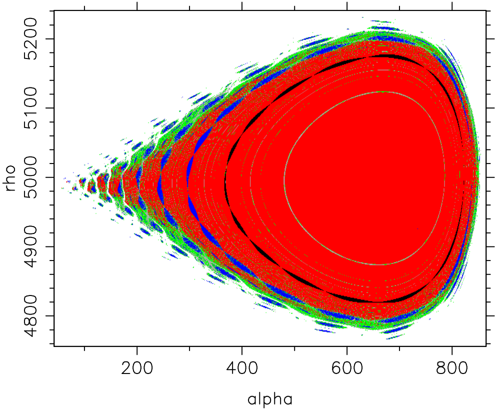
- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

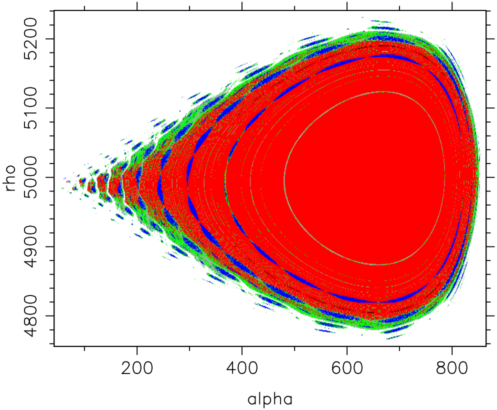
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1

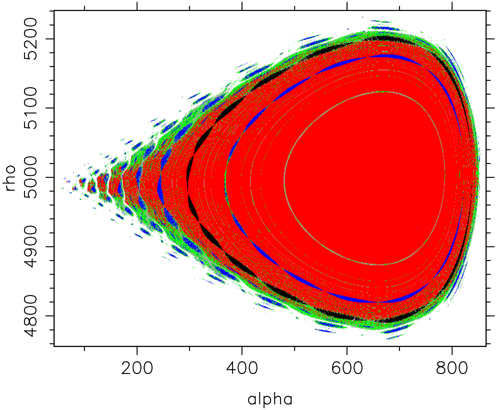
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2

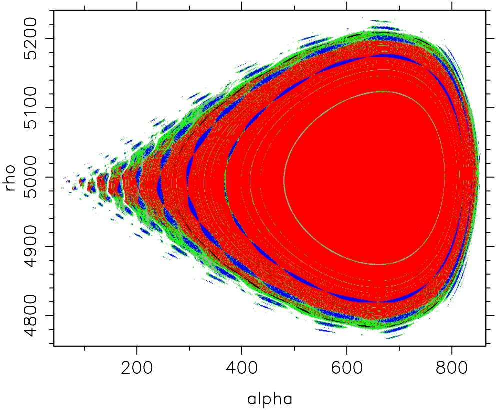
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1

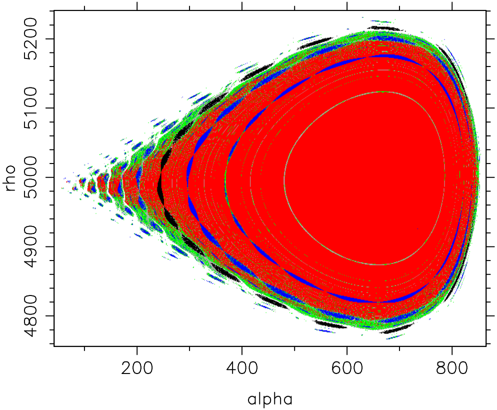
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2

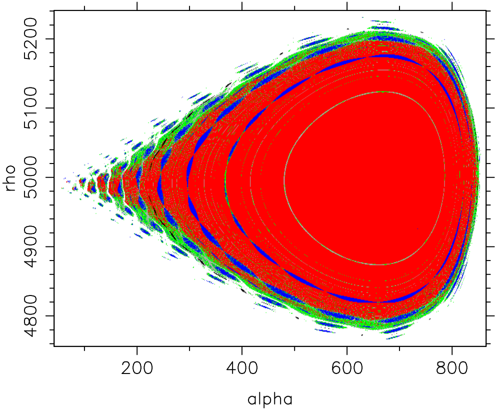
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1

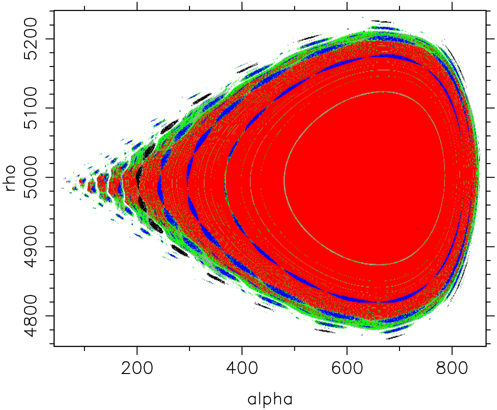
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2

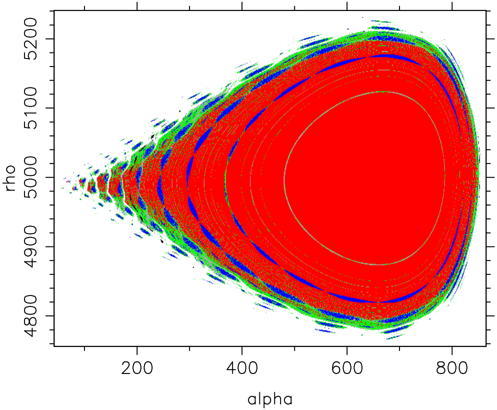
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1

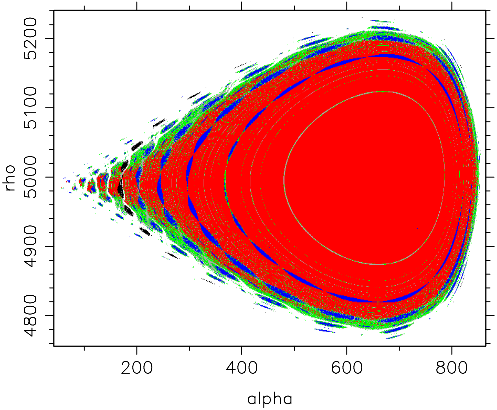
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2

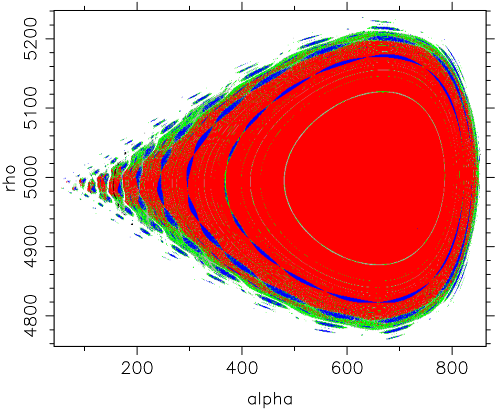
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1

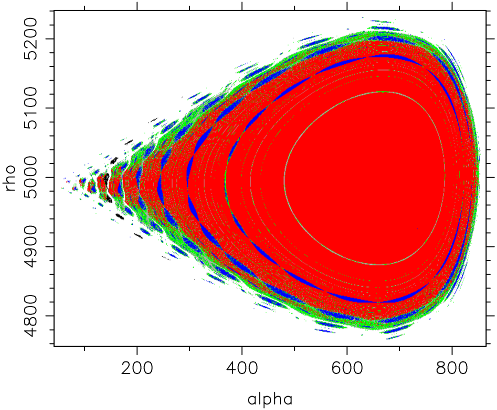
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2

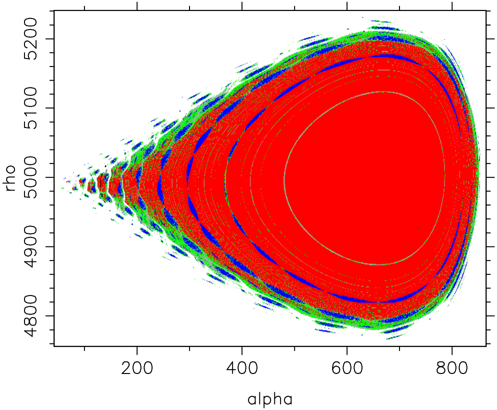
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1

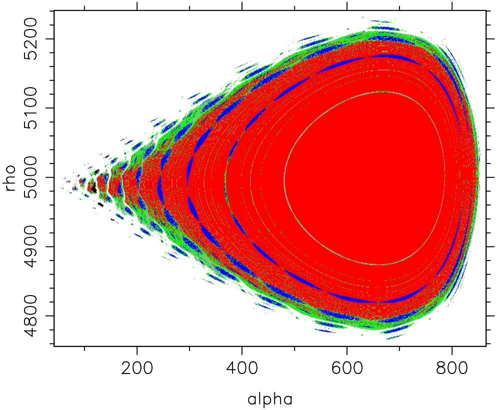
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2

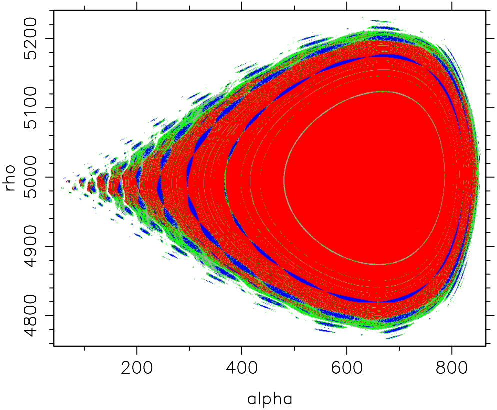
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2, 20:1

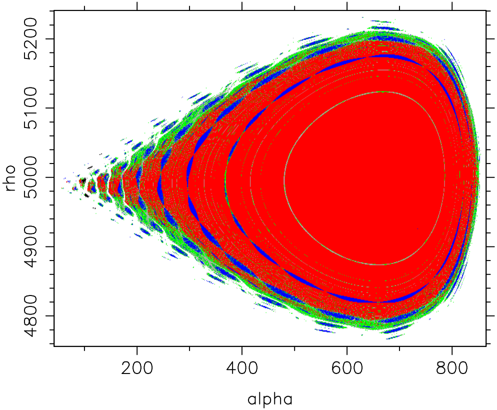
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2, 20:1, 41:2

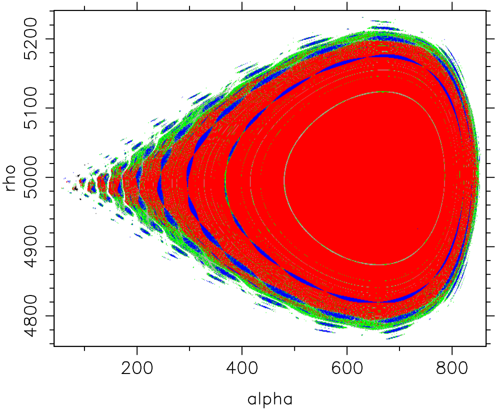
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2, 20:1, 41:2, 21:1

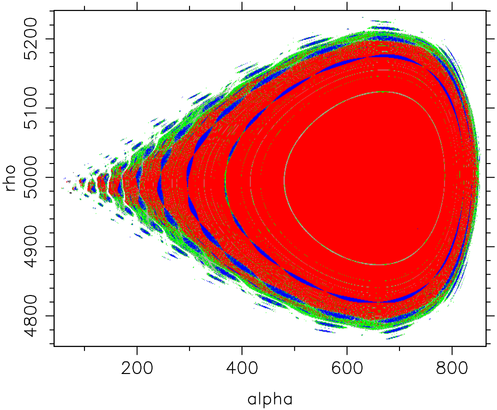
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2, 20:1, 41:2, 21:1, 22:1

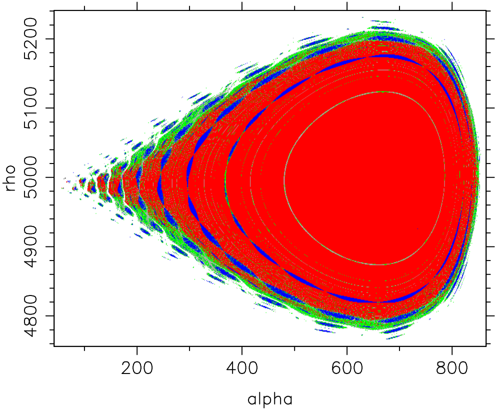
Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2, 20:1, 41:2, 21:1, 22:1, 23:1

Graphical representation



- Blue:
  all the analyses
- Dark gray:
  ended with error code
- Green:
  error $> 10^{-10}$ in determination of linear combinations
- Red:
  ν_{short} not resonant with ν_{long}

Resonances: 14:1, 29:2, 15:1, 31:2, 16:1, 33:2, 17:1, 35:2, 18:1, 37:2, 19:1, 39:2, 20:1, 41:2, 21:1, 22:1, 23:1, 24:1

& that's it

Thank you!!