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# Non-Linear Stability Analysis of Higher Order Dissipative Partial Differential Equations

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**Abstract.** We extend the invariant manifold method for analyzing the asymptotics of dissipative partial differential equations on unbounded spatial domains to treat equations in which the linear part has order greater than two. One important example of this type of equation which we analyze in some detail is the Cahn-Hilliard equation. We analyze the marginally stable solutions of this equation in some detail. A second context in which such equations arise is in the Ginzburg-Landau equation, or other pattern forming equations, near a codimension-two bifurcation.

## 1. Introduction and statement of results

In this paper, we extend the methods developed in [W1], [W2], [EWW], to study the asymptotic behavior of marginally stable non-linear PDE's. These are PDE's such as

$$\partial_t u \,=\, P(-i\nabla_x)u + W'(u) \;,$$

where $u = u(x,t)$, with $x \in \mathbf{R}^d$, and where $P$ is a polynomial. In the papers cited above, we have treated essentially parabolic problems, *i.e.*, the case where $P(\xi) = -\xi^2$. In this paper, we extend the problem to non-parabolic cases such as $P(\xi) = -\xi^4$, where $P(-i\nabla_x)$ has continuous spectrum all the way up to 0. We deal in particular with the stability analysis of the Cahn-Hilliard equation [CH] in an infinite domain. Where appropriate, we indicate how to formulate the assumptions for more general differential operators and non-linearities.

The Cahn-Hilliard equation models the dynamics of a material with the following 3 properties:

i) The material prefers one of two concentrations that can coexist at a given temperature.
ii) The material prefers to be spatially uniform.
iii) The total mass is conserved.

The first point above means that we should consider a potential with 2 minima with equal critical values, and for concreteness, we will choose $W(u) = (1 - u^2)^2$.* The Cahn-Hilliard equation is then

$$\partial_t u \,=\, \Delta\bigl(-\Delta u + W'(u)\bigr) \;, \tag{1.1}$$

or, expanding,

$$\partial_t u \,=\, -\Delta^2 u - 4\Delta u + 4\Delta u^3 \;. \tag{1.2}$$

We will be interested specifically in the *non-linear stability* of the spatially uniform states, $u(x,t) \equiv u_0$.

It is obvious that constants are solutions of (1.2), for any $u_0$. Furthermore, it is easy to check that these solutions are (locally) linearly stable for $|u_0| > 3^{-1/2}$, and linearly unstable for $|u_0| < 3^{-1/2}$. We concentrate our analysis on the remaining case, namely $u_0 = \pm 3^{-1/2}$. In this case, linearizing about $u_0 = 3^{-1/2}$ leads to the linear equation

$$\partial_t v \,=\, -\Delta^2 v \;, \tag{1.3}$$

which has spectrum in $(-\infty, 0]$ and corresponds to the case $P(\xi) = -\xi^4$. For this linearized problem, bounded initial data lead to solutions which tend to 0 as $t \to \infty$ and the purpose of this paper is to study under which conditions the addition of the nonlinear terms does *not* change the stability of the solutions. This is difficult for two reasons: First, as we have said, the spectrum of the linearized problem extends all the way to 0, and second, the nonlinearity does not have a sign.

Considering the Ginzburg-Landau equations (on $\mathbf{R}$),

$$\partial_t u \,=\, \partial_x^2 u + u - u|u|^2 \;, \tag{1.4}$$

* In our example, the curvatures of the two minima are equal. This does not seem to be necessary for our proofs.

we can construct another example of a similar nature. It is provided by those time-independent solutions of (1.4) which are exactly on the borderline between being Eckhaus stable and Eckhaus unstable. These solutions are

$$u_q(x) \;=\; e^{iqx}\sqrt{1-q^2}\;,$$

with $q = 1/\sqrt{3}$, *cf.* [EG]. We believe that the asymptotic behavior of solutions for this problem is of the same nature as that of the Cahn-Hilliard equations. Here, we describe a program which we believe would lead to a proof. The first part of the analysis of this problem would follow rather closely that given in [EWW] for the Swift-Hohenberg equation. Letting $u^* = u_q$ for the critical value $q = 1/\sqrt{3}$, and writing $u = u^* + v$, the equation for $v$ is

$$\partial_t v \;=\; \partial_x^2 v + v - 2v|u^*|^2 - \bar{v}(u^*)^2 + \mathcal{O}(v^2)\;. \tag{1.5}$$

It has a linear part which is like a Schrödinger operator in a periodic potential (the inhomogeneity $u^*$). This can be handled by going to Floquet variables, namely setting

$$v(x,t) \;=\; \int_{-q}^{q} \mathrm{d}k\, e^{ikx} v_k(x,t)\;,$$

where $v_k$ is $\pi/q$-periodic in $x$:

$$v_k(x,t) \;=\; \sum_{m\in\mathbf{Z}} e^{2imqx} v_{k,m}(t)\;.$$

The linear part of (1.5) leaves the subspaces spanned by the $v_k$ invariant, and has discrete spectrum in each such subspace. The spectrum is in $\sigma \le 0$ and the largest eigenvalue is $-\mathcal{O}(k^4)$ when $q$ equals its critical value $q = 1/\sqrt{3}$ (which is the case we discuss here). In this sense, the problem of the marginal Eckhaus instability resembles the problem of the Cahn-Hilliard equation. At this point, the discussion of the problem follows the techniques we developed in [EWW]. We would like to rescale as we will do below for the Cahn-Hilliard equation and its generalizations, but the problem will be more complicated because the Brillouin zone is restricted to $k \in [-q, q]$. We then have to check that the non-linearity is "irrelevant" in the terminology developed below. Again, as in [EWW], we believe that this will not be quite the case, but the saving grace will be that the projection of the potentially non-irrelevant modes onto the eigenstates corresponding to the $-\mathcal{O}(k^4)$ term vanish to some higher degrees because of translation invariance of the original problem, *cf.* [EWW, Section 4], and [S].

In fact, as T. Gallay pointed out to us after a first version of this paper was completed, one can probably avoid the use of Floquet variables in this example by defining a new dependent variable through $u(x,t) = \sqrt{1-q^2}e^{iqx}(1+v(x,t))$. Then the linearized equation for $v(x,t)$ has constant coefficients and one does not need to introduce Floquet variables to study its spectrum. As in the previous argument, the spectrum of this linearized operator behaves like $-\mathcal{O}(k^4)$ when $|k|$ is near zero, (and $q = 1/\sqrt{3}$), and hence we expect that the analysis which follows would allow one to study the long-time behavior of the full nonlinear equation.

We place our examples in the following more general setting. Consider equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{n+1}\Delta^n u + F(u, \{\partial_x^\alpha u\}) \ , \tag{1.6}$$

where the multi-indices $\alpha$ satisfy $|\alpha| \le 2n-1$, and $x \in \mathbf{R}^d$, $t \ge 1$. Furthermore, $F$ is a polynomial in $u$ and its derivatives. We wish to study the asymptotics of the solution $u$ of (1.6) as $t \to \infty$. First, one introduces scaling variables by defining

$$u(x,t) = \frac{1}{t^{d/(2n)}}\, v\big(\frac{x}{t^{1/(2n)}}, \log t\big) \ . \tag{1.7}$$

Introducing new variables $\xi = x/t^{1/(2n)}$ and $\tau = \log t$, the initial value problem (1.6) with initial data at $t=1$ is transformed to the non-autonomous problem

$$\frac{\partial v}{\partial \tau} = (-1)^{n+1}\Delta_\xi^n v + \frac{1}{2n}\xi\cdot\nabla_\xi v + \frac{d}{2n}v + e^{(\frac{2n+d}{2n})\tau} F(e^{-\frac{d\tau}{2n}}v, \{e^{-(\frac{|\alpha|+d}{2n})\tau}\partial_\xi^\alpha v\}) \ , \tag{1.8}$$

with initial data at $\tau = 0$. The analysis of this equation involves two steps:

i) An analysis of the linear operator

ii) A determination of which non-linear terms are relevant.

As we will see, the term $1/(2n)\xi\cdot\nabla_\xi$ plays an important rôle in the analysis of this linear operator as it allows us to push the continuous spectrum of the operator more and more into the stable region by working in Sobolev spaces with higher and higher polynomial weights. These weights force the functions to decrease more and more rapidly near $|x| = \infty$. Taking Fourier transforms on both sides of (1.8) we obtain:

$$\frac{\partial \tilde v}{\partial \tau} = -(p\cdot p)^n \tilde v - \frac{1}{2n}p\cdot\nabla_p \tilde v + e^{(\frac{2n+d}{2n})\tau} F^*(e^{-\frac{d\tau}{2n}}\tilde v, \{e^{-(\frac{|\alpha|+d}{2n})\tau}(-ip)^\alpha \tilde v\}) \ , \tag{1.9}$$

where $F^*$ is the polynomial $F$, written in terms of convolution products, (see the discussion of the non-linearities below).

We will discuss the form of the non-linear terms below, and consider first the linear operator

$$\mathcal{L} = -(p\cdot p)^n - \frac{1}{2n}p\cdot\nabla_p \ . \tag{1.10}$$

A straightforward calculation shows that $\mathcal{L}$ has the countable set of eigenvalues

$$\lambda_j = -\frac{j}{2n} \ , \quad j = 0,1,2,\dots \ , \tag{1.11}$$

with eigenfunctions (written in multi-index notation),

$$\tilde\varphi_\alpha(p) = p^\alpha e^{-(p\cdot p)^n} \ , \tag{1.12}$$

and $|\alpha| = j$.

If we consider $\mathfrak{L}$ as acting on the Sobolev spaces

$$\tilde{H}_{\ell,m} = \left\{ \tilde{v} \; : \; \|p^{\alpha}\partial_p^{\beta}\tilde{v}\|_{\mathrm{L}^2} < \infty \;, \text{ for all } |\alpha| \leq \ell, |\beta| \leq m \right\} \;,$$

there exists a constant $\sigma_m > 0$ such that $\mathfrak{L}$ will have continuous spectrum in the half-plane $\operatorname{Re}\lambda < -\sigma_m$ in addition to the eigenvalues above. Since $\sigma_m$ is increasing to $\infty$ with $m$, we can force the continuous spectrum arbitrarily far into the left half-plane by choosing $m$ appropriately, and the dominant behavior of the linear operator will be dictated by the eigenvalues with the largest real part.

**Remark.** In order to switch back and forth from the Fourier transform representation of $\mathfrak{L}$ to the un-Fourier transformed representation of this operator with ease, we also consider the Sobolev spaces

$$H_{\ell,m} = \left\{ v \; : \; \|\partial_x^{\alpha}x^{\beta}v\|_{\mathrm{L}^2} < \infty \;, \text{ for all } |\alpha| \leq \ell, |\beta| \leq m \right\} \;.$$

Note that Fourier transformation is an isomorphism from $\tilde{H}_{\ell,m}$ to $H_{\ell,m}$.

Note that $\mathfrak{L}$ is *not* sectorial, and therefore we know of no way to bound the semi-group generated by $\mathfrak{L}$ by spectral information alone. However, in Appendix A, we develop an integral representation of the semi-group and we then show that it satisfies the estimates needed for the invariant manifold theorem.

We next discuss which terms in the non-linearity are "relevant." Consider a monomial

$$A = \prod_{j=0}^{s} \left(\partial_x^{\alpha^{(j)}} u\right)^{k_j} \;, \tag{1.13}$$

where the $\alpha^{(j)}$ are distinct multi-indices. After changing variables as in (1.7), and taking Fourier transforms in $x$ this becomes

$$\begin{aligned}\tilde{A} = \exp\left(\left(\frac{2n+d}{2n}\right)\tau\right) \; \exp\left(-\sum_{j=0}^{s}\left(\frac{|\alpha^{(j)}|+d}{2n}\right)k_j\tau\right) \\ \times \left((-ip)^{\alpha^{(0)}}\tilde{v}\right)^{*k_0} * \cdots * \left((-ip)^{\alpha^{(s)}}\tilde{v}\right)^{*k_s} \;.\end{aligned} \tag{1.14}$$

Here, $*$ denotes the convolution product. If we combine the powers of $\tau$ in the exponential, we see that if

$$2n + d < \sum_{j=0}^{s}\left(|\alpha^{(j)}| + d\right)k_j \;, \tag{1.15}$$

then the coefficient of this term will go to zero exponentially fast in $\tau$, and hence it will be irrelevant from the point of view of the long time behavior of the solutions.

**Definitions.** A monomial like (1.14) is called *irrelevant* if it satisfies the inequality (1.15). It is called *critical* if the l.h.s. of (1.15) is equal to the r.h.s, and *relevant* in the remaining case.

These definitions are suggested by the following which is our first main result:

**Theorem 1.1.** *Assume all terms in the non-linearity in (1.6) are irrelevant. For any solution* $u(x,t)$ *of (1.6) with sufficiently small initial conditions in* $H_{\ell,m}$ *(with* $\ell > (2n-1)+d/2$ *and* $m > 2(\ell+d+k+1)$ *), there is a constant* $B^*$*, depending on the initial conditions, such that for every* $\varepsilon > 0$*,*

$$\lim_{t\to\infty} t^{(\frac{d+1}{2n}-\varepsilon)} \left\| u(x,t) - \frac{B^*}{t^{d/(2n)}} f^*(\frac{x}{t^{1/(2n)}}) \right\|_{\mathrm{L}^\infty} = 0 \,.$$

*Here,*

$$f^*(\xi) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{d}^d p \, e^{ip\cdot\xi} e^{-(p\cdot p)^n} \,. \tag{1.16}$$

**Remark.** This theorem is a special case of a more detailed analysis which will be given below. That analysis will allow us to compute, in principle, the *form* of the solutions of (1.6) up to $\mathcal{O}(t^{-k})$, for any $k > 0$. We note that if one only wanted the first order asymptotics of the solution, one could also use the renormalization group analysis of [BKL].

We now apply Theorem 1.1 to the Cahn-Hilliard equation. Writing $u = 3^{-1/2} + w$, the function $w$ is seen to satisfy

$$\frac{\partial w}{\partial t} = -\Delta^2 w + 4\sqrt{3}\Delta(w^2) + 4\Delta(w^3) \,. \tag{1.17}$$

Upon expanding $\Delta(w^2)$ we obtain two types of terms—those of the form $w(\partial_{x_i}^2 w)$ and those of the form $(\partial_{x_i} w)^2$. In both cases,

$$\sum (|\alpha^{(j)}| + d) k_j = 2d + 2 \,.$$

Since $n = 2$ in this example, these terms will be irrelevant if $4+d < 2d+2$, that is in dimensions $d > 2$. Also, the term $\Delta(w^3)$ is irrelevant for $d > 1$. Thus, as a corollary to Theorem 1.1 we get immediately

**Corollary 1.2.** *Solutions of the Cahn-Hilliard equation in dimension* $d \geq 3$*, with initial conditions sufficiently close (in* $H_{\ell,m}$*, with* $\ell > 3 + (d/2)$*, and* $m > 2(\ell+d+2)$*) to the constant solution* $u \equiv 3^{-1/2}$ *behave asymptotically as*

$$u(x,t) = \frac{1}{3^{1/2}} + \frac{B^*}{t^{d/4}} f^*\Big(\frac{x}{t^{1/4}}\Big) + \mathcal{O}\Big(\frac{1}{t^{(d+1)/4-\varepsilon}}\Big) \,. \tag{1.18}$$

**Remark.** We will examine below what happens in the cases $d = 1, 2$. The case $d = 2$ is of particular interest because its non-linearity is critical in the renormalization group terminology.

## 2. Invariant manifolds

Note that spectral subspaces corresponding to eigenvalues of $\mathcal{L}$ are automatically invariant manifolds for the semi-flow defined by the linear part of (1.9). The aim of this section is to demonstrate that the full non-linear problem has similar invariant manifolds in a neighborhood of the origin. This then shows that the conceptual understanding of what is happening can be gained purely from a knowledge of $\mathcal{L}$, (and the scaling behavior of the non-linearity).

We begin with a proposition concerning the linear semi-group generated by $\mathcal{L}$.

**Proposition 2.1.** *Let $P_k$ denote the projection onto the spectral subspace associated with the eigenvalues $\{\frac{-j}{2n}\}_{j=0}^{k}$, and let $Q_k = (1 - P_k)$ (in $H_{\ell,m}$). If $m > 2(\ell + d + k + 1)$, then there exists $C_k > 0$ such that the semi-group generated by $\mathcal{L}$ satisfies*

$$\|Q_k e^{\tau\mathcal{L}} Q_k v\|_{\ell,m} \leq \frac{C_k}{t^{q/2n}} \exp\bigl(-\frac{k+1}{2n}\tau\bigr)\|v\|_{\ell-q,m}\ , q = 0, 1, \ldots, 2n-1\ . \tag{2.1}$$

**Proof.** The proof, which is presented in Appendix A, is modeled on the proof in [EWW] which treats the case $n = 1$.

Given such estimates on the linear evolution, the construction of invariant manifolds is straightforward. Denote by $y$ the coordinates on the (finite-dimensional) range of $P_k$, and let $z = Q_k\tilde{v}$. Finally let $\eta = e^{-\tau/(2n)} = t^{-1/(2n)}$. Then, applying the projection operators $P_k$ and $Q_k$ to (1.9), it can be written as the system of equations

$$\begin{aligned} \dot{y} &= \Lambda_k y + f(\eta, y, z)\ , \\ \dot{z} &= Q_k\mathcal{L}z + g(\eta, y, z)\ , \\ \dot{\eta} &= -\tfrac{1}{2n}\eta\ , \end{aligned} \tag{2.2}$$

where " $\dot{}$ " denotes differentiation w.r.t. $\tau$. We next need a bound on the non-linearity:

**Lemma 2.2.** *Assume $v \in H_{\ell,m}$ with $\ell > 2n - 1 + d/2$, and assume*

$$2n + d \leq \sum_{j=0}^{s}(|\alpha^{(j)}| + d)k_j\ .$$

*Then the non-linear term (1.14) has $H_{\ell-2n+1,m}$ norm bounded by*

$$C\eta^p \prod_{j=0}^{s} \|v\|_{\ell,m}^{k_j} = C\eta^p\|v\|_{\ell,m}^{K}\ ,$$

*with $p = \sum_{j=0}^{s}(|\alpha^{(j)}| + d)k_j - (2n + d)$, $K = \sum_{j=0}^{s} k_j$, and $C = C\bigl(d, \{k_j\}, \{\alpha^{(j)}\}\bigr)$.*

**Proof.** Taking the inverse Fourier transform of (1.14), and substituting $\eta = e^{-\tau/(2n)}$, Eq.(1.14) becomes

$$\eta^p \prod_{j=0}^{s} \left(\partial_\xi^{\alpha^{(j)}} v\right)^{k_j}\ .$$

The result then follows from the Sobolev embedding theorem to each factor, and observing that the choice of $\ell$ guarantees that each factor is in fact in $\mathrm{L}^\infty$. Note that the lemma has the immediate corollary (because $F$ is a polynomial):

**Corollary 2.3.** *Under the hypotheses of Lemma 2.2, for every $r \geq 1$, the non-linear term in (1.9) is a $\mathcal{C}^r$ function from $\mathbf{R} \times H_{\ell,m}$ to $H_{\ell-2n+1,m}$.*

This corollary in turn implies that the terms in (1.14) and (2.2) are all $\mathcal{C}^r$ functions. This, in conjunction with the estimates on the linear semi-group is sufficient to establish the following

**Theorem 2.4.** *Suppose that $\ell > 2n - 1 + d/2$ and $m > 2(\ell + d + k + 1)$. Suppose further that all terms in the nonlinearity satisfy*

$$2n + d \leq \sum_{j=0}^{s} (|\alpha^{(j)}| + d) k_j \, . \tag{2.3}$$

*Then there exists a $\mathcal{C}^{1+\alpha}$ function $h(\eta, y)$, with $\alpha > 0$, defined in some neighborhood of the origin in $\mathbf{R} \times \mathbf{R}^{\dim \operatorname{range}(P_k)}$, such that the manifold $z = h(\eta, y)$ is left invariant by the semi-flow of (2.2). Furthermore, any solution of (2.2) which remains near the origin for all times approaches a solution of (2.2)—restricted to the invariant manifold—at a rate $\mathcal{O}\big(e^{\frac{k+1-\varepsilon}{2n}\tau}\big)$.*

**Proof.** The existence of the invariant manifold, given the assumptions on the linear semi-group and the non-linearity, seems, to our knowledge, not to be explicitly spelled out in the literature. The formulation which comes closest to our needs is the one given in [H], where the assumptions on the non-linearity are those we have in our case, but the semi-group is supposed to be analytic. However, Henry's construction of the invariant manifold only uses certain bounds on the decay of the semi-group, and not the stronger assumption of analyticity. Those bounds *are* true in our case, by Proposition 2.1.

To be more precise about exactly how one constructs the invariant manifold, note that we are looking for a function $h(\eta, y)$, whose graph $\{(\eta, y), h(\eta, y)\}$ is invariant with respect to the semiflow defined by (2.2). A standard calculation then shows that $h$ should satisfy

$$h(\eta, y) = -\int_0^{-\infty} d\tau \, (Q_k e^{-\mathcal{L}\tau} Q_k) g\big(\varphi_\tau(\eta, y), h \circ \varphi_\tau(\eta, y)\big) \, , \tag{2.4}$$

where $\varphi_\tau(\eta, y)$ is the flow defined by

$$\begin{aligned} \dot{\eta} &= -\frac{1}{2n}\eta \, , \\ \dot{y} &= \Lambda_k y + f\big(\eta, y, h(\eta, y)\big) \, . \end{aligned} \tag{2.5}$$

To prove that $h$ exists, one finds a fixed point of the map $(h, \varphi) \mapsto \big(F(h, \varphi), G(h, \varphi)\big)$ defined by

$$\begin{aligned} F(h, \varphi)(\eta, y) &= -\int_0^{-\infty} d\tau \, (Q_k e^{-\mathcal{L}\tau} Q_k) g\big(\varphi_\tau(\eta, y), h \circ \varphi_\tau(\eta, y)\big) \, , \\ G(h, \varphi)_t(\eta, y) &= \left(e^{-t/2n}\eta \, , \; e^{\Lambda_k t} y + \int_0^t ds \, e^{\Lambda_k (t-s)} f\big(\varphi_s(\eta, y), h \circ \varphi_s(\eta, y)\big)\right) \, . \end{aligned} \tag{2.6}$$

From this point we follow closely the argument of Gallay [G, pp.257–258]. Let $d_k$ be the dimension of the range of $P_k$, and define $\mathcal{E}^c$ to be a neighborhood of the origin in $\mathbf{R} \oplus \mathbf{R}^{d_k}$, and let $\mathcal{E}^s$ be a neighborhood of the origin in the range of $Q_k$ (equipped with the $H_{\ell,m}$ norm). We assume that $h$ and $\varphi$ are elements of the metric spaces

$$\begin{aligned} H_\sigma &= \{h : \mathcal{E}^c \to \mathcal{E}^s \mid h(0,0) = 0\,; \\ &\qquad \|h(\eta,y) - h(\tilde\eta,\tilde y)\|_{\ell,m} \le \sigma(|\eta-\tilde\eta| + \|y-\tilde y\|)\}\,, \\ K_\beta &= \{\varphi : \mathbf{R}_+ \times \mathcal{E}^c \to \mathcal{E}^c \mid \varphi_0(\eta,y) = (\eta,y)\,; \\ &\qquad \varphi_t(0,0) = (0,0) \text{ for all } t\,, \varphi_t \text{ is continuous in } t\,; \\ &\qquad \|\varphi_t(\eta,y) - \varphi(\tilde\eta,\tilde y)\| \le De^{\alpha t}(|\eta-\tilde\eta| + \|y-\tilde y\|) \text{ for all } t \in \mathbf{R}\}\,, \end{aligned} \tag{2.7}$$

where we use the ordinary Euclidean norm in $\mathbf{R}^{d_k}$ for $y$, and $\alpha$ is a positive constant, smaller than $(k/2n)$. We now apply the contraction mapping theorem to prove that $F$ and $G$ in (2.7) have fixed points. The only difference with the estimates of [G] are that in the present case the nonlinear terms "loose" derivatives and we must take advantage of the smoothing properties of the semigroup $Q_k e^{\mathcal{L}t} Q_k$ to recover them.

We first show, following the estimate of [G, p.258] that $F(h,\varphi)$ is in $H_\sigma$, paying particular attention to the way in which our case differs from Gallay's. From Corollary 2.3 we see that the nonlinear term $g$ in (2.2) is a Lipshitz function as a map from $\mathcal{E}^c \oplus \mathcal{E}^s$ to $H_{\ell-2n+1,m}$ and if $\ell_g$ is the Lipshitz constant of $g$, one can make $\ell_g$ arbitrarily small by restricting $(\eta,y,z)$ to a sufficiently small neighborhood of the origin. We can estimate

$$\begin{aligned} &\|F(h,\varphi)(\eta,y) - F(h,\varphi)(\tilde\eta,\tilde y)\|_{\ell,m} \\ &\le \int_0^{-\infty} d\tau\, \|(Q_k e^{-\mathcal{L}\tau} Q_k)\big(g\big((\varphi_\tau(\eta,y), h\circ\varphi_\tau(\eta,y)\big) - g\big(\varphi_\tau(\tilde\eta,\tilde y), h\circ\varphi_\tau(\tilde\eta,\tilde y)\big)\big)\|_{\ell,m} \\ &\le C\int_0^{-\infty} d\tau\, |\tau|^{-\frac{2n-1}{2n}} e^{-\frac{k+1}{2n}|\tau|} \\ &\qquad \times \big\|g\big(\varphi_\tau(\eta,y), h\circ\varphi_\tau(\eta,y)\big) - g\big(\varphi_\tau(\tilde\eta,\tilde y), h\circ\varphi_\tau(\tilde\eta,\tilde y)\big\|_{\ell-2n-1,m} \\ &\le C\int_0^{-\infty} d\tau\, |\tau|^{-\frac{2n-1}{2n}} e^{-\frac{k+1}{2n}|\tau|} \\ &\qquad \times \ell_g\big(\|\varphi_\tau(\eta,y) - \varphi_\tau(\tilde\eta,\tilde y)\| + \|h\circ\varphi_\tau(\eta,y) - h\circ\varphi_\tau(\tilde\eta,\tilde y)\|_{\ell,m}\big) \\ &\le C\int_0^{-\infty} d\tau\, |\tau|^{-\frac{2n-1}{2n}} e^{-\frac{k+1}{2n}|\tau|} \ell_g(1+\sigma)e^{\alpha|\tau|}(|\eta-\tilde\eta| + \|y-\tilde y\|) \\ &\le \frac{C(n,k)}{\alpha - (k+1)/(2n)} \ell_g(1+\sigma)(|\eta-\tilde\eta| + \|y-\tilde y\|)\,. \end{aligned} \tag{2.8}$$

Thus, we find that $F(h,\varphi)$ is in $H_\sigma$ provided

$$\frac{C(n,k)}{\alpha - (k+1)/(2n)} \ell_g(1+\sigma) < \sigma\,.$$

which we can insure by taking $(\eta, y, z)$ in a sufficiently small neighborhood of the origin (since as we noted above, this results in $\ell_g$ becoming small).

Note that the only difference between this estimate and the corresponding estimate of [G, p.258] is that we used the smoothing property of the semigroup that comes from Proposition 2.1, while Gallay did not assume that his semigroup was smoothing. The proof that $K(h, \varphi) \in K_\beta$ and the proofs that $F$ and $G$ are contractions follow as in [G, pp.258–259], with the one change that we must use the smoothing property of $Q_k e^{\mathcal{L}\tau} Q_k$ to overcome the loss of derivatives in $g$.

Once we have shown that $F$ and $G$ are contractions, the fixed point $h$ gives the invariant manifold whose existence is asserted in Theorem 2.4, though this shows only that $h$ is Lipshitz, not $\mathcal{C}^{1+\alpha}$ as claimed. To prove that $h$ is in fact $\mathcal{C}^{1+\alpha}$, one follows the proof of [G, Lemma 2.10], again with the sole change that when one estimates the factors of $Q_k e^{\mathcal{L}\tau} Q_k$ which occur in the mappings $\hat{F}$ and $\hat{G}$, one must use the smoothing of the semigroup.

Once one knows that the manifold exists, it is also easy to show that any solution which remains near the origin must approach a solution on the invariant manifold (see, *e.g.* [C]). Note that even though our non-linearity is quite smooth, we cannot hope, in general, to obtain an invariant manifold whose smoothness is greater than $\mathcal{C}^{1+\alpha}$, since this smoothness is related to the gap between the spectrum of $\Lambda_k$, and that of $Q_k \mathcal{L} Q_k$, (see, *e.g.* [LW]).

## 3. Applications

Here, we show how the existence of the invariant manifold implies Theorem 1.1 and related results. To prove Theorem 1.1, we assume that all terms in the non-linearity are irrelevant. This means that (2.3) holds. Suppose further that $k = 1$ and that $\ell > 2n - 1 + d/2$ and $m > 2(\ell + d + k + 1)$. These hypotheses guarantee that Theorem 2.4 applies and hence any solution near the origin must approach a solution on the invariant manifold, at a rate $\mathcal{O}\big(e^{\frac{2-\varepsilon}{2n}\tau}\big)$ in $H_{\ell,m}$.

The equations on the invariant manifold can be written as a system of ordinary differential equations:

$$\begin{aligned} \dot{y}_0 &= \big\langle \varphi_0^* | f\big(y, h(\eta, y), \eta\big)\big\rangle \,, \\ \dot{y}_{1,j} &= -\tfrac{1}{2n} y_{1,j} + \big\langle \varphi_{1,j}^* | f\big(y, h(\eta, y), \eta\big)\big\rangle \,, \quad j = 1, \ldots, d \,, \\ \dot{\eta} &= -\tfrac{1}{2n}\eta \,, \end{aligned} \tag{3.1}$$

where $\varphi_0^*$ and $\varphi_{1,j}^*$ are the projections onto the spectral subspace of $\lambda_0$ and $\lambda_1 = -1/(2n)$, respectively. Note that $\lambda_1$ has a $d$-dimensional spectral subspace.

The important observation to make at this point is that since the non-linearity is assumed to be irrelevant, there exist constants $C_0$ and $C_1$ such that

$$\big|\big\langle \varphi_0^* | f\big(y, h(\eta, y), \eta\big)\big\rangle\big| \le C_0 \eta^p \,, \quad \big|\big\langle \varphi_1^* | f\big(y, h(\eta, y), \eta\big)\big\rangle\big| \le C_1 \eta^p \,,$$

for some $p \ge 1$. Since $\eta(\tau) = e^{-\tau/(2n)}\eta(0)$, this implies immediately that solutions of (3.1) behave as

$$\begin{aligned} y_0(\tau) &= B^* + \mathcal{O}(e^{-\tau/(2n)}) \,, \\ y_{1,j}(\tau) &= \mathcal{O}(e^{-\tau/(2n)}) \,. \end{aligned}$$

Furthermore, because the nonlinear terms are proportional to $\eta^p$, for some $p \geq 1$, the function $h$ whose graph gives the invariant manifold will also be proportional to $\eta^p$. This means that as $\tau \to 0$ (*i.e.*, $\eta \to 0$), the center manifold becomes "flat" —that is, it coincides with the eigendirections corresponding to the eigenvalues $\lambda_0$ and $\lambda_1$. The eigenfunction with eigenvalue 0 of $\mathcal{L}$ is $e^{-(p\cdot p)^n}$, or taking inverse Fourier transform, $f^*$, *cf.* (1.16). Thus, in $H_{\ell,m}$ solutions of (1.9) behave as

$$\tilde{v}(p,\tau) \;=\; B^* e^{-(p\cdot p)^n} + \mathcal{O}(e^{-\tau/(2n)}) \,.$$

Reverting from scaling variables to the unscaled variables $u(x,t)$ and using the Sobolev lemma to estimate the $\mathrm{L}^\infty$ norm in terms of the $H_{\ell,m}$ norm, we obtain Theorem 1.1. Since we observed above that the non-linearity in the Cahn-Hilliard equation is irrelevant when $d \geq 3$, we immediately see in this case that (1.18) holds for initial conditions which are close to $u \equiv 3^{-1/2}$, which yields Corollary 1.2.

## 4. The critical case

We now consider the Cahn-Hilliard equation in dimension $d = 2$, which is the critical case in terms of the renormalization group terminology [BKL]. This means that in some non-linear terms the inequality (1.15) becomes an equality.

In the Cahn-Hilliard equation, when $d = 2$ (and $n = 2$), we see that the quadratic term is critical, and the cubic term is irrelevant. Note that Theorem 2.4 still implies the existence of an invariant manifold tangent at the origin to the eigenspace of $\lambda_0$. This means that when written in the form of (2.2), the non-linearity can be written as the sum of 2 pieces—one quadratic in $y$ and $z$ which is independent of $\eta$ (and hence critical) and a cubic piece in $y$ and $z$ which is linear in $\eta$ (and hence irrelevant). This implies that the Eqs.(3.1), when reduced to the invariant manifold, take the form

$$\begin{aligned}
\dot{y}_0 \;&=\; \langle \varphi_0^* | f^{(2)}(y, h(\eta,y),\eta)\rangle + \langle \varphi_0^* | f^{(3)}(y, h(\eta,y),\eta)\rangle \,, \\
\dot{y}_{1,j} \;&=\; -\tfrac{1}{4} y_{1,j} + \langle \varphi_{1,j}^* | f^{(2)}(y, h(\eta,y),\eta)\rangle + \langle \varphi_{1,j}^* | f^{(3)}(y, h(\eta,y),\eta)\rangle \,, \quad j = 1,2 \,, \\
\dot{\eta} \;&=\; -\tfrac{1}{4}\eta \,.
\end{aligned} \tag{4.1}$$

We now exploit the form of the non-linear term in (1.17), namely $4\sqrt{3}\Delta(w^2) + 4\Delta(w^3)$, plus the fact that the eigenfunction $\varphi_0^* \equiv 1$. Thus if we integrate by parts, we find that

$$\langle \varphi_0^* | f^{(2)}(y, h(\eta,y),\eta)\rangle + \langle \varphi_0^* | f^{(3)}(y, h(\eta,y),\eta)\rangle \;=\; 0 \,,$$

so that in (4.1), $\dot{y}_0 \equiv 0$ and thus $y_0(t) = y_0(0)$. This means that the invariant manifold contains a curve of fixed points, and that any solution near the origin approaches one of these fixed points with a rate $\mathcal{O}(e^{-\tau/4})$. Note further that since the quadratic term in the nonlinearity is independent of $\eta$ in this case, the center manifold will not be "flat" as it was in the case of an irrelevant nonlinearity. Thus, while to lowest order, the fixed points in the invariant manifold will be proportional to the eigenvector with eigenvalue zero, there will be higher order corrections which can be computed perturbatively by computing the terms in the Taylor series for the invariant manifold.

Since from (4.1) we also see that $y_{1,j} = \mathcal{O}(e^{-\tau/4})$, we find upon reverting to the unscaled variables the second main result:

**Theorem 4.1.** *For $d = 2$, if the initial conditions of the Cahn-Hilliard equation are sufficiently close in $H_{5,18}$ to the stationary state $u \equiv 3^{-1/2}$, then the solution behaves asymptotically as*

$$u(x,t) \;=\; \frac{1}{3^{1/2}} + \frac{1}{t^{1/2}}\tilde{f}^*\bigl(\frac{x}{t^{1/4}}\bigr) + \mathcal{O}\bigl(\frac{1}{t^{3/4-\varepsilon}}\bigr) \;,$$

*where $\tilde{f}^*$ is one of the fixed points on the invariant manifold. We denote it by $\tilde{f}^*$ to indicate that to lowest order it is equal to the function $f^*$ which is the eigenfunction of the operator $\mathcal{L}$ with eigenvalue zero, but it will have higher order corrections coming from the curvature of the invariant manifold.*

**Remark.** Note that this result implies that in contrast to the situation in $d \geq 3$, the long-time asymptotics are no longer given be the solution of the linearized equation—nonlinear effects enter even at lowest order.

## 5. The relevant case

Here, we consider the case of $d = 1$ where one term of the non-linearity is relevant. This necessitates a change of strategy, because the quadratic term is proportional to $\eta^{-1}$ and hence the non-linear terms in (2.2) are not smooth enough to apply the invariant manifold theorem. In order to circumvent this difficulty, we choose a scaling different from (1.7). Consider again the Cahn-Hilliard equation, (1.17), with $u = 3^{-1/2} + w$. In $d = 1$, we get

$$\frac{\partial w}{\partial t} \;=\; -\frac{\partial^4}{\partial x^4}w + 4\sqrt{3}\frac{\partial^2}{\partial x^2}\bigl(w^2\bigr) + 4\frac{\partial^2}{\partial x^2}\bigl(w^3\bigr) \;. \tag{5.1}$$

Now let $w(x,t) = t^{-1/2}W(x/t^{1/4}, \log t)$. Then $W$ satisfies

$$\frac{\partial W}{\partial \tau} \;=\; -\partial_\xi^4 W + \frac{1}{4}\xi\cdot\partial_\xi W + \frac{1}{2}W + 3^{1/2}\partial_\xi^2\bigl(W^2\bigr) + e^{-\tau/2}\partial_\xi^2\bigl(W^3\bigr) \;. \tag{5.2}$$

Proceeding as in the other cases, we define the linear operator $-\partial_\xi^4 + \frac{1}{4}\xi\partial_\xi + \frac{1}{2}$, which in Fourier variables becomes

$$\mathcal{L}_1 \;=\; -p^4 - \tfrac{1}{4}p\partial_p + \tfrac{1}{4} \;,$$

so that it has eigenvalues $\mu_j = \frac{1-j}{4}$, $j = 0, 1, \ldots$. Thus, unlike the operator $\mathcal{L}$, we have one eigenvalue lying in the right half-plane. Let $\tilde{\eta} = e^{-\tau/8}$, and let $y_0$ and $y_1$ denote the amplitudes of the eigenvectors with eigenvalues $\mu_0$ and $\mu_1$. Then (5.2) takes the form

$$\begin{aligned}
\dot{y}_0 \;&=\; \tfrac{1}{4}y_0 + f_0(y_0, y_1, \eta, y^\perp) \;,\\
\dot{y}_1 \;&=\; f_1(y_0, y_1, \eta, y^\perp) \;,\\
\dot{\eta} \;&=\; -\tfrac{1}{8}\eta \;,\\
\dot{y}^\perp \;&=\; Q\mathcal{L}_1 y^\perp + f^\perp(y_0, y_1, \eta, y^\perp) \;.
\end{aligned} \tag{5.3}$$

Here, $Q$ is the projection onto the complement of the eigenspaces corresponding to $\mu_0$ and $\mu_1$, $y^\perp = QW$, and $f_0$, $f_1$, and $f^\perp$ are the projections of the non-linearity onto the various subspaces.

Since the spectrum of $Q\mathcal{L}_1 Q$ lies in the half-plane Re $\mu \le \frac{1}{4}$, we can construct an invariant manifold for (5.3) which is the graph of a function $h^\perp(y_0, y_1, \eta)$, and every solution of (5.3) which remains in a neighborhood of the origin will approach this manifold at a rate $\mathcal{O}(e^{-\tau/4})$. What is more, the equations on the invariant manifold are extremely simple in this case, since the projections onto the "0" and "1" components correspond to integrating with respect to the functions 1 and $x$, respectively. Applying these projections to the non-linearity and integrating once, resp. twice by parts, we see that these projections of the non-linear terms vanish. Thus, the equations on the invariant manifold of (5.3) are simply

$$\dot{y}_0 = \tfrac{1}{4} y_0 \,, \quad \dot{y}_1 = 0 \,, \quad \dot{\tilde{\eta}} = -\tfrac{1}{8}\tilde{\eta} \,.$$

Note that these equations again imply that there is a line of fixed points in the invariant manifold corresponding to $y_0 = 0$ and $y_1 = y_1(0)$. Just as in the two dimensional case, these fixed points will be tangent at the origin to the eigenvector of $\mathcal{L}_1$ with eigenvalue zero, and higher order corrections to this first approximation can be computed perturbatively from the equation for the invariant manifold. Thus, *as long as the solution of (5.3) remains in a neighborhood of the origin*, it will be of the form

$$\begin{aligned} y_0(\tau) &= e^{\tau/4} y_0(0) \,, \\ y_1(\tau) &= y_1(0) \,, \\ \tilde{\eta}(\tau) &= e^{-\tau/8}\tilde{\eta}(0) \,, \\ y^\perp(\tau) &= \mathcal{O}(e^{-\tau/4}) \,, \end{aligned} \tag{5.4}$$

and we see that the solution either leaves the neighborhood of the origin, or it approaches one of the fixed points on the invariant manifold. Note that the solutions that remain near the origin must have $y_0 = 0$. Thus:

**Theorem 5.1.** *Suppose that the initial condition of the Cahn-Hilliard equation is of the form $u_0 = 3^{-1/2} + w_0$ with $w_0$ small in the $H_{\ell,m}$ norm for some $\ell \ge 4$ and $m \ge 15$. Assume furthermore that $\int_{-\infty}^{\infty} \mathrm{d}x\, w_0(x) = 0$. Then the solution is of the form*

$$u(x,t) = \frac{1}{3^{1/2}} + \frac{1}{t^{1/2}} f^{**}\Big(\frac{x}{t^{1/4}}\Big) + \mathcal{O}\Big(\frac{1}{t^{3/4-\varepsilon}}\Big) \,,$$

*where $f^{**}$ is one of the fixed points on the invariant manifold.*

**Proof.** The proof is an obvious modification of the one of Theorem 1.1, taking into account the special form of the eigenfunctions corresponding to the eigenvalues $\mu_0$ and $\mu_1$.

## Appendix. Bounds on the linear semi-group

In this appendix, we sketch the proof of Proposition 2.1. The proof is quite similar to the estimates on the linear semi-group in Appendix B of [EWW], (which was given for the case of a one-dimensional Laplacian, or in the present notation $n = d = 1$) so we concentrate only on the points where the present argument differs from the one in [EWW].

We begin with the representation

$$\big(e^{\tau\mathcal{L}}v\big)(x) \;=\; \frac{e^{\frac{\tau d}{2n}}}{2\pi^d}\int \mathrm{d}^d z\, g(z,\tau)v\big(e^{\frac{\tau}{2n}}(x+z)\big)\;, \tag{A.1}$$

where

$$g(z,\tau) \;=\; \int \mathrm{d}^d k\, e^{ik\cdot z}\exp\big(-(k\cdot k)^n(1-e^{-\tau})\big)\;. \tag{A.2}$$

As in [EWW], the action of the semi-group is analyzed by considering separately the behavior of the part far from the origin and that close to the origin. The new difficulty here is that we do not have an explicit representation of $g$ as in the case $n = 1$. However, the technique of estimating the long-time behavior will remain essentially the same. Let $\chi_R$ be a smooth cutoff function which vanishes for $|x| < R$ and is equal to 1 for $|x| > 4R/3$. We start by studying the region far from the origin. The analog of Proposition B.2 of [EWW] is

**Proposition A.1.** *For every $\ell \geq 0$ and every $m \geq 0$, there exist a $\gamma > 0$ and a $C(\ell,m) < \infty$ such that for all $v \in H_{\ell,m}$ one has*

$$\Big\|\chi_R e^{\tau\mathcal{L}}v\Big\|_{\ell,m} \;\leq\; \frac{C(\ell,m)}{\big(a(\tau)\big)^{\frac{q}{2n}}}e^{(\frac{\tau}{2n})(d+\ell)}\Big(e^{-\tau m/2}+e^{-\gamma R^{2n/(2n-1)}}\Big)\,\|v\|_{\ell-q,m}\;, \tag{A.3}$$

*for $q = 0, 1, \ldots, 2n-1$. Here, $a(\tau) = 1 - e^{-\tau}$.*

The crucial step in proving this estimate is to derive the asymptotics of $g(z,\tau)$ for large $z$. This will replace the explicit (Gaussian) estimates for the $d = 1$, $n = 1$ case analyzed in [EWW]. This estimate is provided by the following

**Proposition A.2.** *The kernel $g(z,\tau)$ decays faster than any inverse power of $z$ for $|z|$ large. In fact, one has the estimate*

$$|g(z,\tau)| \;\leq\; Ca(\tau)^{-\frac{d}{2n}}\exp\big(-\gamma(|z|^{2n}/a(\tau))^{\frac{1}{2n-1}}\big)\;, \tag{A.4}$$

*for some $\gamma = \gamma(n,d) > 0$.*

**Remark.** If $n = 1$, we recover the explicit bound on the Green's function:

$$\frac{C}{a(\tau)^{d/(2n)}}e^{-\gamma(2,d)z^2/a(\tau)}\;.$$

**Proof.** We need to estimate the quantity

$$I_{n,d} = \int \mathrm{d}^d k\, e^{-a(\tau)\left(\sum_{j=1}^d k_j^2\right)^n} e^{i\sum_{j=1}^d k_j x_j} . \tag{A.5}$$

By rotational symmetry, it suffices to bound the preceding expression for $x = (x_1, 0, \dots, 0)$ with $x_1 \geq 0$. Setting $x_1 = 2na(\tau)z^{2n-1}$, and $k = (p, q)$, with $p \in \mathbf{R}$, and $q \in \mathbf{R}^{d-1}$, this means that we must bound

$$X = \int \mathrm{d}p\, \mathrm{d}^{d-1}q \exp\bigl(-a(\tau)(p^2 + q \cdot q)^n - 2inpa(\tau)z^{2n-1}\bigr) .$$

If we rescale the variables as $p = zt$, and $q = zs$, then we have

$$X = z^d \int \mathrm{d}t\, \mathrm{d}^{d-1}s \exp\bigl(-a(\tau)z^{2n}\bigl((t^2 + s \cdot s)^n + 2int\bigr)\bigr) .$$

**Remark.** Note that the polynomial $(t^2 + s \cdot s)^n - 2int$ is independent of $z$.

We will bound $X$ by taking advantage of the fact that the integrand is an entire function and translate the contour of integration so that it passes through at least one critical point of the exponent. These critical points occur at $s = 0$ and the roots of $t^{2n-1} = -i$ —that is, at the points $t_k = \exp(i\frac{\pi(4k+1)}{2(2n-1)})$, $k = 0, 1, 2, \dots, 2n-2$.

Inserting this expression into the exponent of the integrand of $X$, we see that the value of the polynomial at the critical points is

$$\begin{aligned} -a(\tau)z^{2n} &\left(\exp(i\frac{2n\pi(4k+1)}{2(2n-1)}) - 2ni\exp(i\frac{\pi(4k+1)}{2(2n-1)})\right) \\ &= (2n-1)a(\tau)z^{2n}\exp(i\frac{\pi(4k+2n)}{2(2n-1)}) . \end{aligned}$$

In particular, if we take $k = 0$, then the real part of the critical value is

$$(2n-1)a(\tau)z^{2n}\cos\bigl(\pi/2 + \pi/(4n-2)\bigr) \approx -a(\tau)z^{2n} \cdot \frac{\pi}{2} ,$$

when $n$ is large (and is negative for all $n > 0$). Integrating over the region $\mathbf{R} + t_0$ and observing that there is only one critical point on this line, we get, using standard techniques of stationary phase:

$$X \approx a(\tau)^{-d/(2n)} e^{-C_n a(\tau) z^{2n}} ,$$

with $C > 0$ and $C_n \to \pi/2$ as $n \to \infty$, when $z \to \infty$. Reverting to the original variables, this leads to

$$I_{n,d} \approx a(\tau)^{-d/(2n)} e^{-D_n x^{2n/(2n-1)}/a(\tau)^{1/(2n-1)}} ,$$

where $D_n = C_n/(2n)^{1/(2n-1)}$. This completes the proof of Proposition A.2.

We now consider the action of the semi-group on functions localized inside a ball of radius $R$. A key observation here is the following lemma. Let $\varphi_0(x)$ denote the eigenfunction (written in position space) of $\mathfrak{L}$ with eigenvalue 0 and let $T(x) = \varphi_0(x)^{1/2}$ (note that $\varphi_0(x) > 0$ for all $x$).

**Lemma A.3.** *The operator* $H = T^{-1}\mathfrak{L}T$ *is self-adjoint on (a dense domain in)* $\mathrm{L}^2(\mathbf{R}^d)$ *and has the same eigenvalues as* $\mathfrak{L}$.

**Remark.** The domains of $\mathfrak{L}$, $T$ and $H$ have to be chosen carefully here. The correct choices have been explained in detail in [EWW, p.197].

**Proof.** The proof is a straightforward calculation. We note further that if $\tilde{\varphi}_\alpha$ are the eigenfunctions of $\mathfrak{L}$ then the eigenfunctions of $H$ are $\psi_\alpha(x) = \big(T^{-1}\varphi_\alpha\big)(x)$.

If we take the inverse Fourier transform of the eigenfunctions $\tilde{\varphi}_\alpha(p)$ of (1.12), we see that

$$|\varphi_\alpha(x)| \;\approx\; C|x|^{|\alpha|}\exp(-\gamma|x|^{\frac{2n}{2n-1}})\;,$$

for some $\gamma > 0$, using the same sort of estimates as those used to bound the kernel $g$ of the semi-group. Thus, for $|x|$ sufficiently large, we get

$$|\psi_\alpha(x)| \;\approx\; C|x|^{|\alpha|}\exp(-\tfrac{1}{2}\gamma|x|^{\frac{2n}{2n-1}})\;.$$

The usefulness of introducing the operator $H$ is that it is sectorial, since it is self-adjoint and bounded below. Therefore, the associated semi-group can be estimated from spectral information alone. In particular, if $P_k^{(0)}$ denotes the projection onto the spectral subspace spanned by the eigenfunctions with eigenvalues 0, $-\frac{1}{2n}$, $\frac{-2}{2n}$,$\dots$, $\frac{-k}{2n}$, and $Q_k^{(0)}$ is defined by $Q_k^{(0)} = 1 - P_k^{(0)}$, then we have a bound on the operator norm of $Q_k^{(0)}e^{\tau H}Q_k^{(0)}$

$$\|Q_k^{(0)}e^{\tau H}Q_k^{(0)}\| \;\le\; C_k e^{-\tau(k+1)/(2n)}\;. \tag{A.6}$$

We can use this information to bound the semi-group associated with $\mathfrak{L}$. Note that if we denote by $P_k$ and $Q_k$ the projection associated with the spectral subspaces of $\mathfrak{L}$ (as we did for $H$), then we have the identity:

$$e^{\tau\mathfrak{L}}Q_k v \;=\; e^{\tau\mathfrak{L}}Q_k(1-\chi_R)v + e^{\tau\mathfrak{L}}Q_k\chi_R v\;.$$

Since $\chi_R v$ is localized away from the origin, it can be studied with the help of Proposition A.1, so we focus on the other term. There we get

$$\begin{aligned}\|e^{\tau\mathfrak{L}}Q_k(1-\chi_R)v\|_{\ell,m} \;&=\; \|TT^{-1}e^{\tau\mathfrak{L}}TT^{-1}Q_kTT^{-1}(1-\chi_R)v\|_{\ell,m}\\ &=\; \big\|T\big(e^{\tau H}Q_k^{(0)}\big)\big(T^{-1}(1-\chi_R)v\big)\big\|_{\ell,m}\\ &\le\; C\exp\big(-\tau\frac{k+1}{2n}\big)\big\|T^{-1}(1-\chi_R)v\big\|_{\ell,m}\;.\end{aligned}$$

Using now the bounds on $\varphi_0$, we see that $|T^{-1}(x)| \le C\exp(\gamma|x|^{\frac{2n}{2n-1}})$ and that $(1-\chi_R v)(x) = 0$ when $|x| > 4R/3$, we get

$$\big\|T^{-1}(1-\chi_R)v\big\|_{\ell,m} \le C\exp\big(\gamma(4R/3)^{\frac{2n}{2n-1}}\big)\|v\|_{\ell,m} ,$$

so that finally

$$\Big\|e^{\tau\mathcal{L}}Q_k(1-\chi_R)v\Big\|_{\ell,m} \le C\exp\big(\gamma(4R/3)^{\frac{2n}{2n-1}}\big)\exp\big(-\tau\frac{k+1}{2n}\big)\|v\|_{\ell,m} .$$

Thus we have proven:

**Proposition A.4.** *Under the hypotheses of Proposition A.1, there exist constants $C(\ell,m) > 0$ and $\gamma > 0$, such that for all $v \in H_{\ell,m}$, one has*

$$\Big\|e^{\tau\mathcal{L}}Q_k(1-\chi_R)v\Big\|_{\ell,m} \le C\exp\big(\gamma(4R/3)^{\frac{2n}{2n-1}}\big)\exp\big(-\tau\frac{k+1}{2n}\big)\|v\|_{\ell,m} .$$

We now return to the:

**Proof of Proposition 2.1.** As in [EWW] it is only necessary to consider the term with highest derivative in $\|\chi_R e^{\tau\mathcal{L}}v\|_{\ell,m}$. All other terms are easier to estimate. Also, as in that paper, we use the fact that

$$\mathrm{D}^\ell e^{\tau\mathcal{L}} = e^{\tau\ell/(2n)}e^{\tau\mathcal{L}}\mathrm{D}^\ell , \tag{A.7}$$

where $\mathrm{D}^\ell$ is a shorthand notation for a product of derivatives w.r.t. the $x_j$ of total degree $\ell$. Thus,

$$\big(e^{\tau\mathcal{L}}\mathrm{D}^\ell v\big)(x) = \frac{e^{\tau d/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\, g(z,\tau)\big(\mathrm{D}^\ell v\big)(e^{\tau/(2n)}(x+z)) . \tag{A.8}$$

First consider the $q=0$ case of (A.3). Then

$$\|\chi_R e^{\tau\mathcal{L}}v\|_{\ell,m} \le \frac{e^{\tau d/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\,|g(z,\tau)|\big\|w^m\chi_R\big(\mathrm{D}^\ell v\big)(e^{\tau/(2n)}(.+z))\big\|_2 , \tag{A.9}$$

where $w$ is the operator of multiplication by $(1+x\cdot x)^{1/2}$. Note that the conclusions of Lemma B.4 of [EWW] do not depend on the exact form of $g$ and so it also holds in the present situation and we have

**Lemma A.5.** *One has the bounds*

$$\big\|w^r\chi_R v(e^{\tau/(2n)}(.+z))\big\|_2^2 \le \begin{cases} Ce^{-\frac{\tau m}{n}}\|v\|_{0,m}^2 , & \text{if } |z| \le 7R/8 , \\ C(1+|z|^2)^r\|v\|_{0,m}^2, & \text{if } |z| > 7R/8 . \end{cases}$$

**Remark.** Note that the proof in [EWW, p.199] is also unaffected by the dimension $d$ in which we work.

Now use Lemma A.5 to bound the integral in (A.9) by writing it as an integral over $|z| \le 7R/8$ and an integral over $|z| > 7R/8$. The integral over $|z| \le 7R/8$ is bounded with the aid of Lemma A.5 as

$$C\frac{e^{\tau d/(2n)}}{(2\pi)^d}\int_{|z|\le 7R/8} |g(z,\tau)|e^{-\tau m/2}\|v\|_{\ell,m} \;\le\; C(n)e^{\frac{\tau d}{2n}}e^{-\frac{\tau m}{2n}}\|v\|_{\ell,m}\;, \tag{A.10}$$

where the last step used the estimates of Theorem 4.1 to show that $\int \mathrm{d}z|g(z,\tau)| \le C$, with $C$ independent of $\tau$.

To estimate the integral in the outer region, we use the second part of Lemma A.5 and then bound it by

$$C\frac{e^{\tau d/(2n)}}{(2\pi)^d}\int_{|z|>7R/8} |g(z,\tau)|(1+|z|^2)^{m/2}\|v\|_{\ell,m} \;\le\; C(n,m)e^{\tau d/(2n)}\exp\bigl(-\gamma R^{\frac{2n}{2n-1}}\bigr)\|v\|_{\ell,m}\;, \tag{A.11}$$

for some $\gamma > 0$, where, again, we have used the estimates of decay in Proposition A.2 both to extract the factor of $\exp\bigl(-\gamma R^{\frac{2n}{2n-1}}\bigr)$ as well as to bound the integral over $z$. Combining (A.1), (A.10), and (A.11), we get the $q = 0$ case of (A.3).

We next indicate how to treat the $q > 0$ cases of (A.3). Consider the case $q = 1$. We can rewrite (A.8) by integrating by parts once w.r.t. one component of $z$, for example $z_1$. Then,

$$\bigl(e^{\tau\mathcal{L}}\mathrm{D}^\ell v\bigr)(x) \;=\; \frac{e^{\tau d/(2n)}e^{-\tau/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\bigl(\mathrm{D}_{z_1}g\bigr)(z,\tau)\bigl(\mathrm{D}^{\ell-1}v\bigr)\bigl(e^{\tau/(2n)}(x+z)\bigr)\;. \tag{A.12}$$

Differentiating (A.2) w.r.t. $z_1$ gives

$$\bigl(\mathrm{D}_{z_1}g\bigr)(z,\tau) \;=\; \int \mathrm{d}^d k\, ik_1\; \exp(iq\cdot z)\exp\bigl((k\cdot k)^n(1-e^{-\tau})\bigr)\;. \tag{A.13}$$

To estimate (A.13), first replace $k$ by $p_j = a(\tau)^{1/(2n)}k_j$, where, as before $a(\tau) = 1 - e^{-\tau}$. Then,

$$\bigl(\mathrm{D}_{z_1}g\bigr)(z,\tau) \;=\; \frac{i}{a(\tau)^{d/(2n)}}\frac{1}{a(\tau)^{1/(2n)}}\int \mathrm{d}^d p\, p_1\exp\bigl((p\cdot p)^n\bigr)\exp\bigl(ip\cdot z/a(\tau)^{1/(2n)}\bigr)\;. \tag{A.14}$$

We estimate this integral by using the method of stationary phase as in the proof of Proposition A.2. The extra factor of $p_1$ does not cause any trouble as it is easily offset by the exponentially decaying terms.

One now uses the Schwarz inequality to rewrite

$$\|\chi_R e^{\tau\mathcal{L}}\mathrm{D}^\ell v\|_{\ell,m} \;\le\; \frac{e^{\tau d/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\,\Bigl|\mathrm{D}_{z_1}g(z,\tau)\Bigr|\;\Bigl\|w^m\chi_R\bigl(\mathrm{D}^{\ell-1}v\bigr)\bigl(e^{\tau/2}(.+z)\bigr)\Bigr\|_2\;, \tag{A.15}$$

and then proceeds as in the case when $q = 0$, breaking the integral over $z$ into the same two pieces as before. These two pieces are then estimated with the aid of Lemma A.5. Note that while the factor $a(\tau)^{-d/(2n)}$ of (A.14) will be absorbed when one integrates w.r.t $z$, the remaining factor of $a(\tau)^{-1/(2n)}$ will remain in the final bound of (A.3). The bounds for $q = 2, 3, \ldots 2n-1$ follow in a similar fashion.

To complete the proof of Proposition 2.1, first rewrite

$$e^{\tau\mathcal{L}}Q_k = e^{\tau\mathcal{L}/2}Q_k e^{\tau\mathcal{L}/2} = e^{\tau\mathcal{L}/2}Q_k\chi_R e^{\tau\mathcal{L}/2} + e^{\tau\mathcal{L}/2}Q_k(1-\chi_R)e^{\tau\mathcal{L}/2}\,. \tag{A.16}$$

The second of these terms involves an estimate of the action of $e^{\tau\mathcal{L}/2}Q_k$ on a function localized near the origin, so by Proposition A.4, we get a bound

$$\|e^{\tau\mathcal{L}/2}Q_k(1-\chi_R)e^{\tau\mathcal{L}/2}v\|_{\ell,m} \le C_q \exp\left(\gamma(\frac{4R}{3})^{2n/(2n-1)} - \tfrac{1}{2}\frac{k+1}{2n}\tau\right)\|v\|_{\ell-q,m}\,. \tag{A.17}$$

We use Proposition A.1 to bound the first term of (A.16):

$$\frac{C(\ell,m)}{a(\tau)^{\frac{q}{(2n)}}}e^{\frac{\tau}{2n}(\ell+d)}\left(e^{-\frac{\tau m}{4n}} + e^{-\gamma R^{2n/(2n-1)}}\right)\|v\|_{\ell-q,m}\,. \tag{A.18}$$

As a preliminary step, we note that if we first choose $m$ and $R$ such that

$$e^{\frac{\tau}{2n}(\ell+d)}\left(e^{-\frac{\tau m}{4n}} + e^{-\gamma R^{2n/(2n-1)}}\right) \le Ce^{-\mu((k+1)/(2n))\tau}\,,$$

then for sufficiently small $\mu$ (roughly speaking $\mu \sim \frac{1}{2}\big(1+(4/3)^{2n/(2n-1)}\big)^{-1}$), the Eqs.(A.17) and (A.18) imply

$$\|e^{\tau\mathcal{L}}Q_k\| \le \frac{C}{a(\tau)^{q/(2n)}}e^{-\mu\frac{k+1}{2n}\tau}\|v\|_{\ell-q,k} = \frac{C}{a(\tau)^{q/(2n)}}e^{-\mu|\lambda_{k+1}|\tau}\|v\|_{\ell-q,k}\,.$$

**Remark.** Note that this requires that we choose $m > 2(\ell+d+\mu(k+1))$, which is where the restriction on $m$ in Proposition 2.1 (and hence Theorem 1.1 ) arises.

This shows that the projection of the semi-group onto the complement of the eigenspace spanned by the first $k$ eigenvalues decays with a rate proportional to the eigenvalue $\lambda_{k+1}$. We can sharpen the decay rate so that we obtain a rate like $\exp\big(-(1-\varepsilon)|\lambda_{k+1}|\big)$ by the techniques of [EWW], (see Eq. B.14 and following) and this completes the proof of Proposition 2.1

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