# Arnold's mechanism of diffusion in the spatial circular Restricted Three Body Problem

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# Outline

# Spatial Circular RTBP

- Two primaries of masses $\mu$, $1 - \mu$ rotate on circles about their common center of mass.
- Sun-Earth system $\mu \approx 3.04 \times 10^{-6}$.
- Infinitesimal particle moves in space under the gravitational influence of primaries.

# Equations of Motion

- Rotating system of coordinates $(x, y, z)$
  $$\ddot{x} = 2\dot{y} + \frac{\partial \omega}{\partial x},$$
  $$\ddot{y} = -2\dot{x} + \frac{\partial \omega}{\partial y},$$
  $$\ddot{z} = \frac{\partial \omega}{\partial z}.$$
- Effective potential: $\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$.
- Energy function
  $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z)$
- Jacobi integral: $C(x, y, z, \dot{x}, \dot{y}, \dot{z}) = -2H(x, y, z, \dot{x}, \dot{y}, \dot{z})$

# Equilibrium Points

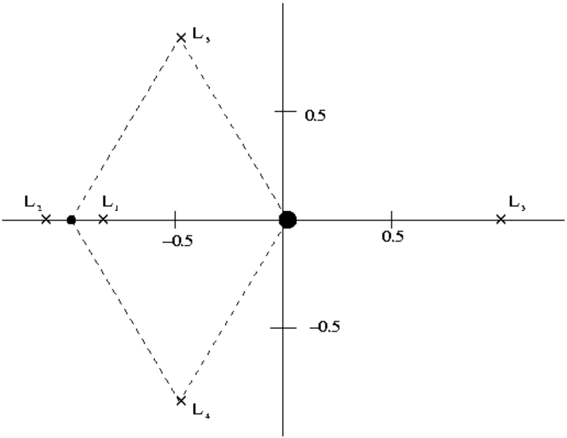



Figure: The five equilibrium points of the RTBP.

# Invariant Manifolds

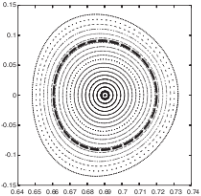
- $L \in \{L_1, L_2, L_3\}$ is center $\times$ center $\times$ saddle.
- Center manifold about $L$.
- Energy manifold $M_c = \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \colon C = c\}$.
- Restriction of center manifold to $M_c$ is a normally hyperbolic invariant manifold $\tilde{\Lambda}$ (3D).
- Stable/unstable invariant manifolds $W^s(\tilde{\Lambda})$, $W^u(\tilde{\Lambda})$ (4D).
- Typically, $W^s(\tilde{\Lambda}) \pitchfork W^u(\tilde{\Lambda})$ along a homoclinic manifold.

# Description of the Problem

- For $C \simeq C_L$, $\tilde{\Lambda}$ is filled with many invariant 2D tori $\mathcal{T}$.
- Normal form on $\tilde{\Lambda}$: action-angle coordinates $(I, J, \phi, \psi)$,
  - $I =$ out-of-plane amplitude,
  - $J =$ in-plane amplitude (implicit from energy condition).
- Arnold's transition chain of invariant tori?

$$\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n: \quad W^u(\mathcal{T}_i) \pitchfork W^s(\mathcal{T}_{i+1}) \quad \forall i.$$

- Shadowing trajectory?
- Symbolic dynamics?



# Main Result

## Theorem (semi-numerical)

- *Given* $0 < I < I' < I_{\max}$ *and* $\epsilon > 0$, *there exists a trajectory along which the action changes from* $\epsilon$*-close to* $I$ *to* $\epsilon$*-close to* $I'$*.*
- *There exist 'chaotic' trajectories, which visit some given level sets of* $I$ *in any prescribed order.*

# Idea of Proof

- $\tilde{\Lambda}$ is a 3D NHIM for the flow $\Phi_t$ with
  - inner dynamics $\Phi_t|_{\tilde{\Lambda}} \colon \tilde{\Lambda} \to \tilde{\Lambda}$,
  - outer dynamics $\tilde{S} \colon \tilde{\Lambda} \to \tilde{\Lambda}$.
- Fix a suitable Poincaré surface $\Sigma$ with first return map $F$.
- Let $\Lambda = \tilde{\Lambda} \cap \Sigma$, a 2D NHIM for the map $F$ with
  - inner dynamics $T = F|_{\Lambda} \colon \Lambda \to \Lambda$,
  - outer dynamics $S \colon \Lambda \to \Lambda$.

## Lemma

*If $\exists$ windows $R_i \in \Lambda$ well aligned under successive iterates of $T$ and $S$, then $\exists$ a true orbit passing close to the windows.*

- Find well aligned windows with increasing action $I \implies$ $\exists$ true orbit of $F$ (hence of $\Phi_t$) with increasing action $I$.

# Local Approximation of Dynamics

- High-order truncated normal form around $L$:

$$H = H_N(x_1 y_1, I = \frac{x_2^2 + y_2^2}{2}, J = \frac{x_3^2 + y_3^2}{2}) + R_{N+1}.$$

- Implementation based on Lie series method [Jorba 1997], can be parallelized.
- Alternatively, use a partial normal form.
- $x_1 = y_1 = 0$: Center manifold.
- Equations of motion on center manifold

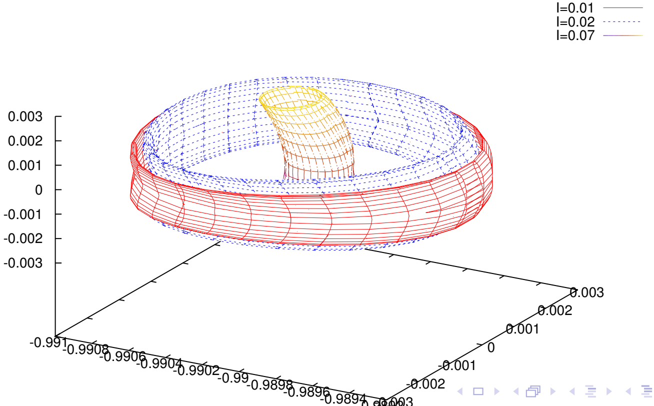
$$\begin{aligned} \dot{I} &= 0, & \dot{\phi} &= \omega(I, J) \\ \dot{J} &= 0, & \dot{\psi} &= \nu(I, J). \end{aligned}$$

# Normally Hyperbolic Invariant Manifold

- $x_1 = y_1 = 0$, energy condition $C = 3.00087$

$$I(t) = I_0, \qquad \phi(t) = \omega t + \phi_0,$$
$$\psi(t) = \nu t + \psi_0.$$

- $\tilde{\Lambda}$ is a family of invariant 2D tori: $\bigcup_{I \in (0,\ 0.072)} \mathcal{T}(I)$.

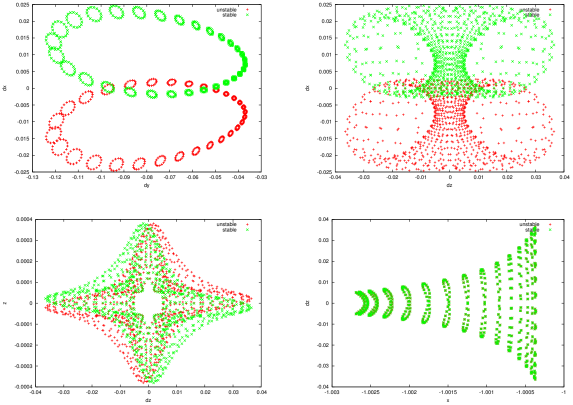


# Stable and Unstable Manifolds

- $x_1 = 0$: Local stable invariant manifold $W^s_{\mathrm{loc}}(\tilde{\Lambda})$.
- $y_1 = 0$: Local unstable invariant manifold $W^u_{\mathrm{loc}}(\tilde{\Lambda})$.
- $x_1 = 0, \quad x_+ = (I, \phi, \psi) \in \tilde{\Lambda}$:
  Local stable preserved foliation $W^s_{\mathrm{loc}}(x_+)$.
- $x_2 = 0, \quad x_- = (I, \phi, \psi) \in \tilde{\Lambda}$:
  Local unstable preserved foliation $W^u_{\mathrm{loc}}(x_-)$.
- Use normal form inside $10^{-5}$-neighborhood of $\tilde{\Lambda}$.
- Use numerical integration outside $10^{-5}$-neighborhood (local error $10^{-14}$).

# Homoclinic Manifold

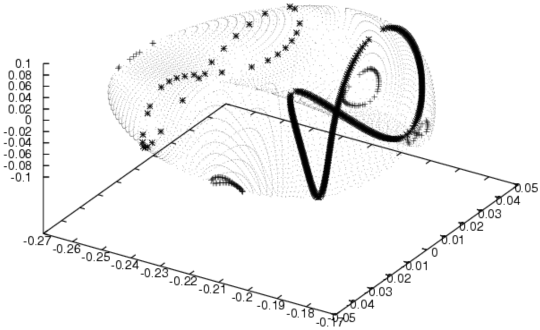
- We follow [Masdemont 2005].
- Integrate st/unst manifolds of tori $W^s(\mathcal{T}_+), W^u(\mathcal{T}_-)$ up to surface of section $\{y = 0\}$.




- Find 'common' points within margin of error $(10^{-9}) \longrightarrow$ 'First cut' homoclinics.

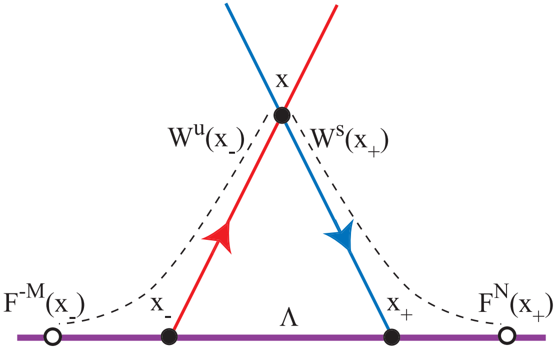
# Homoclinic Manifold

- Repeat varying $\mathcal{T}_-, \mathcal{T}_+$ (parallel computation).



# Scattering Map

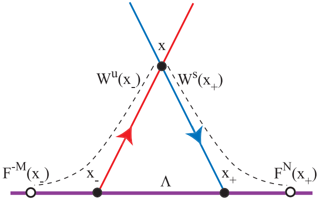
- Introduced by [A. García], [Delshams, de la Llave & Seara].
- $S\colon \Lambda \to \Lambda, \qquad S(x_-) = x_+.$



# Computation of Scattering Map

- Follow [Delshams, Masdemont & Roldán 2007].
- For any point in the intersection, record initial conditions $x_s, x_u \in \tilde{\Lambda}$ and integration times $t_s, t_u$.
- Integrate $x_u$ forward in $\tilde{\Lambda}$ for the time $t_u \longrightarrow x_-$.
  Integrate $x_s$ backwards in $\tilde{\Lambda}$ for the time $t_s \longrightarrow x_+$.
- Scattering map:

$$x_- \xrightarrow{\tilde{S}} x_+ .$$



# Reduced Model

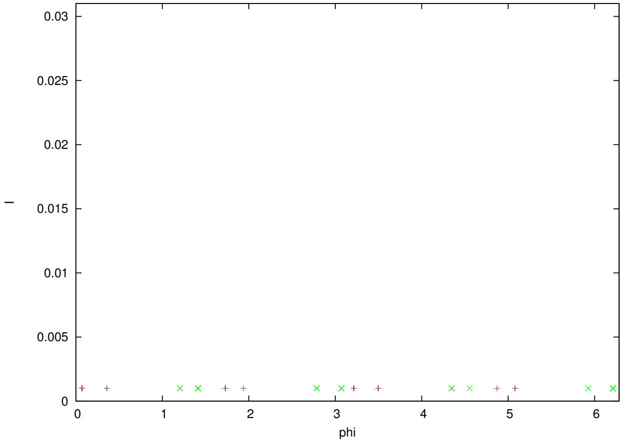
- $\Lambda = \tilde{\Lambda} \cap \Sigma$.
- The dynamics associated to $\tilde{\Lambda}$ for the flow

$$\Phi_t|_{\tilde{\Lambda}} : \tilde{\Lambda} \to \tilde{\Lambda},$$
$$\tilde{S} : \tilde{\Lambda} \to \tilde{\Lambda}$$

  induce dynamics associated to $\Lambda$ for the map:

$$T = F|_{\Lambda} : \Lambda \to \Lambda \qquad \text{twist map} \quad \text{(2D)},$$
$$S : \Lambda \to \Lambda \qquad \text{scattering map} \quad \text{(2D)}.$$

# Effect of Scattering Map on Action Level Sets




- 8 homoclinic orbits → 8 local scattering maps continued to 2 maximal scattering maps.
- Diffusion is non-uniform in *I*.

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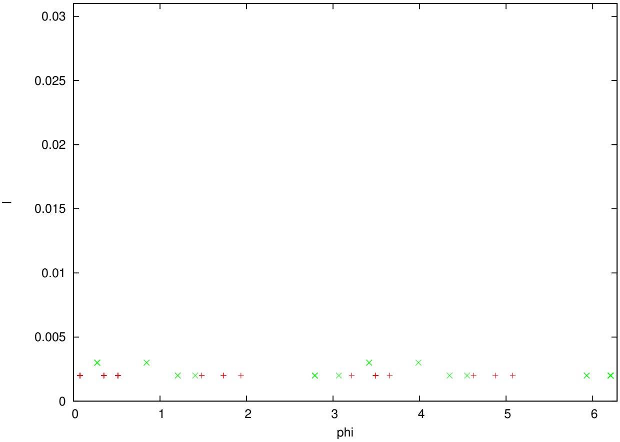


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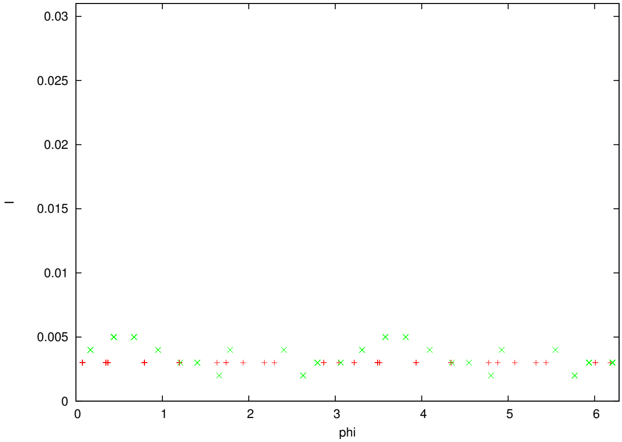


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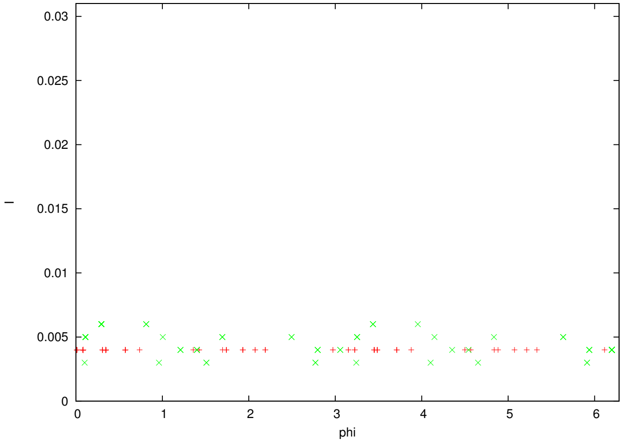


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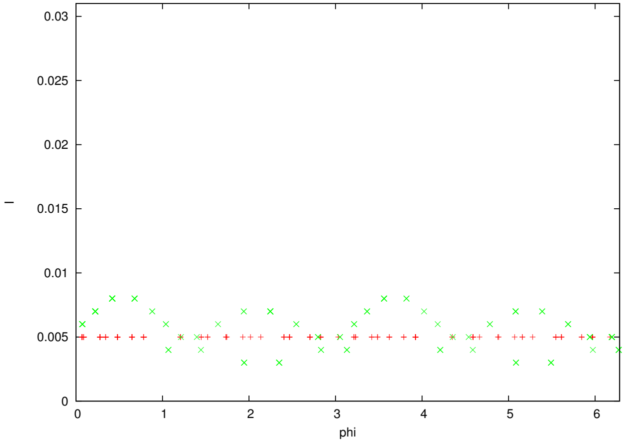


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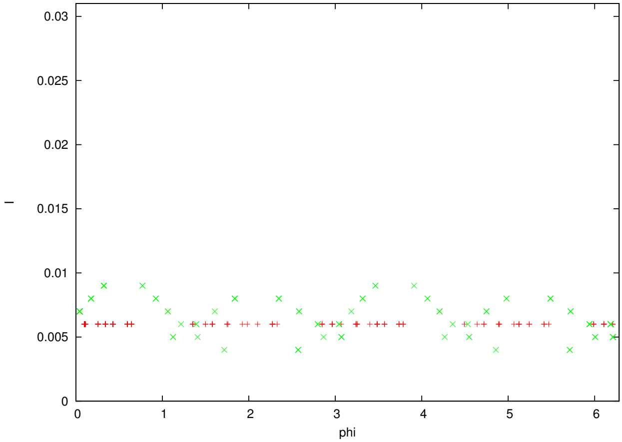


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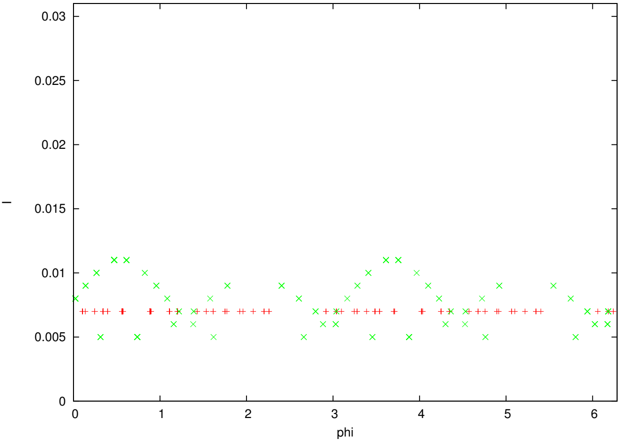


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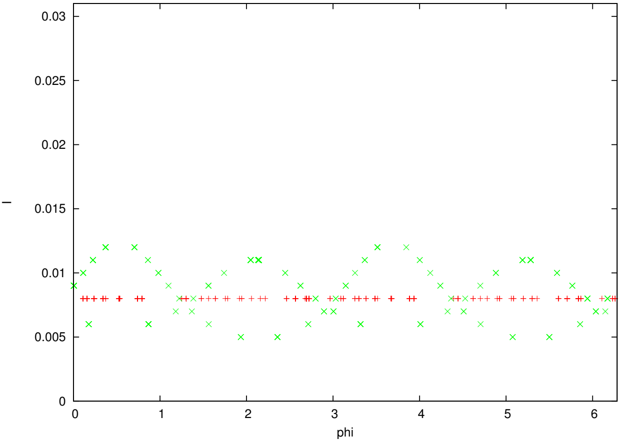


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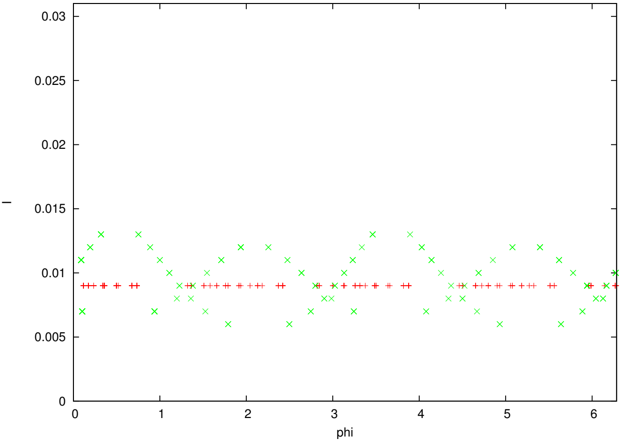


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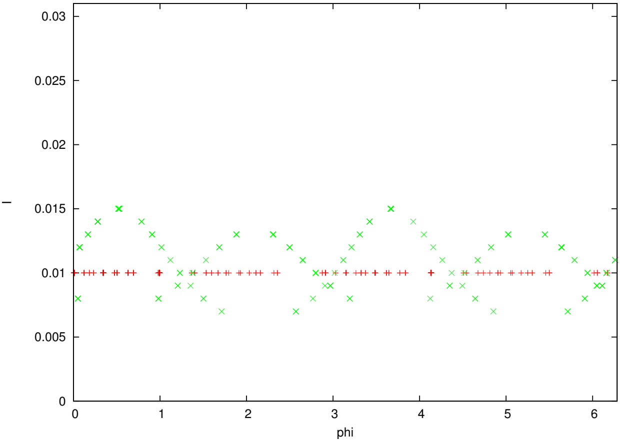


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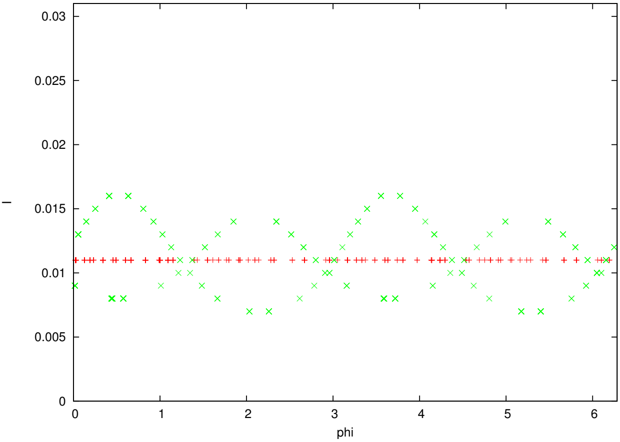


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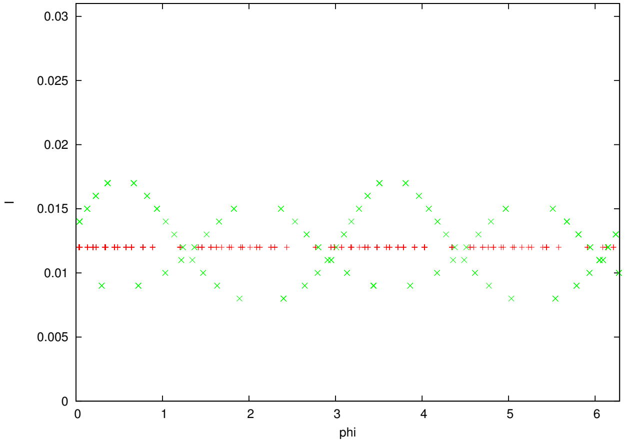


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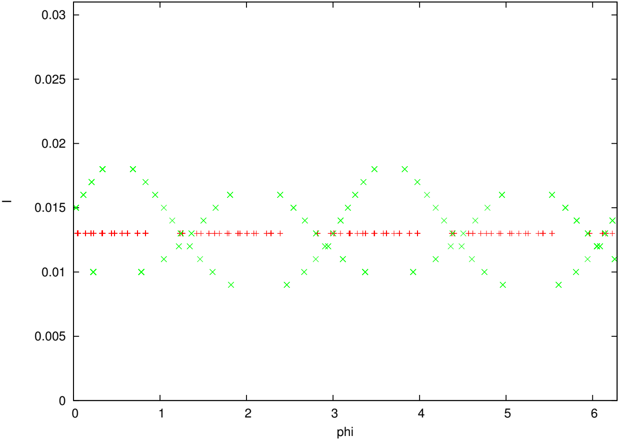


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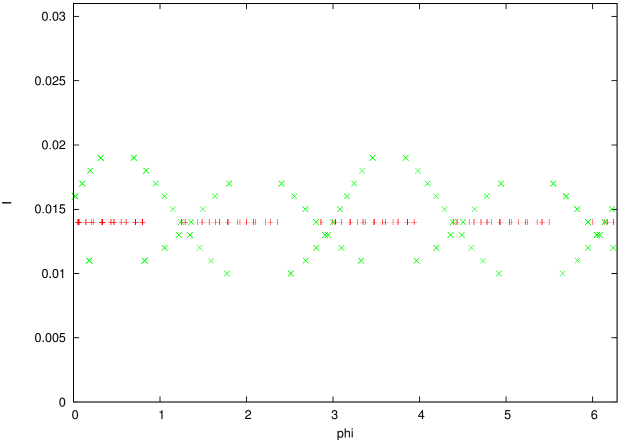


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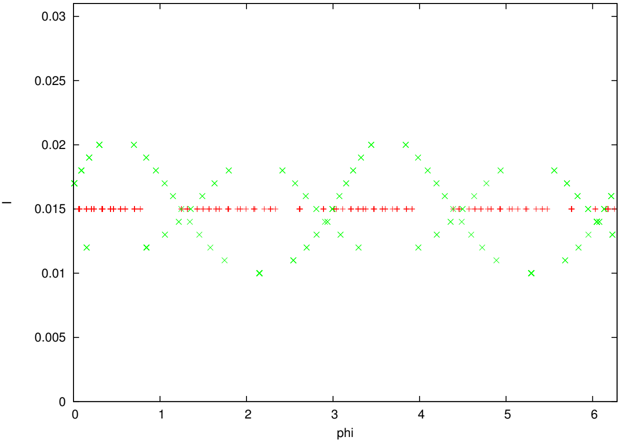


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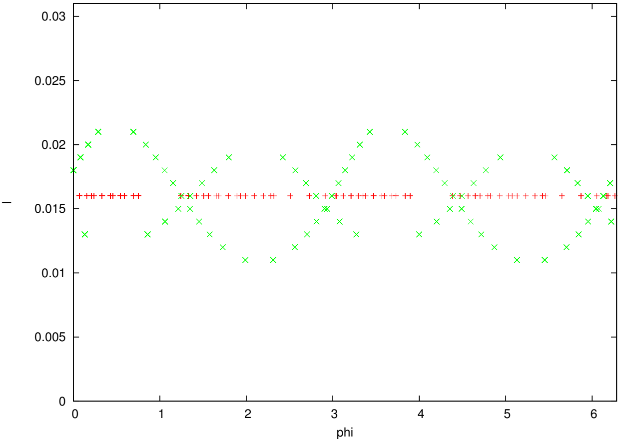


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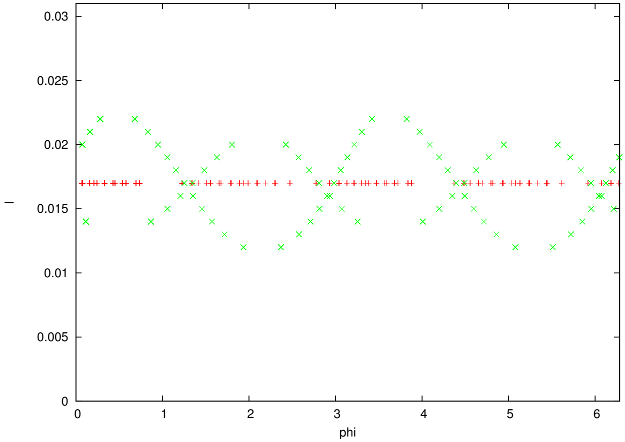


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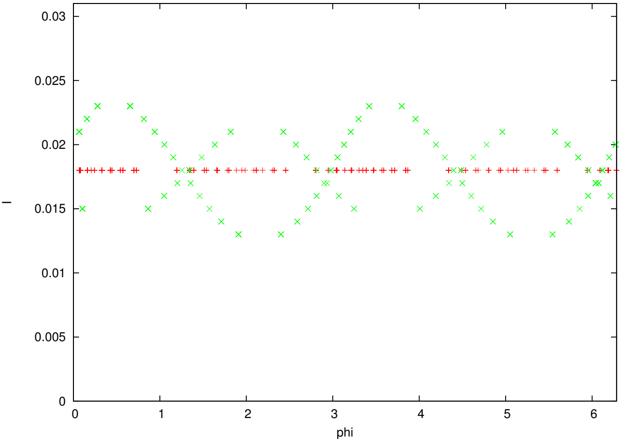


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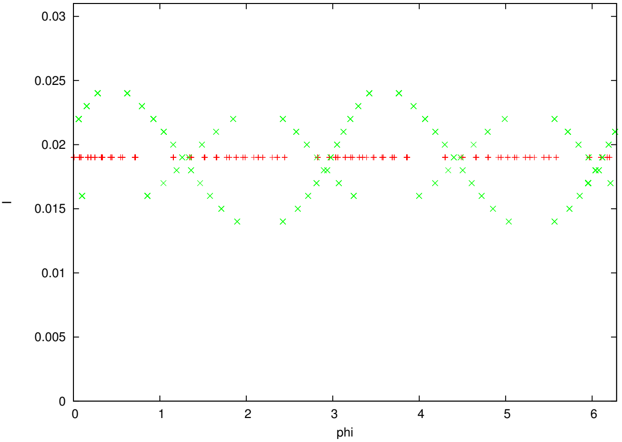


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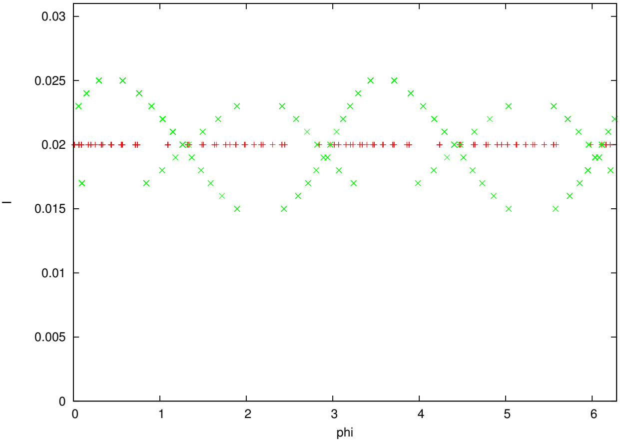


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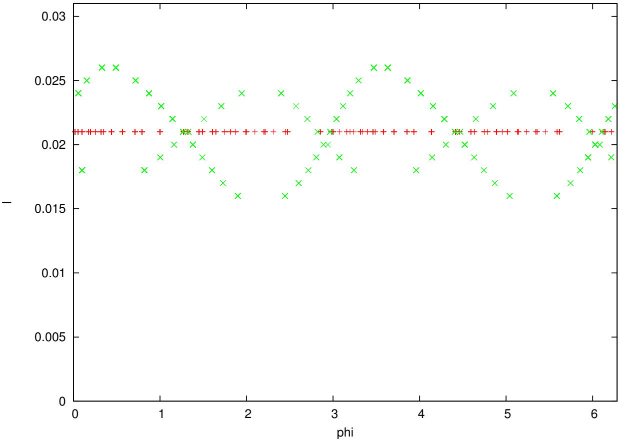


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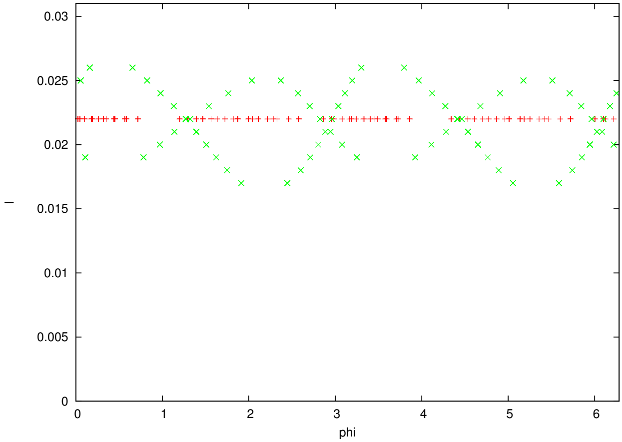


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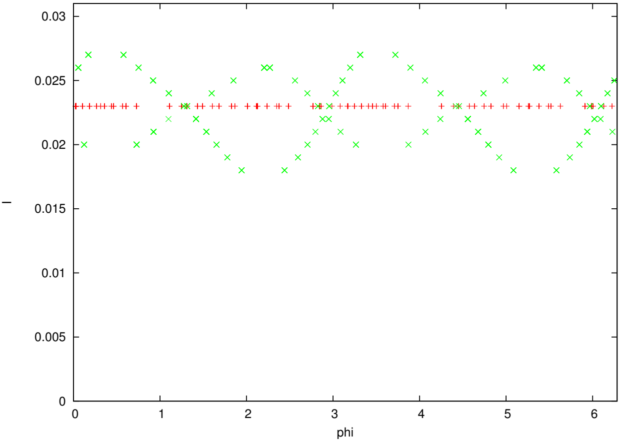


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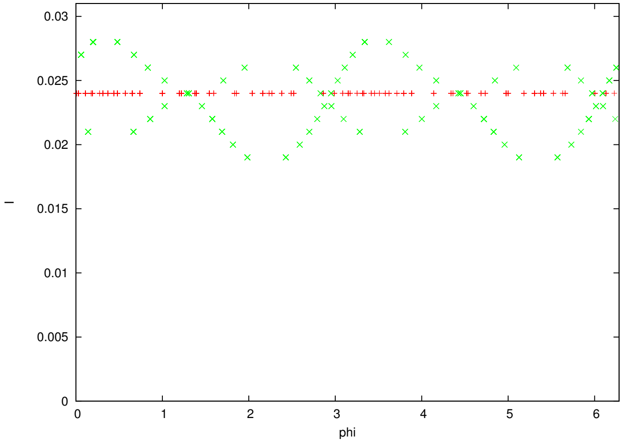


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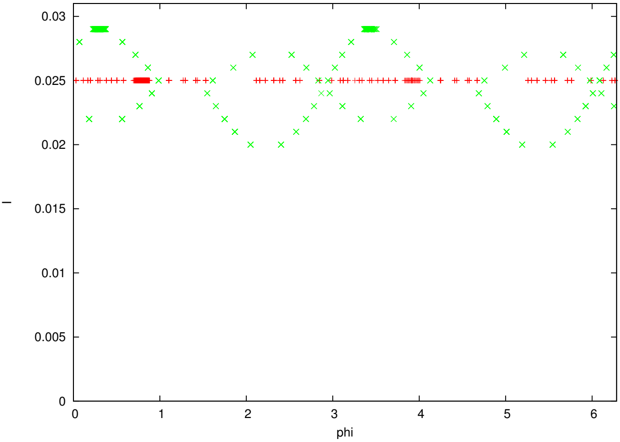


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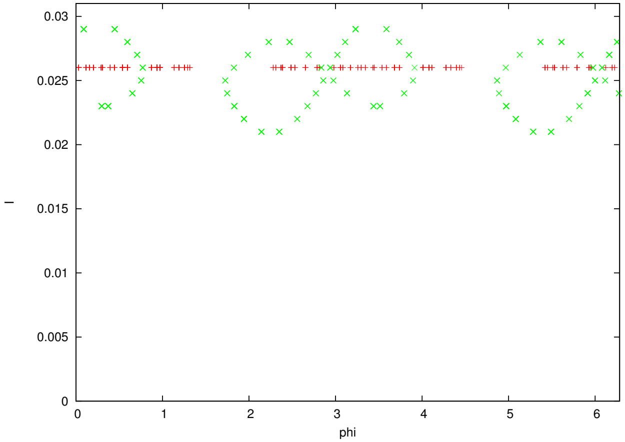


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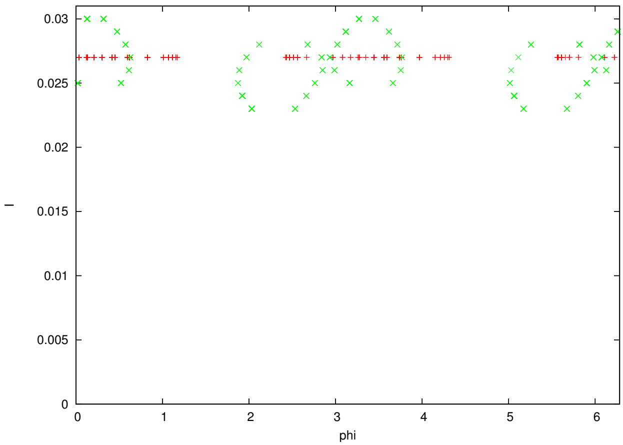


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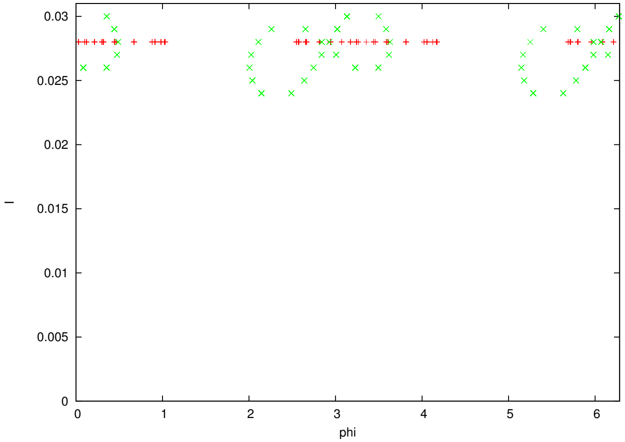


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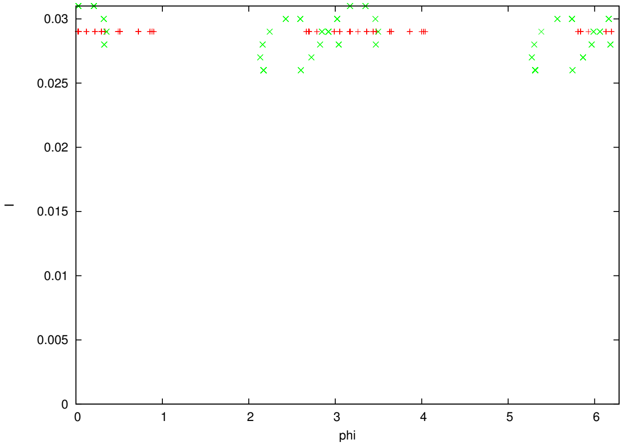


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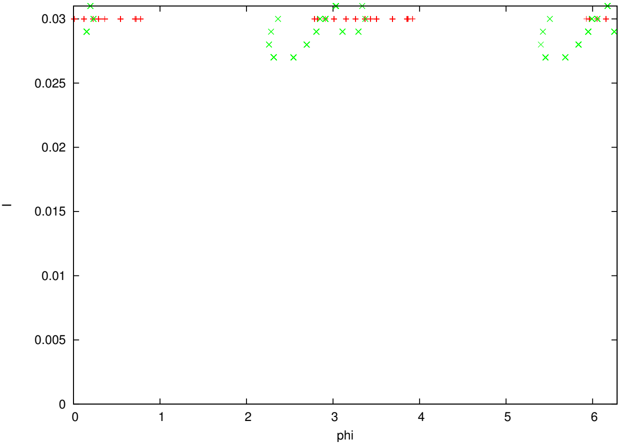


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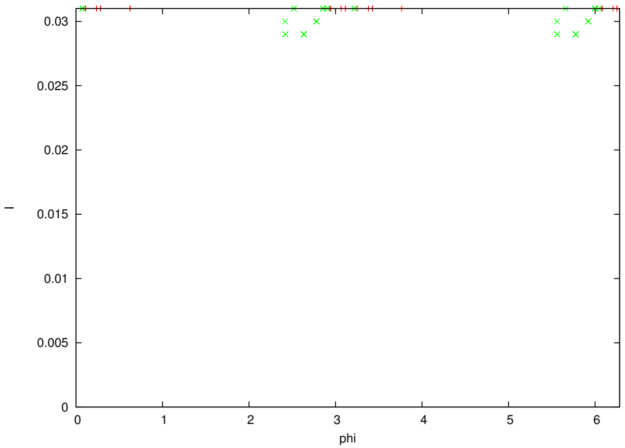


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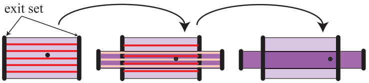
# Method of correctly aligned windows

- [Gidea & Robinson, 2003], [Gidea & Zgliczynski, 2004]
- A window – a homeomorphic copy of a multi-dimensional rectangle
- One window correctly aligns with another – Brouwer degree of the projection in the exit direction is non-zero
- Products of correctly aligned windows are correctly aligned

### Theorem (detection of orbits)

*Given a bi-infinite sequence of windows – if each window is correctly aligned with the next window* $\Rightarrow\ \exists$ *orbit that visits all windows*
*Given finitely many windows with correct alignments between any two of them* $\Rightarrow\ \exists$ *symbolic dynamics.*



# Topological Shadowing Lemma

## Lemma

*Let $\{R_i\}_{i\in\mathbb{Z}}$ be a bi-infinite sequence of 2D windows in $\Lambda$. Assume the following:*

(i) $R_{2i} \subseteq dom(S)$ *and* $R_{2i+1} \subseteq codom(S)$.

(ii) $R_{2i}$ *is correctly aligned with* $R_{2i+1}$ *under the outer (scattering) map* $S$.

(iii) $R_{2i+1}$ *is correctly aligned with* $R_{2i+2}$ *under some iterate* $T^{K_i}$ *of the inner map* $T$*, with* $K_i$ *sufficiently large.*

*Then, for every bi-infinite sequence of positive reals $\{\epsilon_i\}_{i\in\mathbb{Z}}$, there exists a 'true' orbit $F^n(z)$ that gets $(\epsilon_i)$-close to some appropriate iterates of $R_i$.*

# Align Windows by Scattering Map

- $T_i = \{I = I_i\}$, $T_{i+1} = \{I = I_{i+1}\}$, homoclinic orbit $\Longrightarrow$

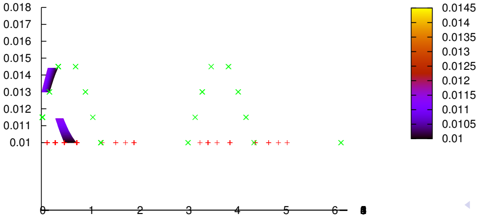
$$x_- \in T_i \xrightarrow{S} x_+ \in T_{i+1}.$$

- Continuation: vary $I_i$ and $I_{i+1}$

$$R_{2i} \subset \mathrm{dom}(S) \xrightarrow{S} R_{2i+1} \subset \mathrm{codom}(S).$$

- Homoclinic excursions

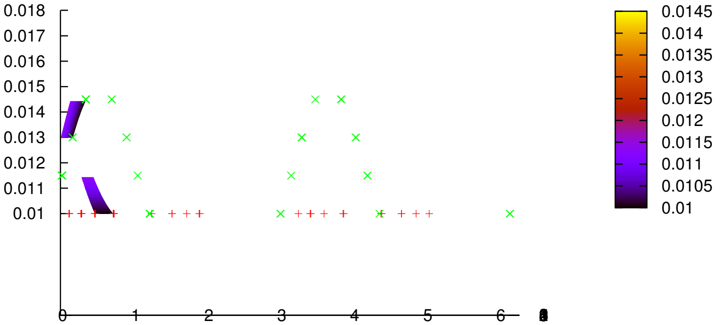
$$D_{2i} = F^{-M_{i-1}}(R_{2i}) \xrightarrow{F^{N_i} \circ S \circ F^{M_{i-1}}} D_{2i+1} = F^{N_i}(R_{2i+1}).$$



# Two Jumps of Scattering Map

- Previous construction for $T_i$ and $T_{i+1}$:

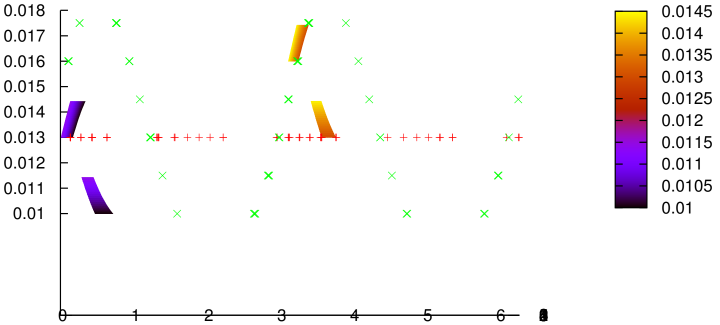
$$R_{2i} \xrightarrow{S} R_{2i+1}.$$



# Two Jumps of Scattering Map

- Repeat the construction for $T_{i+1}$ and $T_{i+2}$:

$$R_{2i+2} \xrightarrow{S} R_{2i+3}.$$



# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows

$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$

- The twist is very weak, so $K_i$ is very large.

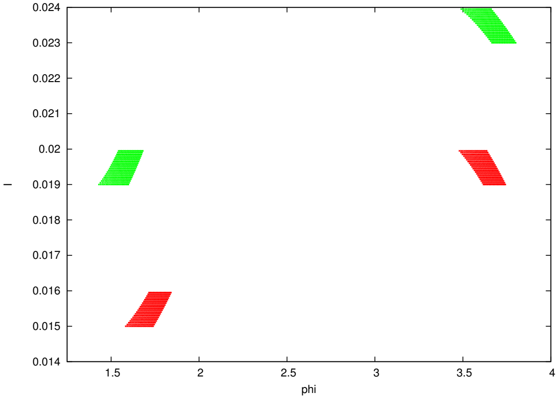



Figure: Windows after $T^0$ twist iterates.

# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows

$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$

- The twist is very weak, so $K_i$ is very large.

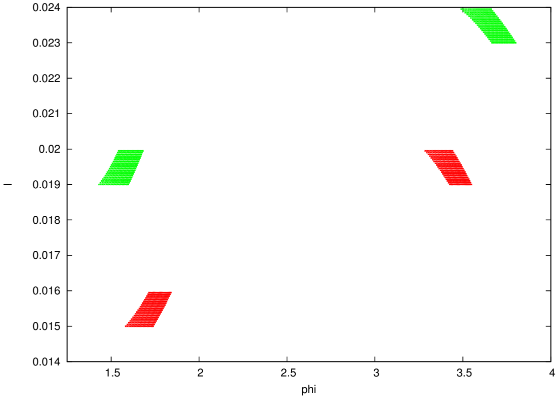



Figure: Windows after $T^1$ twist iterates.

# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows

$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$

- The twist is very weak, so $K_i$ is very large.

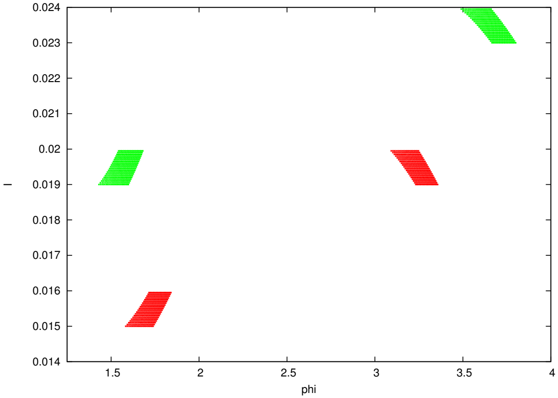



Figure: Windows after $T^2$ twist iterates.

# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows

$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$

- The twist is very weak, so $K_i$ is very large.

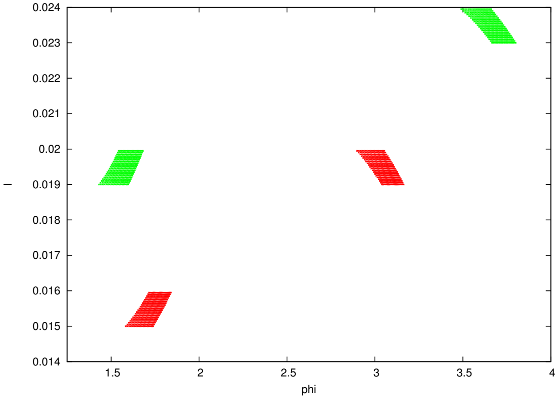



Figure: Windows after $T^3$ twist iterates.

# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows
$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$
- The twist is very weak, so $K_i$ is very large.

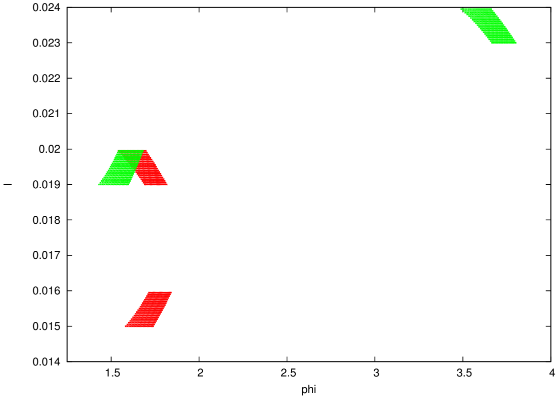



Figure: Windows after $T^{10}$ twist iterates.

# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows

$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$

- The twist is very weak, so $K_i$ is very large.

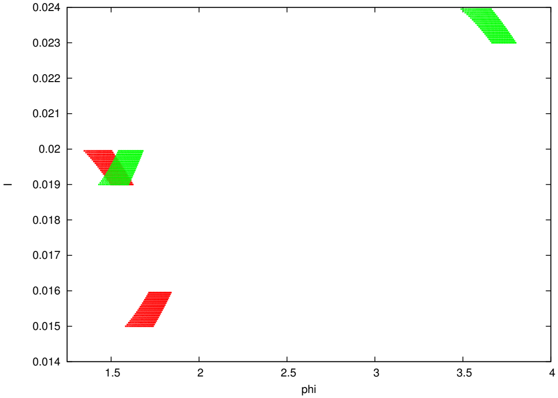



Figure: Windows after $T^{11}$ twist iterates.

# Align Windows by Twist Map

- Use high enough iterate $T^{K_i}$ to align windows
$$R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2}.$$
- The twist is very weak, so $K_i$ is very large.

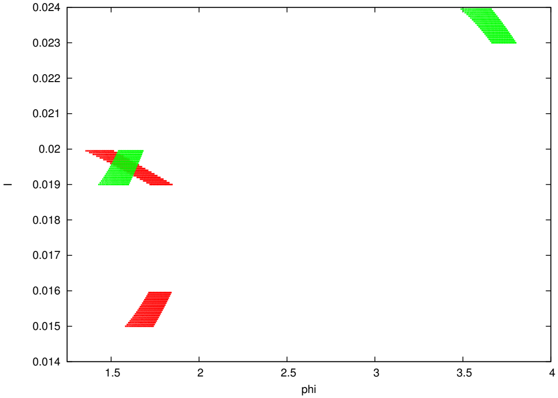
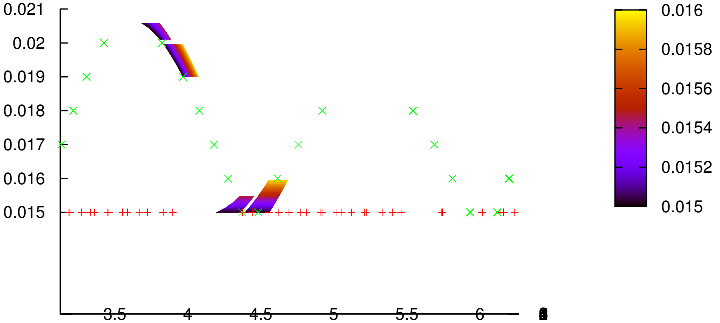



Figure: Windows after $T^{173}$ twist iterates.

# Choose Windows

- Choose the windows to maximize the jump by the scattering map.
- This maximizes the initial tilt of the image window $\Longrightarrow$ fewer iterates of the twist map ($K_i \approx 10$).



# Existence of Diffusion Orbits

- Obtain a sequence of correctly aligned windows

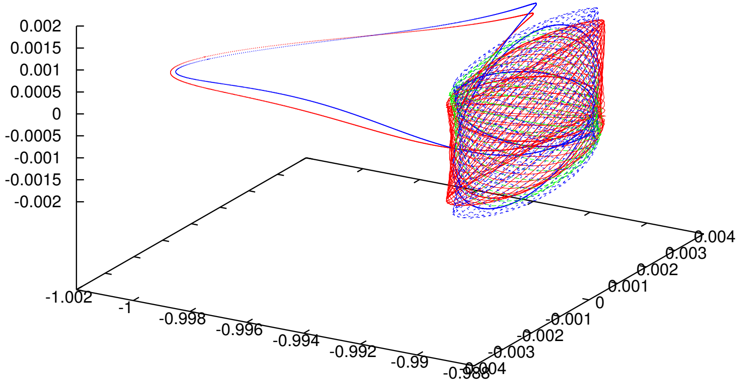
$$\longrightarrow R_{2i} \xrightarrow{S} R_{2i+1} \xrightarrow{T^{K_i}} R_{2i+2} \longrightarrow$$

- By our topological shadowing lemma, there exists a true orbit that goes $\epsilon_i$-close to appropriate iterates of $R_i$ for all $i$.
- End of proof.

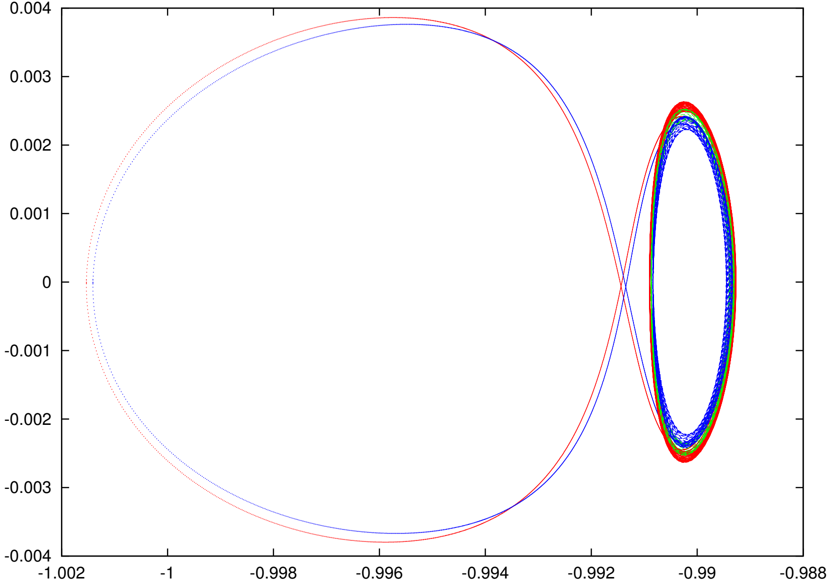
# How to Obtain Diffusive Orbits in Practice

- Recover homoclinic trajectories corresponding to scattering map (they go $10^{-5}$-close to the windows).
- To get correct alignment of windows by the twist, homoclinics are pushed $10^{-50}$-close to $\Lambda$ due to hyperbolicity.
- We obtain a numerical orbit that goes $\epsilon$-close to $T(I_1), T(I_2)$.
- True orbit from the numerical/applications point of view.

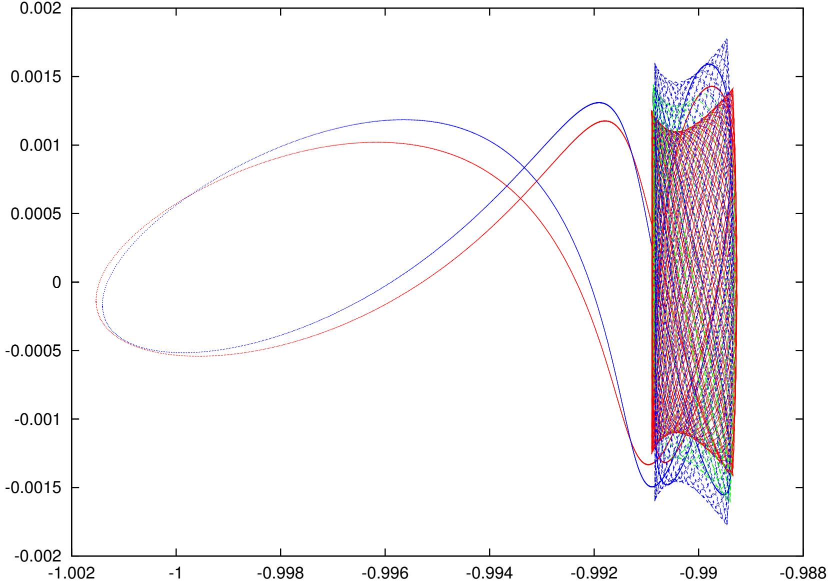
# Example Diffusive Orbit (2 Jumps)



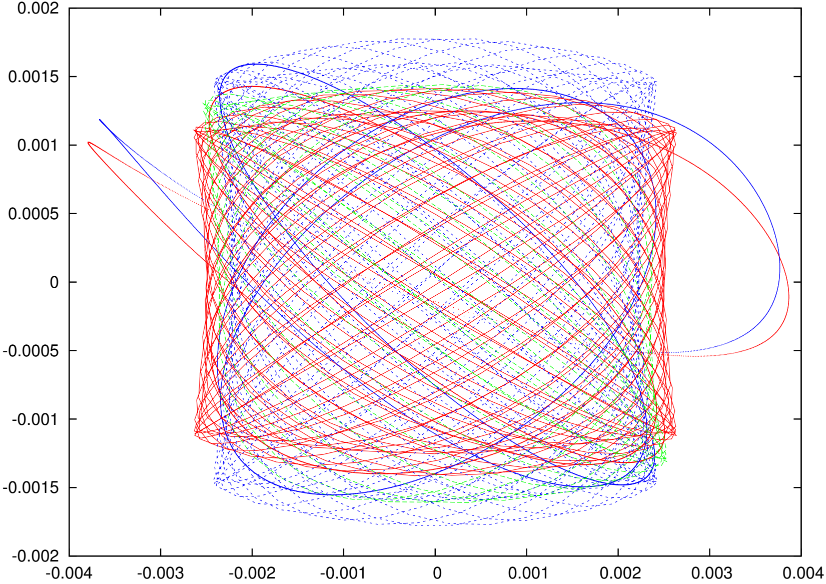
# Example Diffusive Orbit (2 Jumps)



# Example Diffusive Orbit (2 Jumps)



# Example Diffusive Orbit (2 Jumps)



# Conclusions

- Semi-numerical proof of existence of diffusive orbits near equilibrium points $L_1, L_2$ in the spatial circular RTBP.
- Geometrical mechanism similar to [Arnold 1964].
- May be turned into a Computer Assisted Proof (in project).
- Numerical diffusive orbits for practical applications.
- Project: Show diffusion in larger domain that includes "large gaps" (see [Delshams, de la Llave & Seara 2003]).

# Acknowledgements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