Resonance transitions for Oterma comet in Sun-Jupiter system

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Bibliography

KLMR W. S. Koon, M. W. Lo, J. E. Marsden and S. D. Ross, *Heteroclinic Connections between Periodic Orbits and Resonance Transitions in Celestial Mechanics*, Chaos, 10(2000), no. 2, 427–469

WZ1 D. Wilczak, P. Zgliczynski, *Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - A Computer Assisted Proof*, *Comm. Math. Phys.* 234 (2003) 1, 37-75

WZ2 D. Wilczak, P. Zgliczynski, *Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem*, *Comm. Math. Phys.* 259, 561-576 (2005)

The physical problem and the numerical results from KLMR

Observation: Jupiter comets (*Oterma*, *Gehrels 3*) make rapid transition from heliocentic orbits outside Jupiter to heliocentric orbits inside the Orbit of Jupiter and vice versa.

The interior heliocentric orbit is close to the $3:2$ resonance (three revolutions around the Sun in two Jupiter periods) while the exterior heliocentric one is near $2:3$ resonance .

KLMR: PCR3BP (planar restricted three body problem) as a model for the Sun-Jupiter-comet system.

Methods of dynamical system theory: the transitions are the consequence of the existence of several homo- and heteroclinic orbits between the libration points.

In fact the existence of symbolic dynamics on three symbols was claimed.

Symbolic dynamics - definitions

Bernoulli Shift : $\Sigma_k = \{1, 2, \ldots, k\}^{\mathbf{Z}}$, $\sigma : \Sigma_k \to \Sigma_k$

$$\sigma(c)_i = c_{i+1}$$

Bernoulli shifts are dynamical equivalent to a coin tossing.

Definition. $P : X \to X$ - continuous, $S \subset X$, S-compact, we say that P has *a symbolic dynamics* on k symbols on S, when the following conditions are satisfied

- $P(S) = S$, i.e. S is P-invariant
- there exists a continuous map $\pi : S \to \Sigma_k$, such that $\sigma \circ \pi = \pi \circ P$
- $\pi(S) = \Sigma_k$ (or at least $\pi(S)$ is a large subset of Σ_k)

PCR3BP problem

$$\ddot{x} - 2\dot{y} = \Omega_x(x,y), \qquad \ddot{y} + 2\dot{x} = \Omega_y(x,y), \quad (1)$$

$$\begin{aligned} \Omega(x,y) &= \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2} \\ r_1 &= \sqrt{(x+\mu)^2+y^2} \\ r_2 &= \sqrt{(x-1+\mu)^2+y^2} \end{aligned}$$

Jacobi integral:

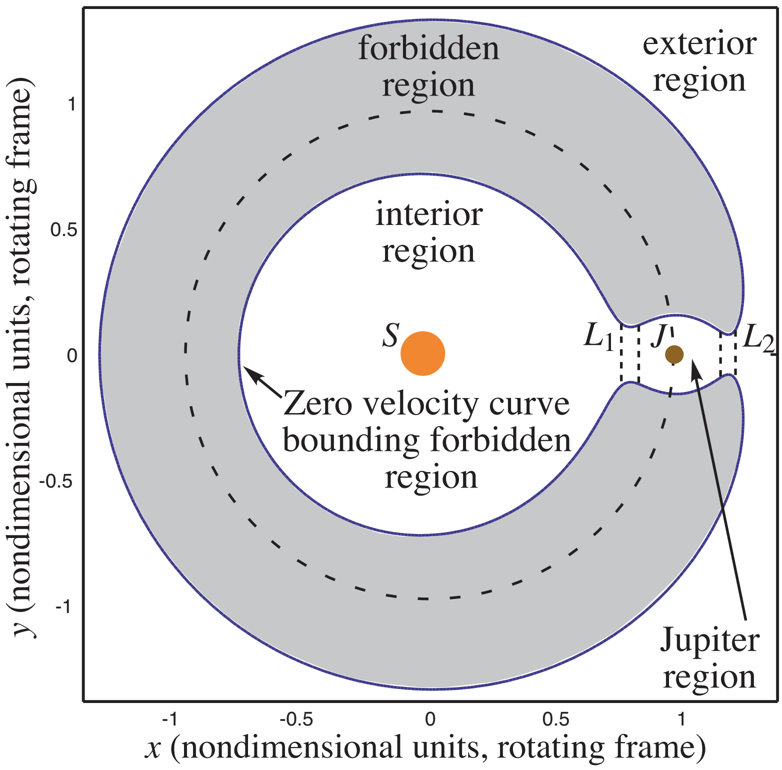
$$C(x,y,\dot{x},\dot{y}) = -(\dot{x}^2+\dot{y}^2) + 2\Omega(x,y) = \mathbf{const}.$$

$$\mathcal{M}(\mu, C) = \{(x,y,\dot{x},\dot{y}) \mid C(x,y,\dot{x},\dot{y}) = C\},$$

$C = 3.03$, $\mu = 0.0009537$ - *Oterma comet* in Sun-Jupiter system.

Hill's Region

Hill's region - the projection of $\mathcal{M}(\mu, C)$ onto position space (coordinates (x, y))



Picture from KLMR

OUR RESULTS FOR PCR3BP

For $C = 3.03$, $\mu = 0.0009537$ - *Oterma* values, the existence of

0. periodic orbits L_1^* and L_2^* around the libration points L_1 and L_2, respectively.

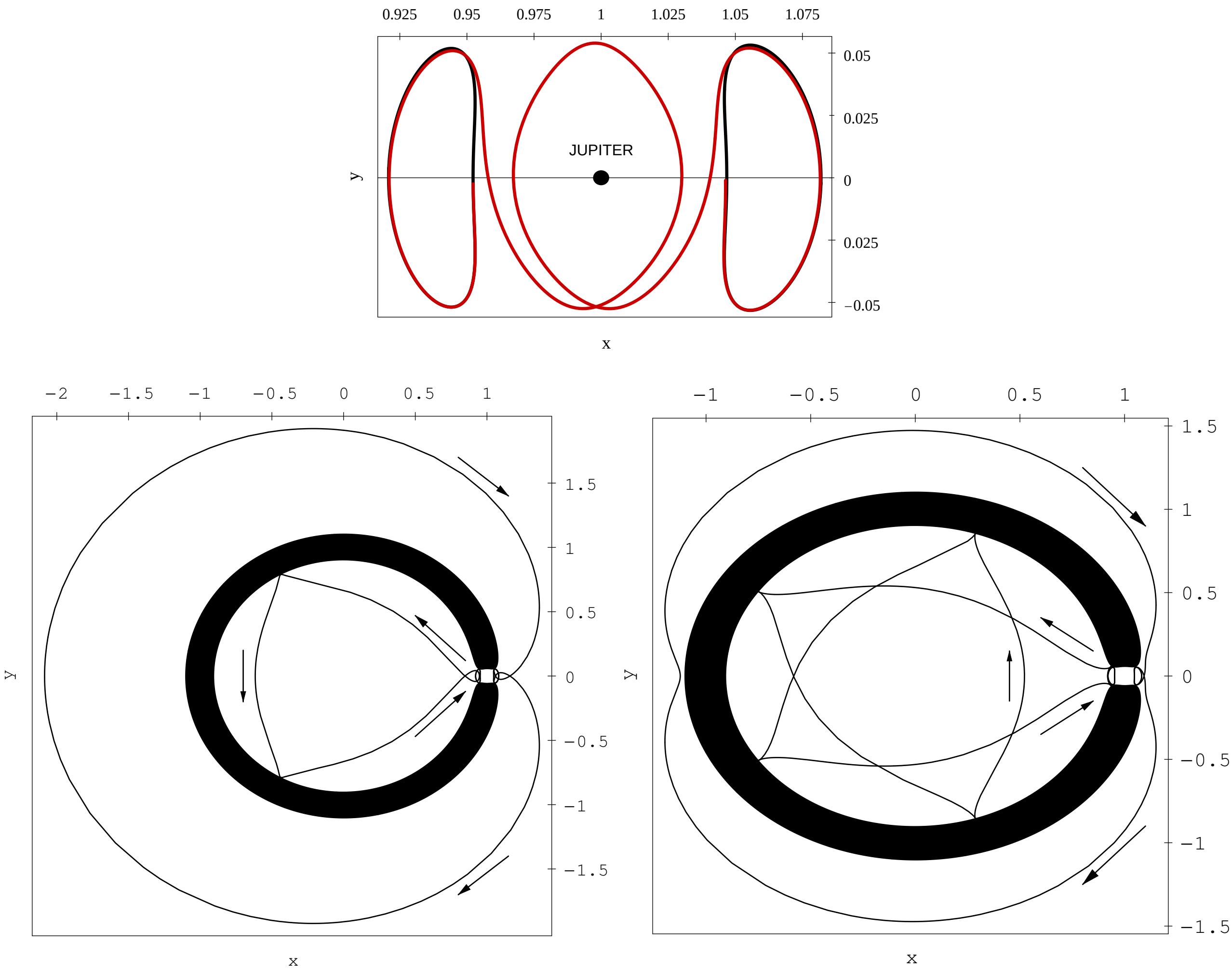
1. topologically transversal heteroclinic orbits connecting L_1^* and L_2^* and vice versa in the Jupiter region.

2. two topologically transversal homoclinic orbit to L_1^* in interior (Sun) region and to L_2^* in exterior region.

3. symbolic dynamics:

$$S \to S, L_1^*, \quad L_1^* \to L_1^*, S, L_2^* \quad L_2^* \to L_1^*, L_2^*, X, \quad X \to X, L_2^*.$$

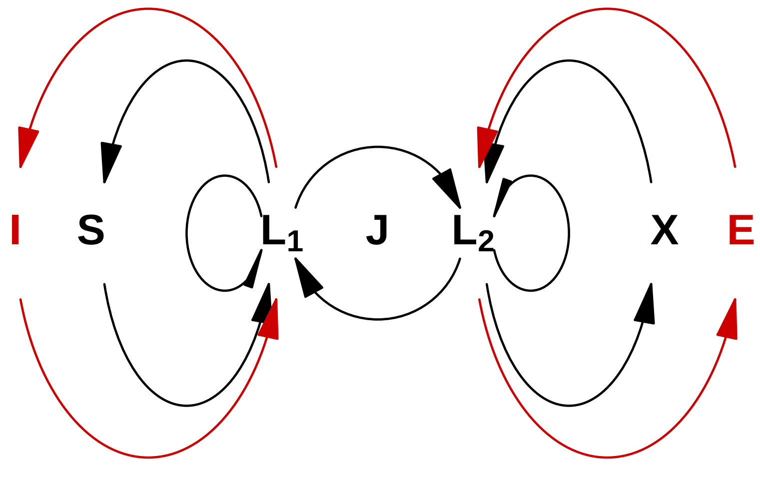
Hetero- and homoclinic orbits



Left: 3:2 - homoclinic (internal region) and 1:2 homoclinic (external reg.)

Right: 5:3 - homoclinic (internal region) and 2:3 homoclinic (external reg.)

Symbolic dynamics - the graph representation



Sections and Poincaré maps

Sections: $\Theta = \{(x, y, \dot{x}, \dot{y}) \in \mathcal{M} \mid y = 0\}$, $\Theta_+ = \Theta \cap \{\dot{y} > 0\}$, $\Theta_- = \Theta \cap \{\dot{y} < 0\}$.

Coordinates on $\Theta_\pm$: $T_\pm : U \subset \mathbf{R}^2 \to \Theta_\pm$

$$T_\pm(x, \dot{x}) = (x, 0, \dot{x}, \pm\sqrt{2\Omega(x, 0) - \dot{x}^2 - C}\ ) \quad (2)$$

Poincaré maps between sections $\Theta_\pm$

$$\begin{aligned} P_+ &: \Theta_+ \to \Theta_+ \\ P_- &: \Theta_- \to \Theta_- \\ P_{\frac{1}{2},+} &: \Theta_+ \to \Theta_- \\ P_{\frac{1}{2},-} &: \Theta_- \to \Theta_+. \end{aligned}$$

$$\begin{aligned} P_+(x) &= P_{\frac{1}{2},-} \circ P_{\frac{1}{2},+}(x), \\ P_-(x) &= P_{\frac{1}{2},+} \circ P_{\frac{1}{2},-}(x) \end{aligned}$$

Symmetries in PCR3BP

If $(x(t), y(t))$ is a trajectory for PCR3BP, then $(x(-t), -y(-t))$ is also a trajectory.

Let $R : \Theta_{\pm} \to \Theta_{\pm}$ $R(x, \dot{x}) = (x, -\dot{x})$ for $(x, \dot{x}) \in \Theta_{\pm}$. We have

$$\text{if } P_{\pm}(x_0) = x_1, \quad \text{then} \quad P_{\pm}(R(x_1)) = R(x_0)$$

$$\text{if } P_{\frac{1}{2},\pm}(x_0) = x_1, \quad \text{then} \quad P_{\frac{1}{2},\mp}(R(x_1)) = R(x_0)$$

Symbolic dynamics for PCR3BP

$$
\begin{aligned}
f_{(1,1)} &= P_{+} \\
f_{(2,1)} &= P_{-} \circ P_{1/2,+} \circ (P_{1/2,-} \circ P_{1/2,+})^{4} \circ P_{+}, \\
f_{(1,2)} &= P_{+} \circ P_{1/2,-} \circ (P_{1/2,+} \circ P_{1/2,-})^{4} \circ P_{-}, \\
f_{(2,2)} &= P_{-}.
\end{aligned}
$$

Theorem. For every $\alpha = \{\alpha_i\} \in \{1,2\}^{\mathbf{Z}}$ there exists $x_0 \in H_{\alpha_0}$ (close to $L^*_{\alpha_0}$), such that

- the trajectory of x_0 is defined for $t \in (-\infty, \infty)$ and stays in the Jupiter region

- $x_n = f_{(\alpha_n,\alpha_{n-1})} \circ \cdots \circ f_{(\alpha_2,\alpha_1)} \circ f_{(\alpha_1,\alpha_0)}(x_0) \in H_{\alpha_n}$ for $n > 0$

- $x_n = f^{-1}_{(\alpha_{n+1},\alpha_n)} \circ \cdots \circ f^{-1}_{(\alpha_{-1},\alpha_{-2})} \circ f^{-1}_{(\alpha_0,\alpha_{-1})}(x_0) \in H_{\alpha_n}$ for $n < 0$.

Moreover,

periodic orbits: If α is k-periodic, then x_0 can be chosen so that $x_k = x_0$ (i.e. x_0 is periodic).

homo- and heterclinic orbits: If $\alpha_k = i_-$ for $k \leq k_-$ and $\alpha_k = i_+$ for $k \geq k_+$, where $i_-, i_+ \in \{1, 2\}$, then

$$\lim_{n \to -\infty} x_n = L^*_{i_-}, \qquad \lim_{n \to \infty} x_n = L^*_{i_+}$$

h-sets on the plane - definition

h-set N on the plane:

- $c, u, s \in \mathbf{R}^2$, u, s - linearly independent

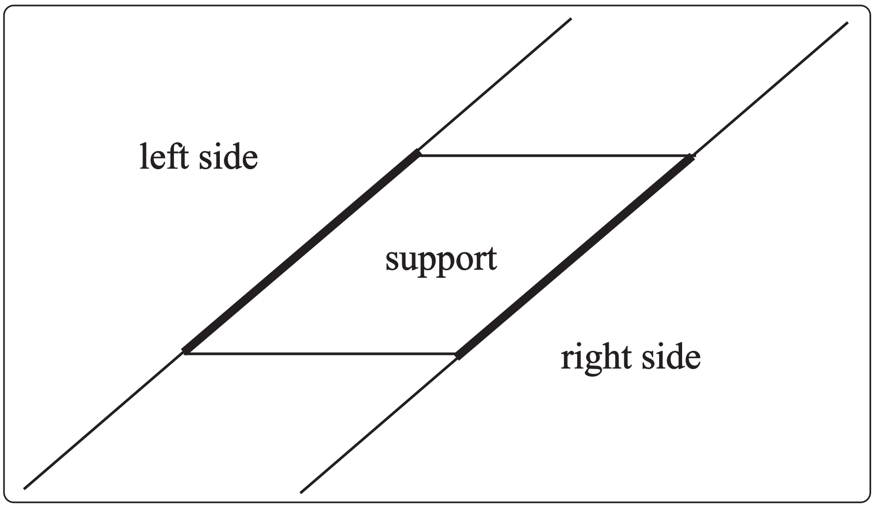
- $|N| = c + [-1, 1]u + [-1, 1]s$ - the support of N

- $N^+ = c + [-1, 1]u + \{-1, 1\}s$ - horizontal edges N

- $N^{le} = c - u + [-1, 1]s$, $N^{re}c + u + [-1, 1]s$ - 'left' and 'right' edfe of N

- $S(N)_l = c + (-\infty, 1)u + (-\infty, \infty)s$, $S(N)_r = c + (1, \infty)u + (-\infty, \infty)s$ - 'left' and 'right' side of N

H-set on the plane



Covering relation - Definition

N, M - h-sets, $f : |N| \to \mathbf{R}^2$ - continuous

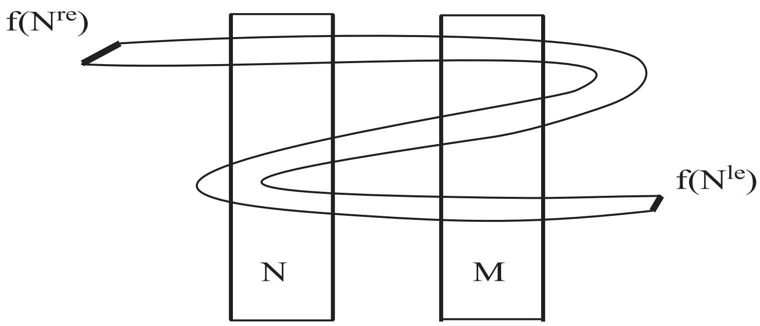
We say, that $N \stackrel{f}{\Longrightarrow} M$ (N f-covers M) if

• $f(|N|) \subset \text{int}(S(M)_l \cup |M| \cup S(M)_r)$

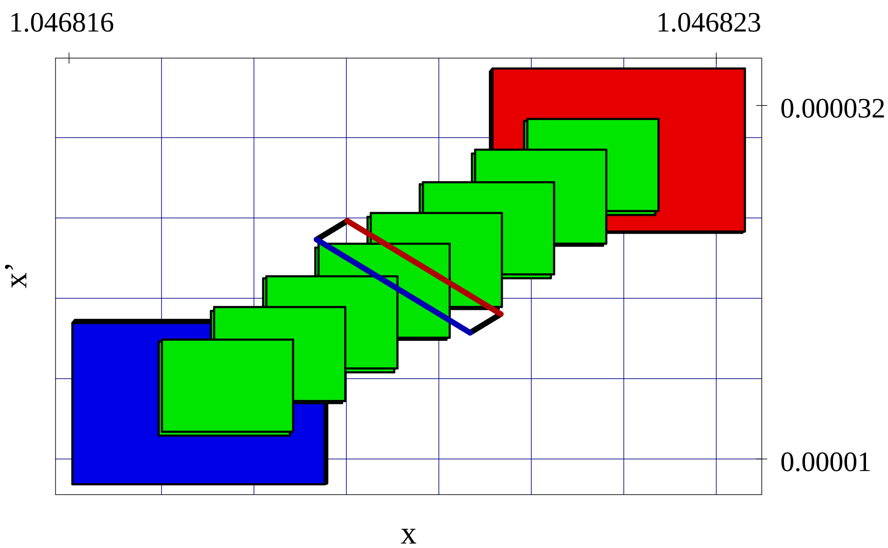
• one of the conditions (O) or (R) is satisfied
(O) $f(N^{le}) \subset S(M)_l$ i $f(N^{re}) \subset S(M)_r$

(R) $f(N^{le}) \subset S(M)_r$ i $f(N^{re}) \subset S(M)_l$

Covering relation - Example



$$N \overset{f}{\Longrightarrow} N \text{ and } N \overset{f}{\Longrightarrow} M$$



Example from the proof for PCR3BP

Main theorem on covering relations

Theorem.(P.Z.)
$N_0, N_1, \ldots, N_k$ - h-sets. $f_i : |N_i| \to \mathbf{R}^2$ -continuous for $i = 0, \ldots, k-1$. Assume, that

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \ldots \overset{f_{k-1}}{\Longrightarrow} N_k.$$

Then there exists $x \in \text{int}|N_0|$ such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_0(x) \in \text{int}|N_{i+1}|, \qquad i = 0, \ldots, k-1.$$

If moreover $N_k = N_0$, then x can be chosen so that

$$f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0(x) = x.$$

Local hyperbolicity - cone conditions

$f : \mathbf{R}^2 \to \mathbf{R}^2$ - $\mathcal{C}^1$ maps. $f(0) = 0$. U - convex, $0 \in U$

$$Df(U) := \begin{pmatrix} \boldsymbol{\lambda_1(U)} & \boldsymbol{\varepsilon_1(U)} \\ \boldsymbol{\varepsilon_2(U)} & \boldsymbol{\lambda_2(U)} \end{pmatrix}.$$

$$f(x) \in Df(U) \cdot x, \quad \text{for } x \in U$$

$$\begin{aligned} \varepsilon_1'(U) &= \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon_1(U)}\}, \\ \varepsilon_2'(U) &= \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon_2(U)}\}, \\ \lambda_1'(U) &= \inf\{|\lambda_1| : \lambda_1 \in \boldsymbol{\lambda_1(U)}\}, \\ \lambda_2'(U) &= \sup\{|\lambda_2| : \lambda_2 \in \boldsymbol{\lambda_2(U)}\}. \end{aligned}$$

Definition Let x_* be a fixed point for f. We say that f is *hyperbolic (satisfies cone conditions)* on $N \ni x_*$, if there exists a local coordinate frame on N, such that (in this new coordinates)

$$\begin{aligned} x_* &= 0 \\ \varepsilon_1'(N)\varepsilon_2'(N) &< (1-\lambda_2'(N))(\lambda_1'(N)-1). \\ N &= [-\alpha_1,\alpha_1]\times[-\alpha_2,\alpha_2], \end{aligned}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are such that the following inequalities are satisfied

$$\frac{\varepsilon_1'(N)}{\lambda_1'(N)-1} < \frac{\alpha_1}{\alpha_2} < \frac{1-\lambda_2'(N)}{\epsilon_2'(N)}. \tag{3}$$

Theorem Assume that f is hyperbolic on N.

1. if $f^k(x) \in N$ for $k \geq 0$, then $\lim_{k\to\infty} f^k(x) = x_*$,

2. if $y_k \in N$ and $f(y_{k-1}) = y_k$ for $k \leq 0$, then $\lim_{k\to-\infty} y_k = x_*$.

Theorem. Assume that g is hyperbolic on N_m and f is hiperboliczny na N_0. Let $x_g = g(x_g) \in N_m$ and $x_f = f(x_f) \in N_0$. Assume that

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \dots$$
$$\overset{f_{m-1}}{\Longrightarrow} N_m \overset{g}{\Longrightarrow} N_m,$$

then there exists a sequence $(x_k)_{k=-\infty}^{0}$ (*this is a backward orbit*), $f(x_k) = x_{k+1}$ for $k < 0$ such that

$$x_k \in N_0, \quad k \leq 0,$$
$$f_{i-1} \circ f_{i-2} \circ \dots \circ f_0(x_0) \in N_i \quad \text{for } i = 1, \dots, m,$$
$$g^n \circ f_{m-1} \circ \dots \circ f_0(x_0) \in N_m \quad \text{for } n > 0,$$
$$\lim_{k \to -\infty} x_k = x_f,$$
$$\lim_{k \to \infty} g^k \circ f_{m-1} \circ \dots \circ f_0(x_0) = x_g.$$

What did we proved with computer assistance

$$
\begin{aligned}
H_1 \overset{P_+}{\Longrightarrow} H_1 \overset{P_+}{\Longrightarrow} H_1^2 \overset{P_{1/2,+}}{\Longrightarrow} N_0 \\
\overset{P_{1/2,-}}{\Longrightarrow} N_1 \overset{P_{1/2,+}}{\Longrightarrow} N_2 \overset{P_{1/2,-}}{\Longrightarrow} N_3 \overset{P_{1/2,+}}{\Longrightarrow} N_4 \\
\overset{P_{1/2,-}}{\Longrightarrow} N_5 \overset{P_{1/2,+}}{\Longrightarrow} N_6 \overset{P_{1/2,-}}{\Longrightarrow} N_7 \\
\overset{P_{1/2,+}}{\Longrightarrow} H_2^2 \overset{P_-}{\Longrightarrow} H_2 \overset{P_-}{\Longrightarrow} H_2.
\end{aligned}
$$

From symmmetry

$$
\begin{aligned}
H_2 = R(H_2) \overset{P_-}{\Longrightarrow} R(H_2^2) \overset{P_{1/2,-}}{\Longrightarrow} R(N_7) \\
\overset{P_{1/2,+}}{\Longrightarrow} R(N_6) \overset{P_{1/2,-}}{\Longrightarrow} R(N_5) \overset{P_{1/2,+}}{\Longrightarrow} R(N_4) \\
\overset{P_{1/2,-}}{\Longrightarrow} R(N_3) \overset{P_{1/2,+}}{\Longrightarrow} R(N_2) \overset{P_{1/2,-}}{\Longrightarrow} R(N_1) \\
\overset{P_{1/2,+}}{\Longrightarrow} R(N_0) \overset{P_{1/2,-}}{\Longrightarrow} R(H_1^2) \overset{P_+}{\Longrightarrow} R(H_1) = H_1
\end{aligned}
$$

What about symmetry of L_1^*, L_2^*, periodic orbits, homo- and heteroclinic connections?

We proved that L_1^*, L_2^* and the 'basic' homoclinic orbits to L_1^* and L_2^* are symmetric.

Moreover, we proved that there exist an infinite number of symmetric periodic orbits and symmetric homoclinic orbits to L_1^* and L_2^*, which can be described by symbolic sequences.

The method of proof: It is enough to look for intersections of Fix$(R) = \{x \mid x = R(x)\}$ with $P^k(Fix(R))$ - this is the Fixed Set Iteration method (also known as DeVogelaere method).

How to get an <u>infinite</u> number of symmetric orbits ?

Theorem. Assume R is a reversing symmetry for P and

$$N_0 \overset{P}{\Longrightarrow} N_1 \overset{P}{\Longrightarrow} N_2 \ldots \overset{P}{\Longrightarrow} N_k.$$

and $\text{Fix}(R) \cap N_0$ is a horizontal disk in N_0 and $\text{Fix}(R) \cap N_k$ is a vertical disk in N_k.

Then there exists $x \in \text{int} N_0$, such that $P^{2k}(x) = x$ and for $i = 0, \ldots, k$ holds

$$P^i(x) \in \text{int} N_k, \quad P^{k+i} \in \text{int} N_{k-i}$$

If $N_1 = N_0$ and N_0 is hyperbolic (with a fixed point x^*) then there exist a symmetric homoclinic to x^* orbit to $P^i(x)$ such that for $i = 0, \ldots, k$ holds

$$P^i(x) \in \text{int} N_k, \quad P^{k+i} \in \text{int} N_{k-i}$$

and $P^i(x) \in N_0$ for $i < 0$ or $i > 2k$.

How to get an infinite number of symmetric orbits ? continuation

We have an infinite number of chains described in the previous theorem.

Future work

- *Does there exists a symbolic dynamics for 3D problem such the corresponding orbits are not all contained in the Sun-Jupiter plane?*

- *Does the symbolic dynamics persist if the Jupiter orbit become an ellipse with a small eccentricity (which is the case in nature)? This means considering PER3BP instead of PCR3BP*. This is work in progress with Maciej Capinski.

Problem: Fixing C in PCR3BP have made our problem hyperbolic and 'easy'. In PRE3BP C is no longer conserved, we have KAM-tori. This becomes a problem of the Arnold diffusion for an a priori-unstable system.

PRE3BP

We want computer assisted proof. What do we need:

• rigorous normally hyperbolic invariant manifold build from Lapunov periodic orbits (see yesterday talk of Maciej Capinski). Not done yet

• the verification of twist condition - should be an easy C^1-computation

• the application of the KAM - probably very difficult to get reasonable size of bounds, but always ok for sufficiently small eccentricity

• the Melnikov type computation, this I'm not sure how to do at this moment, but hopefully standard tools plus rigorous numerics should suffice