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Strict Concavity of the Intersection Exponent for Brownian Motion in Two and Three Dimensions

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Abstract

The intersection exponent for Brownian motion is a measure of how likely Brownian motion paths in two and three dimensions do not intersect. We consider the intersection exponent $\xi(\lambda) = \xi_d(k, \lambda)$ as a function of λ and show that ξ has a continuous, negative second derivative. As a consequence, we improve some estimates for the intersection exponent; in particular, we give the first proof that the intersection exponent $\xi_3(1, 1)$ is strictly greater than the mean field prediction. The results here are used in a later paper to analyze the multifractal spectrum of the harmonic measure of Brownian motion paths.

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1 Introduction

Let $B^1, \ldots, B^k, W^1, \ldots, W^j$, be independent Brownian motions starting at the origin in $\mathbb{R}^d$. Let

$$T_n^i = \inf\{t : |B_t^i| = e^n\},$$

$$S_n^i = \inf\{t : |W_t^i| = e^n\},$$

and let Λ_n, Θ_n be the random sets

$$\Lambda_n = B^1[T_0^1, T_n^1] \cup \cdots \cup B^k[T_0^k, T_n^k],$$

$$\Theta_n = W^1[S_0^1, S_n^1] \cup \cdots \cup W^j[S_0^j, S_n^j].$$

If $d \geq 4$ [10],

$$\mathbf{P}\{\Lambda_n \cap \Theta_n = \emptyset\} = 1,$$

while for $d < 4$,

$$\lim_{n\to\infty} \mathbf{P}\{\Lambda_n \cap \Theta_n = \emptyset\} = \mathbf{P}\{\Lambda_\infty \cap \Theta_\infty = \emptyset\} = 0.$$

The intersection exponent $\xi = \xi_d(k, j)$ is defined for $d < 4$ by the relation

$$\mathbf{P}\{\Lambda_n \cap \Theta_n = \emptyset\} \approx e^{-n\xi},$$

where $\approx$ means that the logarithms of both sides are asymptotic. It can be shown by subadditivity (see e.g. [12]) that the exponent ξ is well defined, and in fact [15, 16], that $\approx$ can be replaced with $\asymp$ where $\asymp$ means that either side is bounded by a constant multiple of the other. Sometimes, $\zeta = \xi/2$ is defined to be the intersection exponent; it can be shown that as $t \to \infty$,

$$\mathbf{P}\{(B^1[0,t] \cup \cdots \cup B^k[0,t]) \cap (W^1[0,t] \cup \cdots \cup W^j[0,t]) = \emptyset\} \asymp t^{-\zeta}.$$

If $d = 1$, it is easy to see that

$$\begin{aligned}\mathbf{P}\{\Lambda_n \subset (-\infty, 0), \Theta_n \subset [0, \infty)\} &\leq \mathbf{P}\{\Lambda_n \cap \Theta_n = \emptyset\} \\ &\leq 2\mathbf{P}\{\Lambda_n \subset (-\infty, 1], \Theta_n \subset [-1, \infty)\}.\end{aligned}$$

Both sides can easily be estimated using the "gambler's ruin estimate" to show that

$$\mathbf{P}\{\Lambda_n \cap \Theta_n = \emptyset\} \asymp e^{-(j+k)n},$$

i.e., $\xi_1(j,k) = j+k$. This paper will consider dimensions 2 and 3 where the values of the intersection exponents are not known.

One of the reasons that properties of intersections of Brownian motions have been studied is that the properties are analogous to those in a number of models in equilibrium statistical physics. There is a critical dimension, four, above which the problem is easy and below which there are interesting nontrivial critical exponents, in this case the intersection exponents. Methods from (mostly, but not exclusively, nonrigorous) mathematical physics, such as renormalization group [1, 7, 11] and conformal field theory [8] have been applied to this problem, and the problem seems like an interesting, nontrivial problem that might be tractable, and hence is a good test problem for these techniques.

We restrict ourselves to dimensions two and three in this paper because these are the only dimensions in which the intersection exponent is defined and nontrivial. However, in $d = 4$ there is an interesting problem if we consider the probability

$$\mathbf{P}\{\mathrm{dist}(\Lambda_n, \Theta_n) \geq 1\}.$$

For $d > 4$, this probability tends to a positive constant as $n \to \infty$, while for $d = 4$ it tends to zero. The power of decay is given by

$$\mathbf{P}\{\mathrm{dist}(\Lambda_n, \Theta_n) \geq 1\} \asymp (\log n)^{-kj/2}.$$

Actually, we do not know if a proof of this fact has been written down. However, Albeverio and Zhou [2] have given a proof in the case $j = k = 1$ and the analogous problem for simple random walk for all j, k has been solved [13]. What happens in four dimensions is that the events

$$U_n = \{\mathrm{dist}(B^1[T_0^1, T_n^1], W^1[T_0^1, T_n^1]) \geq 1\}$$

and

$$V_n = \{\mathrm{dist}(B^2[T_0^1, T_n^1], W^1[T_0^1, T_n^1]) \geq 1\}$$

are independent up to a constant in the sense that

$$\mathbf{P}(U_n \cap V_n) \asymp \mathbf{P}(U_n)\mathbf{P}(V_n).$$

This is analogous to the "mean field" property of models in statistical physics; in general, one expects in critical phenomenon for the mean field property to hold at the critical dimension, but not below. One of the purposes of this paper is to show rigorously that the mean field property does not hold below the critical dimension.

There is a slightly different way of looking at the intersection exponent that will be useful for this paper. Fix k and suppose $B^1, \ldots, B^k$ are independent Brownian motions defined on the probability space $(\Omega, \mathbf{P})$, again starting at the origin. Suppose B is another Brownian motion starting at the origin, defined on the probability space $(\Omega_1, \mathbf{P}_1)$ with stopping times

$$T_n = \inf\{t : |B_1| = e^n\}.$$

We write $(\bar{\Omega}, \bar{\mathbf{P}})$ for $(\Omega \times \Omega_1, \mathbf{P} \times \mathbf{P}_1)$, so that $B^1, \ldots, B^k, B$ are independent on $(\bar{\Omega}, \bar{\mathbf{P}})$. Define the random variable Z_n on Ω by

$$Z_n = \mathbf{P}_1\{B[T_0, T_n] \cap \Lambda_n = \emptyset\}.$$

Note that if Θ_n is defined as above,

$$\mathbf{P}\{\Lambda_n \cap \Theta_n = \emptyset\} = \mathbf{E}[Z_n^j],$$

and hence the intersection exponent $\xi = \xi_d(k, j)$ satisfies

$$\mathbf{E}[Z_n^j] \approx e^{-n\xi}.$$

We can define the intersection exponent $\xi = \xi(k, \lambda)$ for any positive $\lambda > 0$ by the relation $\mathbf{E}[Z_n^\lambda] \approx e^{-n\xi}$. It is also well defined for $\lambda = 0$ if we make the convention $0^0 = 0$, i.e.,

$$\mathbf{P}\{Z_n > 0\} \approx e^{-n\xi(k,0)}.$$

For $d = 3$ the probability on the left hand side is one, so $\xi_3(k, 0) = 0$. For $d = 2$, this is not true; the exponent $\xi_2(k, 0)$ is called the disconnection exponent ($Z_n = 0$ if and only if Λ_n disconnects the origin from infinity.)

Other than $\xi_3(k, 0)$, the only value of the intersection exponent that is known rigorously is $\xi_d(2, 1) = \xi_d(1, 2) = 3 - d$. Some bounds are known. For the disconnection exponent $\xi = \xi_2(k, 0)$, we have [4, 25]

$$1/2\pi \leq \xi < .469, \quad k = 1,$$

$$1/\pi \leq \xi < .985, \quad k = 2.$$

Duplantier (see [9]) conjectured using nonrigorous conformal field theory that $\xi_2(1, 0) = 1/4$. The exponent $\xi_2(2, 0)$ is related to the Hausdorff dimension of the frontier or outer boundary of Brownian motion; it has been proved [17] that the dimension of the frontier is given by $2 - \xi_2(2, 0)$. Mandelbrot [22] conjectured that this dimension is 4/3 and hence has conjectured that $\xi_2(2, 0) = 2/3$. This is consistent with numerical simulation, see e.g. [23].

For $\xi = \xi_d(1, 1)$, the following rigorous bounds are known [4].

$$\xi \in [1 + \frac{1}{4\pi}, \frac{3}{2}), \quad d = 2,$$

$$\xi \in [\frac{1}{2}, 1), \quad d = 3.$$

The lower bounds are obtained from interpolating the exact values and bounds for $\xi_d(1, 2)$ and $\xi_d(1, 0)$ using concavity as we now describe. If $r > 1$, then

$$\mathbf{E}[Z_n^\lambda] \leq \mathbf{E}[Z_n^{r\lambda}]^{1/r},$$

and by Hölder's inequality, if $\lambda = a\lambda_1 + (1 - a)\lambda_2$ with $0 < a < 1$, then

$$\mathbf{E}[Z_n^\lambda] \leq (\mathbf{E}[Z_n^{\lambda_1}])^a (\mathbf{E}[Z_n^{\lambda_2}])^{1-a}.$$

Hence,

$$\xi_d(k, \lambda) \geq r^{-1} \xi_d(k, r\lambda), \tag{1}$$

$$\xi_d(k, \lambda) \geq a\xi_d(k, \lambda_1) + (1 - a)\xi_d(k, \lambda_2). \tag{2}$$

The upper bound for two dimensions uses the Beurling projection theorem (see Lemma 3.2). Nonrigorous conformal field theory predicts that $\xi_2(1, 1) = 5/4$ [8, 9], and this is consistent with simulations [5, 20]. For $d = 3$, there is no reason to believe that $\xi_3(1, 1)$ is rational. Simulations [5, 19] indicate that it is slightly less than .58.

The values of $\xi_d(k, \lambda)$ for $k = 2$ are closely related to the harmonic measure of a Brownian motion path, i.e., the hitting measure of another Brownian motion starting away from the path. To see this intuitively, note that at a typical point $x = B_t$ of a single Brownian path, there are two independent Brownian motions starting at x — the path of B_s for $s < t$ and the path of B_s for $s > t$. Probabilities of starting a Brownian motion near x and avoiding the path can be seen to be related to probabilities that another Brownian motion hits the first path near x (reverse the time). The exact value for $\xi_d(2, 1)$ can be obtained by studying a simple property of the harmonic measure. One other exponent inequality which was proved using this relationship [14] is

$$\xi_2(2, 2) \leq 3.$$

This is obtained by using the $\xi_2(2,1) = 2$ value and a theorem of Makarov [21] about harmonic measure of connected subsets of the plane. This theorem states that the harmonic measure is carried on a set of Hausdorff dimension one. The results in this paper are used in [18] to establish that the frontier of a Brownian path is multifractal; moreover, the multifractal spectrum can be derived from $\xi_d(2,\lambda)$.

In this paper, we fix d and k and consider the function $\xi(\lambda) = \xi_d(k,\lambda)$. By (2), ξ is a concave function of λ. One of the main goals of this paper is to show that ξ is *strictly* concave. In fact we show that for $\lambda > 0$, ξ has two continuous derivatives and

$$\xi''(\lambda) < 0.$$

In particular, (1) and (2) can be replaced with

$$\xi_d(k,\lambda) > r^{-1}\xi_d(k,r\lambda), \quad r > 1, \tag{3}$$

$$\xi_d(k,\lambda) > a\xi_d(k,\lambda_1) + (1-a)\xi_d(k,\lambda_2), \quad a \in (0,1), \;\; a\lambda_1 + (1-a)\lambda_2 = \lambda. \tag{4}$$

As a corollary we improve three of the known inequalities for the exponent to strict inequalities:

$$\xi_2(1,1) > 1 + \frac{1}{4\pi}, \quad \xi_3(1,1) > \frac{1}{2}, \quad \xi_2(2,2) < 3.$$

The first two of these two inequalities follow immediately from the known estimates and (3) and (4). The third is not so immediate, and we will discuss it in the next section where we derive the more general estimate

$$\xi_2(2,\lambda) < 1 + \lambda, \quad d = 2, \; \lambda \neq 1. \tag{5}$$

The inequality for $\xi_3(1,1)$ is important. This is the first rigorous proof that the exponent does not take on its "mean field" values of $1/2$. Mean field is not a well-defined notion, but if the mean field theory were to hold, then we would expect

$$\bar{\mathbf{P}}\{B[0,T_n] \cap (B^1[0,T_n^1] \cup B^2[0,T_n^1]) = \emptyset\}$$
$$\approx \bar{\mathbf{P}}\{B[0,T_n] \cap B^1[0,T_n^1] = \emptyset\}\bar{\mathbf{P}}\{B[0,T_n] \cap B^2[0,T_n^2] = \emptyset\}.$$

Note that if this were to be true, then $\xi(2)$ would equal $2\xi(1)$. As previously noted, the intersection exponent is analogous to a number of critical exponents in statistical physics for which mean field values are not taken on below the critical dimension.

We will make the following convention about constants in this paper. We will start by choosing $0 < \lambda_1 < \lambda_2 < \infty$ and considering $\lambda \in [\lambda_1, \lambda_2]$. Constants $c, c_1, c_2, \ldots$ and β are positive constants that depend only on $d, k, \lambda_1, \lambda_2$. The value of c, c_1, c_2, β may change from line to line; the values of other constants will not change.

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2 Main Results

In this section we give an overview of the main results of the paper. As we will see, analysis of the intersection exponent is essentially the same as determining the large deviation behavior of the random variable $-\log Z_n$. It will be convenient to look at a slightly different random variable than the Z_n discussed in the first section. (In this paper we will look at a number of slightly different versions of Z_n; they all will have the same exponents.) Let u denote the unit vector in $\mathbb{R}^d$ whose first component is 1 and let $\bar{\gamma}$ denote the set

$$\bar{\gamma} = \{-tu : 0 \leq t \leq 1\},$$

Assume we have k Brownian motions $B^1, \ldots, B^k$ starting at $-u$ defined on the probability space $(\Omega, \mathbf{P})$. As before we set

$$\Lambda_n = B^1[0, T_n^1] \cup \cdots \cup B^k[0, T_n^k].$$

Suppose we have another Brownian motion B_t defined on $(\Omega_1, \mathbf{P}_1)$. Let

$$Z_n = \mathbf{P}_1\{B(0, T_n] \cap (\bar{\gamma} \cup \Lambda_n) = \emptyset \mid B(0, T_0] \cap \bar{\gamma} = \emptyset\}.$$

The conditioning is on a set of probability 0; however, it is not difficult using the theory of h-processes (see, e.g, [6]) to make rigorous sense of this quantity. Let

$$\xi_n(\lambda) = -\frac{1}{n} \log \mathbf{E}[Z_n^\lambda].$$

Our first theorem, proved in Section 4, shows that $\xi_n(\lambda)$ converges to $\xi(\lambda)$, the intersection exponent as defined in the previous section.

Theorem 2.1 *For every $0 < \lambda_1 < \lambda_2 < \infty$, there exist $0 < c_1, c_2 < \infty$ such that for all n and all $\lambda_1 \leq \lambda \leq \lambda_2$,*

$$c_1 e^{-n\xi} \leq \mathbf{E}[Z_n^\lambda] \leq c_2 e^{-n\xi}.$$

In order to make the notation more typical of that in large deviations let

$$\Psi_n = -\log Z_n,$$

so that

$$\mathbf{E}[e^{-\lambda \Psi_n}] \asymp e^{-\xi(\lambda) n},$$

and

$$\xi_n(\lambda) = -\frac{1}{n} \log \mathbf{E}[e^{-\lambda \Psi_n}].$$

For a fixed n, we can see by direct computation that

$$\xi_n'(\lambda) = \frac{1}{n} \mathbf{E}_n[\Psi_n],$$

$$\xi_n''(\lambda) = -\frac{1}{n} \mathrm{Var}_n[\Psi_n],$$

$$\xi_n'''(\lambda) = \frac{1}{n}[\mathbf{E}_n[\Psi_n^3] - 3\mathbf{E}_n[\Psi_n^2]\mathbf{E}_n[\Psi_n] + 2(\mathbf{E}_n[\Psi_n])^3],$$

where $\mathbf{E}_n$ and Var_n denote expectation and variance, respectively, with respect to the probability measure $\mathbf{Q}_n = \mathbf{Q}_{n,\lambda}$ with density

$$(\mathbf{E}[e^{-\lambda\Psi_n}])^{-1}e^{-\lambda\Psi_n}.$$

We will show in Section 8 that there exist $a = a(\lambda)$ and $\sigma^2 = \sigma^2(\lambda)$ such that

$$\xi_n'(\lambda) = a + O(\frac{1}{n}),$$

$$\xi_n''(\lambda) = -\sigma^2 + O(\frac{1}{n}),$$

$$\xi_n'''(\lambda) \leq c.$$

The uniform bound on the third derivative allows us to take the limit in the first and second derivative and conclude

$$\xi'(\lambda) = a,$$

$$\xi''(\lambda) = -\sigma^2.$$

While it will not be obvious from the expressions for $\sigma^2(\lambda)$ that it is strictly positive, we will be able to show more directly that

$$\mathrm{Var}_n[\Psi_n] \to \infty,$$

and this will imply $\sigma^2(\lambda) > 0$.

The way to obtain such convergence results is to analyze the measure $\mathbf{Q}_n$. One of the main goals of this paper is to show that the measures $\mathbf{Q}_n$ converge to an invariant measure. We describe the result here briefly; see Sections 5 and 7 for more details. Let $\mathcal{C}_0$ be the set of continuous functions

$$\gamma : [0, s] \to \mathbb{R}^d,$$

with $\gamma(0) = 0$, $|\gamma(s)| = 1$, and $0 \leq |\gamma(t)| < 1, 0 \leq t < s$. We allow s to take on any positive value. Given $\bar{\gamma}_0 = (\gamma_0^1, \cdots, \gamma_0^k) \in \mathcal{C}_0^k$, we start Brownian motions $B^1, \ldots, B^k$ on the unit sphere at the endpoints of these curves. Let

$$Z_n = \mathbf{P}_1\{B(0, T_n] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset \mid B(0, T_0] \cap \bar{\gamma}_0 = \emptyset\}.$$

Here we have written $\bar{\gamma}_0$ for the union of the images of the γ_0^j, and we assume that the $\bar{\gamma}_0$ is sufficiently nice so that the conditioning on B makes sense as an h-process (this is a very weak assumption, and unions of Brownian paths satisfy this). If we start with any distribution μ_0 on $\mathcal{C}_0^k$, concentrated on such nice paths, and consider the probability measure

$$\frac{Z_n^\lambda}{\mathbf{E}[Z_n^\lambda]},$$

we can obtain another measure μ_n on $\mathcal{C}_0^k$. In this measure, the paths are γ_0^j with $B^j[0, T_n^j]$ attached, scaled (using Brownian scaling) so that they are elements of $\mathcal{C}_0^k$. What we show is that there is a unique measure μ on $\mathcal{C}_0^k$ that is invariant (that is, if $\mu_0 = \mu$ then $\mu_n = \mu$ for all n). In fact, if we start with any μ_0, then very quickly the measure on paths is very close to μ. In particular, we show that for any initial μ_0, there is a $C = C(\mu_0)$ such that

$$\mathbf{E}_{\mu_0}[Z_n^\lambda] = Ce^{-n\xi}[1 + O(e^{-\beta\sqrt{n}})].$$

Let $\mathcal{F}_n$ be the σ-algebra generated by

$$\bar{\gamma}_0 \cup \{B^j_t : j = 1, \ldots, k;\ 0 \le t \le T^j_n\}.$$

We define

$$R_n = \lim_{m \to \infty} \frac{e^{m\xi} \mathbf{E}[Z^\lambda_{n+m} \mid \mathcal{F}_n]}{Z^\lambda_n},$$

and let $\bar{\mu}$ be the measure on paths whose density with respect to μ is R_0. Then $\bar{\mu}$ is the invariant measure for the Markov chain on $\mathcal{C}^k_0$ with transition density

$$R_n R_0^{-1} e^{n\xi} Z^\lambda_n.$$

In other words, $\bar{\mu}$ gives a stationary measure. We will see that

$$a = \mathbf{E}_{\bar{\mu}}[\Psi_1],$$

$$\mathrm{Var}_{\bar{\mu}}[\Psi_n] = \sigma^2 n + O(1).$$

Note that $\sigma^2(\lambda) > 0$ implies that $a(\lambda)$ is a strictly decreasing function of λ. In particular, the measures $\mathbf{Q}_{n,\lambda}$ for different λ become singular with respect to each other. We will also write a_λ for $a(\lambda)$. We define

$$a_0 = \lim_{\lambda \downarrow 0} a_\lambda,$$

$$a_\infty = \lim_{\lambda \to \infty} a_\lambda.$$

Since a is a decreasing function, these limits exist and are nonnegative.

Proposition 2.2

$$a_\infty = \begin{cases} 1/2, & d = 2, \\ 0 & d = 3. \end{cases}$$

Proof. The Beurling projection theorem (see Lemma 3.2) implies that there is a constant c such that for $d = 2$,

$$\Psi_n \ge \frac{n}{2} - c.$$

Since

$$a(\lambda) = \mathbf{E}_n[\Psi_n],$$

we see that $a_\lambda > 1/2$, if $d = 2$. In [4], it was shown that for every $\epsilon > 0$ there exists C, M such that

$$\mathbf{P}\{\Psi_n \le (\frac{1}{2} + \epsilon)n\} \ge Ce^{-Mm}, \quad d = 2,$$

$$\mathbf{P}\{\Psi_n \le \epsilon n\} \ge Ce^{-Mn}, \quad d = 3.$$

Hence,

$$\xi(\lambda) \le (\frac{1}{2} + \epsilon)n + M, \quad d = 2,$$

$$\xi(\lambda) \le \epsilon n + M, \quad d = 3.$$

This and the fact that $a_\lambda = \xi'(\lambda)$ is a decreasing function give the lemma. $\square$

In Section 9 we prove that for $d = 2$,

$$a_0 < \infty.$$

For $d = 3$, we have no proof, but we conjecture that $a_0 < \infty$.

We have a one-to-one function

$$a : (0, \infty) \to (a_\infty, a_0).$$

We can invert this function and consider the inverse as a function

$$\lambda : (a_\infty, a_0) \to (0, \infty).$$

We will write either a_λ or λ_a depending on whether we are considering the function or its inverse. Consider another function

$$b : (a_\infty, a_0) \to (0, \infty),$$

defined by saying if $b = b(a)$, then

$$\mathbf{P}\{\Psi_n \leq an\} \approx e^{-bn}.$$

The existence of such a b will be shown in the following proposition.

Proposition 2.3 *If $a_\infty < a < a_0$, then*

$$b(a) = \xi(\lambda_a) - a\lambda_a.$$

Proof. By Chebyshev's inequality, for any $\lambda > 0$,

$$\mathbf{P}\{\Psi_n \leq an\} \leq e^{a\lambda n}\mathbf{E}[e^{-\lambda\Psi_n}] \leq ce^{-n(\xi(\lambda)-\lambda a)}.$$

Hence,

$$b(a) \geq \sup_\lambda \xi(\lambda) - a\lambda.$$

The supremum is taken on when $\xi'(\lambda) = a$, i.e., when $\lambda = \lambda_a$.

Now let $\lambda = \lambda_a$. By the estimate of the variance and Chebyshev's inequality, we see that there exist c_1, c_2 such that

$$\mathbf{E}\left[e^{-\lambda\Psi_n}; an - c_1\sqrt{n} \leq \Psi_n \leq an + c_1\sqrt{n}\right] \geq c_2 e^{-\xi\lambda}. \tag{6}$$

Hence,

$$\mathbf{P}\{\Psi_n \leq an\} \geq \exp\{-n\lambda - c_1\sqrt{n}\lambda + an\}. \quad \square$$

We note that we do not expect for $a < a_\infty$ that

$$\mathbf{P}\{\Psi_n \leq an\} \asymp e^{-bn},$$

but rather we conjecture that

$$\mathbf{P}\{\Psi_n \leq an\} \asymp n^{-1/2}e^{-bn}.$$

This would follow if we could prove the following weak form of the *local* central limit theorem,

$$\mathbf{E}[e^{-\lambda_a \Psi_n}; an \leq \Psi_n \leq an+1] \asymp \frac{1}{\sqrt{n}} e^{-n\xi}.$$

This is very plausible from (6), and, in fact, central limit theorems for stationary processes can be used to show that $n^{-1/2}(\Psi_n - an)$ under the measure

$$\mathbf{E}[Z_n^\lambda]^{-1} Z_n^\lambda,$$

approaches a nontrivial normal distribution. However, we do not know at this point how to go from the central limit theorem to a local central limit theorem. Note that

$$b'(a) = \xi'(\lambda_a)\lambda'(a) - \lambda_a - a\lambda'(a).$$

But $\xi'(\lambda_a) = a$. Hence

$$b'(a) = -\lambda_a.$$

Proposition 2.3 is a "Legendre transform" relation. The following gives the dual relation. It follows immediately from Proposition 2.3.

Proposition 2.4

$$\xi(\lambda) = b(a_\lambda) + \lambda a_\lambda.$$

The case $k = 2$ is related to the multifractal spectrum of harmonic measure of a region bounded by a Brownian path. The multifractal spectrum $F(a)$ is the dimension of the set of points at which the harmonic measure looks like an a-dimensional measure (see [18] for the appropriate definitions). In [18] it is shown that the multifractal spectrum is given by

$$F(a+d-2) = 2 - b(a),$$

at least for all a with $\xi(\lambda_a) \leq 2$. From Makarov's theorem [21] we know that for any subset of the plane $F(1) = 1$. Hence for $k = 2, d = 2$.

$$b(1) = 1.$$

In this case we also know that $\xi(1) = 2$, and

$$\xi'(1) = a_1 = 1.$$

Since ξ is strictly concave this means that for $d = 2, k = 2$,

$$\xi(\lambda) < 1 + \lambda, \quad \lambda \neq 1.$$

This gives (5).

Let us give a summary of the remaining chapters. Section 3 consists of preliminary lemmas that will be used in this paper; it is recommended that this section be skipped on a first reading and referred to as necessary. The main purpose of the next section is to derive Theorem 2.1 and some necessary consequences for the convergence result. The basic structure of this section is similar to that found in the corresponding sections of [15, 16, 17]; unfortunately, we cannot use directly the results of those papers and instead derive the necessary propositions and lemmas from scratch. The next three sections discuss the convergence to equilibrium. The basic idea is similar to that in [15] where an invariant distribution is given on Brownian motions starting at the origin conditioned to have no intersection. The idea is to approximate by subMarkov chains that depend only on the last part of the path Λ_n. A technical complication arises in this approximation. Let $0 < m < n$, and suppose we want to approximate the random variable

$$\mathbf{P}_1\left\{B[0,T_{n+1}] \cap (\bar{\gamma}_0 \cup \Lambda_{n+1}) = \emptyset \mid B[0,T_n] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset\right\}$$

by a random variable that depends only on

$$\Lambda(m,n+1) = B^1[T^1_m, T^1_{n+1}] \cup \cdots \cup B^k[T^k_m, T^k_{n+1}].$$

The error in replacing Λ_{n+1} with $\Lambda(m,n+1)$ is not too difficult to estimate using estimates derived in Section 4. However replacing Λ_n with $\Lambda(m,n)$ in the conditioning is a bit tricky. Section 6 is devoted to proving a lemma which allows for this substitution. We call it a Harnack-type inequality, because it is similar to boundary Harnack principles (unfortunately, a region bounded by a Brownian motion path is too rough to allow one to quote existing boundary Harnack results). Sections 5 and 7 describe and prove the main results about the invariant measure. With the invariant measure in hand, Section 8 does the necessary moment calculations. Section 9 proves that the intersection exponent is continuous at $\lambda = 0$; this is not difficult for $d = 3$, but requires work for $d = 2$.

3 Preliminaries

We collect here a number of facts about random variables and Brownian motion that we will use. Much of this section may be skipped in a first reading and referred to as necessary. Let $B(t)$ denote a Brownian motion in $\mathbb{R}^d$ and as before let

$$T_r = \inf\{t : |B_t| = e^r\}.$$

Let $\mathcal{S}_s$ denote the sphere $\{|x| = e^s\}$. The first lemma is essentially a version of the gambler's ruin estimate.

Lemma 3.1 *There exist constants c_1, c_2 such that if $0 < \epsilon < 1/2$, $x \in \mathcal{S}_{-\epsilon}$, then*

$$c_1\epsilon \leq \mathbf{P}^x\{T_{-1} < T_0\} \leq c_2\epsilon.$$

Proof. Let $G(x) = \log|x|$ if $d = 2$ and $G(x) = |x|^{-1}$ if $d = 3$. Apply optional sampling to the martingale $G(B_{t\wedge\tau})$ where $\tau = T_{-1} \wedge T_0$. $\Box$

The next lemma gives two estimates about planar Brownian motion that we will refer to as the Beurling estimates, see, e.g., [3, V.4].

Lemma 3.2 *Suppose $d = 2$. There exist constants c_1, c_2 such that the following holds. Suppose $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous function with $|\gamma(0)| = 1, |\gamma(1)| = e^r, r \geq 2$. Then if $|y| = 1$,*

$$\mathbf{P}^y\{B[0,T_1] \cap \gamma[0,1] \neq \emptyset\} \geq c_1,$$

$$\mathbf{P}^y\{B[0,T_r] \cap \gamma[0,1] = \emptyset\} \leq c_2 e^{-r/2}.$$

The unique harmonic function in $\{x \in \mathbb{R}^3 : |x| > 1\}$ with boundary value 1 on the sphere of radius 1 and boundary value 0 at infinity is $f(x) = |x|^{-1}$. The next lemma follows immediately from this.

Lemma 3.3 *If $d = 3$, $y \in \mathcal{S}_r, r > 0$,*

$$\mathbf{P}^y\{T_0 < \infty\} = e^{-r}.$$

For any $\delta > 0$ let A_δ be the closed infinite cone

$$A_\delta = \{0\} \cup \{x \in \mathbb{R}^d : |\frac{x}{|x|} - u| \leq \delta\}, \tag{7}$$

where u is the unit vector whose first component is 1.

Lemma 3.4 *For every $\delta > 0$, there exist $a, b > 0$ such that for all $0 < |x| < 1, ||x|^{-1}x - u| \leq \delta/2$,*

$$\mathbf{P}^x\{B[0, T_0] \subset A_\delta\} \geq b|x|^a. \tag{8}$$

Proof (sketch). Let $\tau = \tau_\delta$ be the first time that the Brownian motion reaches $\mathcal{S}_0 \cap \{y : |y - u| \leq \delta/2\}$. Let $q = q(\delta) > 0$ be the infimum over all $z \in \mathcal{S}_{-1} \cap \{y : |ey - u| \leq \delta/2\}$ of

$$\mathbf{P}^z\{B[0, \tau_\delta] \subset A_\delta\}.$$

If we let $p(r) = p(r, \delta)$ be the infimum of the probability on the left hand side of (8) over all $x \in \mathcal{S}_{-r} \cap \{y : |\epsilon^r y - u| \leq \delta/2\}$, then $p(r) \geq qp(r-1)$. Iterating this gives the lemma. $\Box$

The exact value $a = a(\delta)$ can be calculated exactly for $d = 2$ using conformal mapping. It is not so easy to calculate the value in $d = 3$, but we will not need it. We will use the easily verified fact that $a(\delta) \to \infty$ as $\delta \to 0$. The following lemma handles the complement.

Lemma 3.5 *For every $0 < \delta < 1$, there exist $a, b > 0$ such that*

$$\mathbf{P}^{-u}\{B[0, T_n] \cap A_\delta = \emptyset\} \geq be^{-an}.$$

Moreover, $a = a(\delta)$ can be chosen so that

$$\lim_{\delta \to 0} a(\delta) = \begin{cases} 1/2, & d = 2, \\ 0, & d = 3. \end{cases}$$

Proof. The existence of such an a can be proved in the same way as the previous lemma. For $d = 2$, the Beurling estimate gives that $a(0) = 1/2$, and the limit above can be proved by conformal mapping. For $d = 3$ let $q = q(\delta)$ be defined by

$$q = \mathbf{P}^0\{B[T_0, T_1] \cap A_\delta \neq \emptyset\},$$

It is easy to see that $q \to 0$ as $\delta \to 0$; in fact, one can see that for every $\epsilon > 0$ we can find $\delta = \delta(\epsilon)$ sufficiently small such that for all $x \in \mathcal{S}_0$ with $|x - u| \geq \epsilon$,

$$\mathbf{P}^x\{B[0, T_1] \cap A_\delta \neq \emptyset\} < \epsilon.$$

Let V_n be the event $\{B[T_0,T_n]\cap A_\delta=\emptyset\}$ and let $q_n=\mathbf{P}(V_n^c \mid V_{n-1})$. Note that

$$\mathbf{P}\{|B(T_n)-nu|\le \epsilon n; V_n \mid V_{n-1}\}\le \mathbf{P}\{|B(T_n)-nu|\le \epsilon n \mid V_{n-1}\}\le c\epsilon^2,$$

and hence

$$\mathbf{P}\{|B(t_n)-nu|\le \epsilon n \mid V_n\}\le \frac{c\epsilon^2}{1-q_n}.$$

Therefore,

$$q_{n+1}\le \epsilon + c\epsilon^2(1-q_n)^{-1}.$$

In particular, for all ϵ sufficiently small we can show that $q_n<2\epsilon$ implies $q_{n+1}<2\epsilon$, and hence

$$\mathbf{P}^0\{B[T_0,T_n]\cap A_\delta=\emptyset\}\ge (1-2\epsilon)^n. \quad \Box$$

The following two lemmas can be proved in a similar way; we omit the proofs. Let $-A_\delta=\{x: -x\in A_\delta\}$.

Lemma 3.6 *There exist c_1,c_2 such that the following hold. For every $0<\delta<1/2$, let $V=V_\delta$ be the translated infinite cone*

$$V=A_\delta+(1-\delta)u=\{x\in\mathbb{R}^d : x-(1-\delta)u\in A_\delta\}.$$

Then if $x\in\mathcal{S}, |x-u|\le\delta/2$,

$$\mathbf{P}^x\{T_1<T_{-\delta}; B[0,T_1]\subset V\}\ge c_1\delta^{c_2}.$$

Moreover, if $x\in\mathcal{S}, |x-u|\ge 2\delta$,

$$\mathbf{P}^x\{T_1<T_{-\delta}; B[0,T_1]\cap V=\emptyset\}\ge c_1\delta^{c_2}.$$

Lemma 3.7 *There exist c_1,c_2 such that the following is true. Suppose $x\in\mathcal{S}_{1/8}\setminus A_{2\delta}$. Then*

$$\mathbf{P}^x\left\{B[0,T_{1/4}]\cap A_\delta=\emptyset; |e^{-1/4}B(T_{1/4})+u|\le\frac{1}{20}\right\}\ge c\delta^{c_2},$$

$$\mathbf{P}^x\left\{B[0,T_{1/4}]\cap A_\delta=\emptyset; T_{1/4}<T_0; B[T_{1/4},T_1]\subset -A_{1/10}\right\}\ge c\delta^{c_2}.$$

The following lemma contains standard exponential estimates for binomial and geometric random variables. Since the proofs are short, we present them here.

Lemma 3.8 *(a) If X is a binomial random variable with parameters n and p and $k>1$, then*

$$\mathbf{P}\{X\ge knp\}\le \left(\frac{e^{k-1}}{k^k}\right)^{np},$$

$$\mathbf{P}\{X\le k^{-1}np\}\le \left(k^{1/k}\exp\{\frac{1}{k}-1\}\right)^{np}.$$

(b) For every $M<\infty, \epsilon>0$, there exists a $\rho>0$ such that if $X_1,X_2,\ldots$ are independent nonnegative integer random variables with

$$\mathbf{P}\{X_i=j\}\le M\rho^j,$$

then

$$\mathbf{P}\{X_1+\cdots+X_n\ge\epsilon n\}\le M^{-1}e^{-Mn}.$$

Proof. (a) For any t,

$$\mathbf{E}[e^{tX}] = [1+p(e^t-1)]^n \leq (\exp\{e^t-1\})^{np}.$$

Hence if $t>0$,

$$\mathbf{P}\{X \geq knp\} \leq e^{-tknp}\mathbf{E}[e^{tX}] \leq (\frac{\exp\{e^t-1\}}{e^{kt}})^{np},$$

$$\mathbf{P}\{X \leq k^{-1}np\} \leq e^{tnp/k}\mathbf{E}[e^{-tX}] \leq (e^{t/k}\exp\{e^{-t}-1\})^{np}.$$

Letting $t=\log k$ gives the result.

(b) For any $t>0$,

$$\mathbf{P}\{X_1+\cdots+X_n \geq \epsilon n\} \leq e^{-t\epsilon n}\mathbf{E}[e^{t(X_1+\cdots+X_n)}] \leq e^{-t\epsilon n}(\sup_j \mathbf{E}[e^{tX_j}])^n.$$

Let $t=2M/\epsilon$ and then choose ρ sufficiently small so that

$$\mathbf{E}[e^{tX_j}] \leq \min\{1, M^{-1}\}e^M. \quad \Box$$

The following is a simple estimate of binomial coefficients.

Lemma 3.9 *There exists $c<\infty$ such that for all $a \in (0,1/c)$,*

$$\binom{n}{an} \leq ca^{-an}e^{2an},$$

provided an is a positive integer.

Proof. By Stirling's formula, for all $a<1/2$,

$$\begin{aligned}\binom{n}{an} &\leq c\frac{e^{-n}n^{n+(1/2)}}{(an)^{an+(1/2)}e^{-an}[(1-a)n]^{(1-a)n+(1/2)}e^{-(1-a)n}} \\ &\leq ca^{-an}(1-a)^{-(1-a)n}.\end{aligned}$$

As $a\to 0$,

$$(1-a)^{1-a} = e^{(1-a)\ln(1-a)} = e^{-a+O(a^2)}.$$

In particular, for all a sufficiently small,

$$(1-a)^{-(1-a)} \leq e^{2a}. \quad \Box$$

We will not prove the following standard estimates which follow from concavity ($\lambda \leq 1$) and Minkowski's inequality ($\lambda \geq 1$).

Lemma 3.10 *Let X,Y be random variables. If $0<\lambda\leq 1$,*

$$E[|Y|^\lambda] \geq \mathbf{E}[|X|^\lambda] - \mathbf{E}[|X-Y|^\lambda].$$

In particular, if $\mathbf{E}[|X-Y|^\lambda] \le a\mathbf{E}[|X|^\lambda]$,

$$\mathbf{E}[|Y|^\lambda] \ge (1-a)\mathbf{E}[|X|^\lambda].$$

If $1 \le \lambda < \infty$,

$$\mathbf{E}[|Y|^\lambda] \ge [(\mathbf{E}[|X|^\lambda])^{1/\lambda} - (\mathbf{E}[|X-Y|^\lambda])^{1/\lambda}]^\lambda.$$

In particular, if $\mathbf{E}[|X-Y|^\lambda] \le a\mathbf{E}[|X|^\lambda]$,

$$\mathbf{E}[|Y|^\lambda] \ge (1-a^{1/\lambda})^\lambda \mathbf{E}(|X|^\lambda).$$

Hence, for every $\lambda_0 \ge 1$, *there exists a* $C = C(\lambda_0)$ *such that for all* $a \le 1/2, 1 \le \lambda \le \lambda_0$,

$$\mathbf{E}[|Y|^\lambda] \ge [1 - Ca^{1/\lambda}]\mathbf{E}(|X|^\lambda).$$

We will now prove a series of lemmas that will be needed in Section 6. For any $\epsilon \in (0, 1/10)$ let L be the line segment with endpoints u and e^4u, where again u is the unit vector whose first component is 1, and let

$$R_\epsilon = \{x \in \mathbb{R}^d : |x-y| \le \epsilon \text{ for some } y \in L\}.$$

Let $\mathcal{U}_\epsilon$ be the collection of subsets D of $\mathbb{R}^d$ such that

$$D \cap \{1 \le |x| \le e^4\} \subset R_\epsilon$$

and such that for every $x \in R_\epsilon$, the probability that a Brownian motion starting at x hits D before leaving the ball of radius ϵ centered at x is at least 1/100. The second condition implies that D fills up a reasonable fraction of R_ϵ. Let $\tau = \tau_D$ be the first time that the Brownian motion hits the set D and let $\rho = \rho_D = \min\{T_3, \tau\}$. For $|x| < e^3, |y| = e^3$, let $H(x, y; D)$ denote the density of the hitting measure of $B(\rho)$. This is a density of a subprobability measure; in fact, if $\sigma = \sigma_3$ denotes normalized surface measure on $\mathcal{S}_3$, the sphere of radius e^3,

$$\int_{\mathcal{S}_3} H(x, y; D)\, d\sigma(y) = \mathbf{P}^x\{\tau > T_3\}.$$

The next lemma will give an estimate for $H(x, y; D)$. Assume that $D \in \mathcal{U}_\epsilon; |x| \le e^2; |y| = e^3$; and $y \notin R_{2\epsilon}$. It is immediate from the strong Markov property that

$$H(x, y; D) \le \mathbf{P}^x\{T_2 < \tau\} \sup_{|z|=e^2} H(z, y; D).$$

Let $\mathbb{R}^d_-$ denote the points in $\mathbb{R}^d$ whose first component is nonpositive. Then the Harnack principle can be combined with the strong Markov property to conclude

$$H(x, y; D) \ge c\mathbf{P}^x\{T_2 < \tau; B(T_2) \in \mathbb{R}^d_-\} H(-e^2u; y).$$

The boundary Harnack principle can be used to see that for $|z| \le e^2$,

$$H(z, y; D) \asymp H(z, \tilde{y}; D), \quad |\tilde{y}| = e^3, |y - \tilde{y}| \le \epsilon/2.$$

Hence,

$$H(z,y;D) \asymp \epsilon^{1-d}\mathbf{P}^z\{|B(\rho)-y| \leq \epsilon/2\}.$$

Let $\eta(y,\delta)$ be the first time that the Brownian motion reaches the ball of radius δ about y. Then it is easy to see that

$$c_1\mathbf{P}^z\{\eta(y,\epsilon/4) < \tau\} \leq \mathbf{P}^z\{|B(\rho)-y| \leq \epsilon/2\} \leq c_2\mathbf{P}^z\{\eta(y,\epsilon/2) < \tau\}.$$

Let $G(z,w;D)$ denote the Green's function of the Brownian motion killed at time ρ. Then it is standard to see that

$$\mathbf{P}^z\{\eta(y,\epsilon/4) < \tau\} \asymp \mathbf{P}^z\{\eta(y,\epsilon/2) < \tau\} \asymp \epsilon^{d-2}G(z;(1-\epsilon e^{-3}/2)y;D).$$

The Green's function satisfies $G(z_1,z_2;D) = G(z_2,z_1;D)$, and hence

$$H(z,y;D) \asymp \epsilon^{-1}G((1-\epsilon e^{-3}/2)y,z;D).$$

Let $\bar{y} = (1-\epsilon e^{-3}/2)y$. Then again using standard arguments, if $|z| = e^2$

$$G(\bar{y};z;D) \leq c\mathbf{P}^{\bar{y}}\{T_{5/2} < \rho\},$$

$$G(\bar{y};-e^2u;D) \geq c\mathbf{P}^{\bar{y}}\{T_2 < \rho; B(T_2) \in \mathbb{R}^d_-\}.$$

Combining all of the estimates in this paragraph, we see that if $|x| \leq e^2, |y| = e^3$, $x,y \notin R_{2\epsilon}$, and $\bar{y}$ is defined as above

$$H(x,y;D) \leq c\epsilon^{-1}\mathbf{P}^x\{T_2 < \tau\}\mathbf{P}^{\bar{y}}\{T_{5/2} \leq \rho\},$$

$$H(x,y;D) \geq c\epsilon^{-1}\mathbf{P}^x\{T_2 < \tau; B(T_2) \in \mathbb{R}^d_-\}\mathbf{P}^{\bar{y}}\{T_2 < \rho; B(T_2) \in \mathbb{R}^d_-\}.$$

The probabilities can be estimated in a standard fashion to give the following lemma.

Lemma 3.11 *There exist constants c_1, c_2 such that the following is true. Suppose $D \in \mathcal{U}_\epsilon$ satisfies*

$$D \cap \{1 \leq |z| \leq e^4\} \subset R_\epsilon.$$

Let $\alpha_x = \operatorname{dist}(x,R_\epsilon), \alpha_y = \operatorname{dist}(y,R_\epsilon)$. Then for $x,y \notin R_{2\epsilon}$

$$c_1\alpha_x\alpha_y \leq H(x,y;D) \leq c_2\alpha_x\alpha_y, \quad d=2,$$

$$c_1\frac{\log\alpha_x - \log\epsilon}{-\log\epsilon}\frac{\log\alpha_y - \log\epsilon}{-\log\epsilon} \leq H(x,y;D) \leq c_2\frac{\log\alpha_x - \log\epsilon}{-\log\epsilon}\frac{\log\alpha_y - \log\epsilon}{-\log\epsilon}, \quad d=3.$$

We will not need the exact form of the probabilities in the previous lemma, but just the following corollary.

Corollary 3.12 *There exists a c such that the following is true. Let $0 < \epsilon < 1/5$, and let $D = D_\epsilon$ be the open region*

$$D = \{1 < |z| < e^3\} \setminus R_{2\epsilon}.$$

For $x \in D, y \in \partial D \cap \mathcal{S}_3$, let $H(x,y)$ denote the density (in y, with respect to normalized surface measure on $\mathcal{S}_3$) of the distribution of $B(T_3)$ given $B_0 = x$ and $B[0,T_3) \subset D$. Then for all $x_1, x_2 \in D \cap \{|z| \leq e^2\}$, $y \in \mathcal{S}_3$,

$$H(x_1,y) \leq cH(x_2,y).$$

Now suppose we have an open region D contained in $\{1 < |z| < e^n\}$. Let $x_1, x_2 \in D$ and let Y_t^1, Y_t^2 denote h-processes conditioned to leave D at $\mathcal{S}_n$, starting at x_1, x_2 respectively. (This will be conditioning with respect to a set of positive probability, so we really do not need to use h-processes, but it is convenient to use the term anyway.)

In the analysis below, we will use the following coupling of Brownian motions.

Lemma 3.13 *There exists a c such that the following is true. If $x, y \in \mathcal{B}(0, e^{-1})$, there exist Brownian motions B^1, B^2, defined on the same probability space $(\Omega, \mathbf{P})$, with corresponding stopping times T_r^1, T_r^2, and also stopping times τ^1, τ^2 (coupling times) such that:*

$$\mathbf{P}\{\tau^1 < T_{-1/2}^1\} \geq c; \quad \mathbf{P}\{\tau^2 < T_{-1/2}^2\} \geq c;$$

and if $\tau^1 < T_0^1$,

$$B_{t+\tau^1}^1 = B_{t+\tau^2}^2, \quad 0 < t < T_0^1 - \tau^1 = T_0^2 - \tau^2.$$

Moreover if g^i, h^i denote the density of $B^i(T_0^i)$ (with respect to normalized surface measure on $\mathcal{S}$), conditioned on the events $\{\tau^i < T_{-1/2}^i\}$ and $\{\tau^i \geq T_{-1/2}^i\}$, respectively, then

$$c \leq g^i, h^i \leq c^{-1}.$$

Proof. We will only give the coupling rule, leaving it to the reader to verify that it satisfies the conditions of the lemma. We will do the $d = 2$ case; the $d = 3$ case is essentially the same. Let $x = (x^1, x^2), y = (y^1, y^2)$ and let $X_t^1, X_t^2, Y_t^1, Y_t^2$ be independent one dimensional Brownian motions starting at x^1, x^2, y^1, y^2, respectively. Let $B_t^1 = (X_t^1, X_t^2)$ for all t. Let

$$\rho^1 = \inf\{t : Y_t^1 = X_t^1\},$$

and let

$$\rho^2 = \inf\{t \geq \rho^1 : Y_t^2 = X_t^2\}.$$

Then we let

$$B_t^2 = \begin{cases} (Y_t^1, Y_t^2), & 0 \leq t \leq \rho^1, \\ (X_t^1, Y_t^2), & \rho^1 \leq t \leq \rho^2, \\ (X_t^1, X_t^2), & \rho^2 \leq t < \infty. \end{cases} \qquad \Box$$

A corollary is a coupling for h-processes. Suppose D is a bounded open set and suppose h is a positive harmonic function on D. Then the h-process killed upon hitting ∂D is the process with transition density

$$q_t(x, y) = p_t(x, y; D) h(y)/h(x), \quad x, y \in D$$

where $p_t(x, y : D)$ is the density of usual Brownian motion killed upon hitting ∂D. (See [6, Chapter 10] for a detailed discussion of h-processes.) One particular case that we will use is when $V \subset \partial D$, and $h(x)$ is the probability that Brownian motion starting at x first hits ∂D some place in the set V. The following follows immediately from Lemma 3.13 .

Corollary 3.14 *There exists a c such that the following is true. Let h be any positive harmonic function on the interior of $\mathcal{B}(0, e)$ and let $x, y \in \mathcal{B}(0, e^{-1})$. There exist h-processes Y^1, Y^2, defined* *on the same probability space* $(\Omega, \mathbf{P})$, *with corresponding stopping times* T^1_r, T^2_r, *and also stopping times* τ^1, τ^2 *(coupling times) such that:*

$$\mathbf{P}\{\tau^1 < T^1_{-1/2}\} \geq c; \quad \mathbf{P}\{\tau^2 < T^2_{-1/2}\} \geq c;$$

and if $\tau^1 < T^1_0$,

$$Y^1_{t+\tau^1} = Y^2_{t+\tau^2}, \quad 0 < t < T^1_0 - \tau^1 = T^2_0 - \tau^2.$$

Now let A be a closed subset of $\mathbb{R}^d$ and let $D = D_n(A)$ be the open region

$$D = \{|z| < e^n\} \setminus A.$$

We will assume that D is connected. Let $\tau = \tau_A$ be the first hitting time of A and let h be the harmonic function on D,

$$h(x) = \mathbf{P}^x\{\tau_A \geq T_n\}.$$

Let $x, y \in D$. We want to define a coupling of h-processes starting at x, y, i.e., we want to define h-processes Y^1_t, Y^2_t on the same probability space $(\Omega, \mathbf{P})$ such that $Y^1_0 = x, Y^2_0 = y$, and such that with high probability

$$Y_1(T^1_n) = Y_2(T^2_n).$$

Let $J = J_n(A)$ be the set of even integers m less than n such that

$$\mathcal{B}(e^m u, \frac{1}{5}e^m) \subset D.$$

Start by taking independent h-processes starting at x, y, say X^1_t, X^2_t. We let

$$\sigma^i_m = \inf\{t : X^i_t \in \mathcal{B}(e^m u, \frac{1}{30}e^m)\}.$$

Let l_1 be the smallest integer m in J such that

$$\sigma^1_m < T^1_{m+1}, \;\; \sigma^2_m < T^2_{m+2}.$$

Let

$$Y^i_t = X^i_t, \quad 0 \leq t \leq \sigma^i_{l_1}.$$

Let

$$\rho^i_m = \inf\{t \geq \sigma^i_m : Y^i_t \in \partial\mathcal{B}(e^m u, \frac{1}{5}e^m u)\}.$$

We can use Corollary 3.14 to define Y^i_t for

$$\sigma^i_{l_1} \leq t \leq \rho^i_{l_1}$$

in such a way so that with probability at least c,

$$Y^1(\rho^1_{l_1}) = Y^2(\rho^2_{l_2}).$$

We now start up independent h-processes, X^1_t, X^2_t at $Y^1(\rho^1_{l_1}), Y^2(\rho^2_{l_1})$. On the event

$$\{Y^1(\rho^1_{l_1}) = Y^2(\rho^2_{l_1})\},$$

we set

$$Y_t^i = X^1(t - \rho_{l_1}^i),$$

for all t until the boundary is reached. Otherwise, we set

$$Y_t^i = X^i(t - \rho_{l_1}^i),$$

for all t until there exists an l_2, defined similarly as above. If that happens, we do the coupling procedure again, in a way so that the conditional probability of the processes being coupled, given that they were not already coupled is c. This defines the processes Y_t^1, Y_t^2 for $0 \le t \le T_n^i$.

Let $\mathbb{R}_\epsilon(m) = e^m R_\epsilon = \{e^m w : w \in R_\epsilon\}$. Suppose for a given integer m, $\epsilon < 1/10$,

$$A \cap \{e^{m-2} \le |z| \le e^{m+1}\} \subset -R_\epsilon(m-2), \tag{9}$$

and

$$Y^i[T_{m-1}^i, T_{m+1}^i] \subset \{e^{m-2} \le |z| \le e^{m+1}\} \setminus -R_{2\epsilon}(m-2).$$

By the estimate in Lemma 3.11, given this information there is a positive probability that the processes, if they have not already coupled, will couple during this time interval. Suppose we have an $\epsilon < 1/10$ and a collection K of $3j$ integers satisfying (9). Let Y^1, Y^2 be h-processes starting at x, y (not necessarily independent, in fact, we will be concerned mainly with the coupled precesses), and let p_i be the probability that there exists at least j integers m in K with

$$Y^i[T_{m-1}^i, T_{m+1}^i] \not\subset \{e^{m-2} \le |z| \le e^{m+1}\} \setminus -R_{2\epsilon}(m-2).$$

The probability that the coupling does not occur is bounded by

$$p_1 + p_2 + ce^{-\beta j}.$$

Note that p_i depends only on Y^i, so this quantity is independent of the coupling procedure used. This coupling procedure allows us to conclude the following lemma.

Lemma 3.15 *There exist c, β such that the following is true. Let A be a closed subset of $\mathbb{R}^d$, and let*

$$D = D_n(A) = \{|z| < e^n\} \setminus A.$$

Let $x, y \in D \cap \mathcal{S}$ and let Y_t^1, Y_t^2 be h-processes conditioned to hit $\mathcal{S}_n$ before hitting A. Let $H^i(z)$ denote the density (with respect to normalized surface measure σ_n on $\mathcal{S}_n$) of $Y^i(T_n^i)$. Suppose there exists an $\epsilon > 0$, and a subset K of even integers $m \le n$ such that (9) holds, and such that the cardinality of K is at least $3j$. Let p^i be the probability that there exist at least j elements of K such that

$$Y^i[T_{m-1}^i, T_{m+1}^i] \cap \{e^{m-2} \le |z| \le e^{m+1}\} \cap -L_{2\epsilon} \ne \emptyset.$$

Then

$$\int_{\mathcal{S}_n} |H^1(z) - H^2(z)| \, d\sigma_n(z) \le p^1 + p^2 + ce^{-\beta j}.$$

4 Intersection Exponent

In the section we will prove Theorem 2.1 and develop some of the technical results needed for the convergence to a stationary distribution. Let us review our convention about constants. We fix k, d and $0 < \lambda_1 < \lambda_2 < \infty$ and choose $\lambda \in [\lambda_1, \lambda_2]$. Constants $c, c_1, c_2, \ldots$ are positive constants that may depend on $k, d, \lambda_1, \lambda_2$, but may not depend on λ. The values of c, c_1, c_2 may vary from place to place, but the values of $c_3, c_4, \ldots$ will not. We let $\mathcal{B}(x, r)$ denote the closed ball of radius r about x, and we let $\mathcal{S}_r$ denote the circle or sphere of radius e^r centered at the origin. We write $\mathcal{S}$ for $\mathcal{S}_0$ and $\mathcal{S}^k$ for k-tuples of points in $\mathcal{S}$. We let $u = (1, 0)$ or $(1, 0, 0)$ be the unit vector with first component 1.

Let $B^1, \ldots, B^k, B$ be independent Brownian motions, not all necessarily starting at the same point. We will assume that $B^1, \ldots, B^k$ are defined on the probability space $(\Omega, \mathbf{P})$ and B is defined on the probability space $(\Omega_1, \mathbf{P}_1)$. We write $(\bar{\Omega}, \bar{\mathbf{P}})$ for $(\Omega \times \Omega_1, \mathbf{P} \times \mathbf{P}_1)$. We use $\mathbf{E}, \mathbf{E}_1$, and $\bar{\mathbf{E}}$ for expectations with respect to $\mathbf{P}, \mathbf{P}_1$, and $\bar{\mathbf{P}}$, respectively. Define, as before, the stopping times

$$T_r^j = \inf\{t : |B_t^j| = e^r\} = \inf\{t : B_t^j \in \mathcal{S}_r\},$$

$$T_r = \inf\{t : |B_t| = e^r\} = \inf\{t : B_t \in \mathcal{S}_r\}.$$

Let $\mathcal{F}_r$ be the σ-algebra

$$\mathcal{F}_r = \sigma\{B_t^j; 1 \leq j \leq k, t \leq T_r^j\}.$$

Let Λ_r denote the random set

$$\Lambda_r = B^1[0, T_r^1] \cup \cdots \cup B^k[0, T_r^k],$$

and for $r < s$ let

$$\Lambda(r, s) = B^1[T_r^1, T_s^1] \cup \cdots \cup B^k[T_r^k, T_s^k].$$

Note that Λ_r is a set determined by $\mathcal{F}_r$-measurable random variables.

For any $r > 0$ and $y \in \mathcal{S}$, define the following $\mathcal{F}_r$-measurable random variables:

$$Z_{r,y} = \mathbf{P}_1^y\{B[0, T_r] \cap \Lambda_r = \emptyset\},$$

$$Z_{r,\cdot} = \mathbf{P}_1^0\{B[T_0, T_r] \cap \Lambda_r = \emptyset\} = \int_{\mathcal{S}} Z_{r,y}\, d\sigma(y),$$

$$\bar{Z}_r = \sup_{y \in \mathcal{S}} Z_{r,y}.$$

Here $\sigma = \sigma_0$ denotes the normalized surface measure on $\mathcal{S}$.

Let

$$\phi(r) = \phi(r, \lambda, k, d) = \sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[Z_{r,y}^\lambda].$$

$$\tilde{\phi}(r) = \tilde{\phi}(r, \lambda, k, d) = \int_{\mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[Z_{r,\cdot}^\lambda]\, d^k\sigma(\mathbf{x}),$$

$$\bar{\phi}(r) = \bar{\phi}(r, \lambda, k, d) = \sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[\bar{Z}_r^\lambda].$$

If $r, s > 0$, then

$$\bar{\phi}(r + s) = \sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[\bar{Z}_{r+s}^\lambda] \leq \sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[\bar{Z}_r^\lambda Y_{r,s}^\lambda],$$

where

$$Y_{r,s} = \sup_{y \in \mathcal{S}_r} \mathbf{P}_1^y\{B[0, T_{r+s}] \cap \Lambda(r, r+s) = \emptyset\}.$$

By Brownian scaling,

$$\mathbf{E}^{\mathbf{x}}[Y_{r,s}^\lambda \mid \mathcal{F}_r] \leq \bar{\phi}(s),$$

and hence

$$\bar{\phi}(r+s) \leq \sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[\mathbf{E}(\bar{Z}_r^\lambda Y_{r,s}^\lambda \mid \mathcal{F}_r)] \leq \bar{\phi}(r)\bar{\phi}(s).$$

By standard arguments using the subadditivity of $\log \bar{\phi}$ we can see that there is a

$$\xi = \xi(\lambda) = \xi(\lambda, k, d),$$

such that as $r \to \infty$,

$$\bar{\phi}(r) \approx e^{-\xi r},$$

where $\approx$ means that the logarithms of both sides are asymptotic. Moreover, $\bar{\phi}(r) \geq e^{-\xi r}$. The first major goal of this section is to establish the inequality

$$c_1 e^{-\xi r} \leq \phi(r), \tilde{\phi}(r), \bar{\phi}(r) \leq c_2 e^{-\xi r}. \tag{10}$$

In order to prove (10), it suffices to show the following:

$$\bar{\phi}(r) \leq c\tilde{\phi}(r), \tag{11}$$

$$\tilde{\phi}(r) \leq c\phi(r), \tag{12}$$

and for all r, s,

$$\phi(r+s) \geq c_1 \phi(r)\phi(s). \tag{13}$$

To see this, note that the first two of these equations and the trivial inequality $\phi(r) \leq \bar{\phi}(r)$ imply that

$$\tilde{\phi}(r) \asymp \phi(r) \asymp \bar{\phi}(r), \tag{14}$$

where $\asymp$ means that each side is bounded by a constant times the other side. Also, (13) implies that $f(r) = \log \phi(r) + \log c_1$ is a superadditive function and hence

$$-\xi = \lim_{r \to \infty} \frac{f(r)}{r} = \sup_{r \to \infty} \frac{f(r)}{r},$$

i.e.,

$$\phi(r) \leq c_1^{-1} e^{-r\xi}.$$

If will follow from one of our first lemmas (see (23)) that

$$\phi(r+1) \geq c\phi(r), \quad \tilde{\phi}(r+1) \geq c\tilde{\phi}(r), \quad \bar{\phi}(r+1) \geq c\bar{\phi}(r) \tag{15}$$

We will assume these inequalities now and use them to derive (11) and (12). By the Harnack inequality applied to B,

$$\bar{Z}_r \leq cY_r,$$

where

$$Y_r = \mathbf{P}_1\{B[T_1, T_r] \cap \Lambda(1,r) = \emptyset\}.$$

By applying the Harnack inequality to $B^1, \ldots, B^k$, and using Brownian scaling, we can see that for any $\mathbf{x} \in \mathcal{S}^k$,

$$\mathbf{E}^{\mathbf{x}}[Y_r^\lambda] \leq c\tilde{\phi}(r-1).$$

This gives that

$$\bar{\phi}(r) \leq c\tilde{\phi}(r-1), \tag{16}$$

and then (11) follows from (15). For (12), we first note that if $\lambda \geq 1$

$$\int_{\mathcal{S}} Z_{r,y}^\lambda \, d\sigma(y) \geq [\int_{\mathcal{S}} Z_{r,y} \, d\sigma(y)]^\lambda,$$

and hence

$$\phi(r) = \int_{\mathcal{S}} (\sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[Z_{r,y}^\lambda]) \, d\sigma(y) \geq \int_{\mathcal{S}} \int_{\mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[Z_{r,y}^\lambda] \, d\sigma^k(\mathbf{x}) \, d\sigma(y) \geq \tilde{\phi}(r).$$

Therefore, if suffices to prove (12) for $\lambda < 1$. For any $\delta > 0$, let

$$\bar{Z}_r^{(1)} = \bar{Z}_{r,\delta}^{(1)} = \sup\{Z_{r,y} : y \in \mathcal{S}, \mathrm{dist}(y, \Lambda_r) \leq \delta\},$$

$$\bar{Z}_r^{(2)} = \bar{Z}_{r,\delta}^{(2)} = \sup\{Z_{r,y} : y \in \mathcal{S}, \mathrm{dist}(y, \Lambda_r) \geq \delta\}.$$

Note that $\bar{Z}_r \leq \bar{Z}_r^{(1)} + \bar{Z}_r^{(2)}$, and since $\lambda < 1$,

$$\bar{Z}_r^\lambda \leq (\bar{Z}_r^{(1)})^\lambda + (\bar{Z}_r^{(2)})^\lambda.$$

Below we will show that there exists a δ_0 such that for all $\mathbf{x} \in \mathcal{S}^k$,

$$\mathbf{E}^{\mathbf{x}}[(\bar{Z}_r^{(1)})^\lambda] \leq \frac{1}{2}\bar{\phi}(r). \tag{17}$$

This implies for some $\mathbf{x}$,

$$\mathbf{E}^{\mathbf{x}}[(\bar{Z}_r^{(2)})^\lambda] \geq \frac{1}{2}\bar{\phi}(r).$$

By Harnack, there exists a set of $y \in \mathcal{S}$ of surface measure at least c_1 such that

$$Z_{r,y} \geq \frac{1}{2}\bar{Z}_r^{(2)}.$$

(This c_1 depends on δ_0, but we have fixed δ_0.) In particular,

$$\int_{\mathcal{S}} Z_{r,y}^\lambda \, d\sigma(y) \geq c\bar{Z}_r^{(2)},$$

and hence

$$\phi(r) \geq \mathbf{E}^x[\int_{\mathcal{S}} Z_{r,y}^\lambda \, d\sigma(y)] \geq c\bar{\phi}(r).$$

To derive (17), we first assume $d = 2$. Let

$$Y_r = \sup_{y \in \mathcal{S}_2} \mathbf{P}^y\{B[0, T_r] \cap \Lambda(2, r) = \emptyset\},$$

$$X_r = X_{r,\delta} = \sup \mathbf{P}^z\{B[0, T_1] \cap \Lambda_r = \emptyset\},$$

where the supremum is over all $z \in \mathcal{S}$ with $\mathbf{dist}(z, \Lambda_r) \leq \delta$. Note that

$$\bar{Z}_r^{(1)} \leq X_r Y_r,$$

and by the Beurling estimates (Lemma 3.2), $X_r \leq c\delta^{1/2}$. Hence

$$\mathbf{E}^{\mathbf{x}}[(\bar{Z}_r^{(1)})^\lambda] \leq c\delta^{\lambda/2}\mathbf{E}^{\mathbf{x}}[Y_r^\lambda] \leq c\delta^{\lambda/2}\bar{\phi}(r-2) \leq c\delta^{\lambda/2}\bar{\phi}(r).$$

In particular for δ sufficiently small,

$$\mathbf{E}^{\mathbf{x}}[(\bar{Z}_r^{(1)})^\lambda] \leq \frac{1}{2}\bar{\phi}(r).$$

For $d = 3$, let

$$\rho^j = \rho^j(r) = \inf\{t : B_t^j \in \mathcal{S}_2; B^j[t, T_r] \cap \mathcal{B}(0, 1) = \emptyset\},$$
$$\hat{\Lambda}_r = B^1[\rho^1, T_r^1] \cup \cdots \cup B^k[\rho^k, T_r^k].$$

Let

$$Y_r = \sup_{y \in \mathcal{S}_2} \mathbf{P}^y\{B[0, T_r] \cap \hat{\Lambda}(2, r) = \emptyset\},$$
$$X_r = X_{r,\delta} = \sup \mathbf{P}^z\{B[0, T_1] \cap \Lambda_r = \emptyset\},$$

where the supremum is over all $z \in \mathcal{S}$ with $\mathbf{dist}(z, \Lambda_r) \leq \delta$. One can easily check that the distribution of Y_r given

$$\Lambda_r \cap \mathcal{B}(0, e) = \emptyset,$$

is bounded above and below by a constant times the unconditioned distribution. Note also that X_r, Y_r are independent and $\bar{Z}_r^{(1)} \leq X_r Y_r$; hence

$$\mathbf{E}^{\mathbf{x}}[(\bar{Z}_r^{(1)})^\lambda] \leq c\mathbf{E}^{\mathbf{x}}[X_r^\lambda]E^{\mathbf{x}}[Y_r^\lambda].$$

It is not difficult (using, say, Lemma 4.4 below as well as the transience of Brownian motion) to show

$$\mathbf{E}^{\mathbf{x}}[X_r^\lambda] \leq \epsilon(\delta),$$

for some $\epsilon(\delta) \to 0$. Also, the transience of the Brownian motion implies that with positive probability (independent of r),

$$\hat{\Lambda}_r = \Lambda(2, r).$$

Hence, it is not difficult to see that

$$E^{\mathbf{x}}[Y_r^\lambda] \leq c\bar{\phi}(r-2) \leq c\bar{\phi}(r).$$

This finishes (17) and hence we have derived (11) and (12).

The next lemmas are important technical lemmas that make precise the notion that Brownian motions conditioned not to intersect have a reasonable chance of being far apart. It will be convenient for us to set up some notation. Let $\bar{\gamma} = (\gamma^1, \ldots, \gamma^k)$ be a k-tuple of continuous functions

$$\gamma^j[0, b^j] :\to \mathbb{R}^d$$

with $|\gamma^j(b^j)| = 1$ and $|\gamma^j(t)| < 1$ for $t < b^j$ (we allow trivial functions with $b^j = 0$). As before, we start independent Brownian motions $B^1, \ldots, B^k$, at $\gamma^1(b^1), \ldots, \gamma^k(b^k)$, respectively, defined on the probability space $(\Omega, \mathbf{P})$, and let

$$\bar{\gamma}_r = \bar{\gamma} \cup \Lambda_r.$$

We also have another Brownian motion B defined on the probability space $(\Omega_1, \mathbf{P}_1)$. Depending on context, we will start B at: the origin; uniformly on $\mathcal{S}_r$ for some $r \leq 0$; or at some $|x| \leq 1$. We will write conditioning with respect to the event

$$B(0, T_0] \cap \bar{\gamma}_0 = \emptyset.$$

In the latter two cases, this conditioning will be on a set of positive probability. In the first case, we will mean by this conditioning that B, up through time T_0, is an h-process conditioned to not hit $\bar{\gamma}_0$. In particular, we will assume that $\bar{\gamma}_0$ is sufficiently nice so that this conditioning is well-defined. Below we will use the shorthand expression "$\bar{\gamma}_0$ is an h-set", to be mean that it is sufficiently nice. Note that for each r, $\bar{\gamma}_r$, appropriately scaled, is an allowable k-tuple of functions, and we can talk of probabilities given

$$B(0, T_r] \cap \bar{\gamma}_r = \emptyset.$$

For $r \geq 0$, we let

$$\delta_r = e^{-r} \min\{\text{dist}(B(T_r), \bar{\gamma}_r), \text{dist}(B^1(T_r^1), B[0, T_r]), \ldots, \text{dist}(B^k(T_r^k), B[0, T_r])\},$$

and let D_ϵ be the $\mathcal{F}_r$-measurable random variable

$$D_\epsilon = \mathbf{P}_1\{\delta_r \geq \epsilon \mid B(0, T_r] \cap \bar{\gamma}_r = \emptyset\}.$$

Let $A = A_{1/10}$ as in (7) and $-A = \{x : -x \in A\}$. For $s \leq r$, let $U(s, r), \tilde{U}(s, r)$ be the events

$$U(s, r) = \{B^j[T_s^j, T_r^j] \subset -A, j = 1, \ldots, k\},$$

$$\tilde{U}(s, r) = \{B[T_s, T_r] \subset A\}.$$

By an initial configuration we will mean a random allowable k-tuple $\bar{\gamma}$, defined on the probability space (Ω, P), where the $\bar{\gamma}$ is independent of the Brownian motions $B^1, \ldots, B^k$ (except for the fact that the Brownian motions are rotated so that they start at $\gamma(b^1), \ldots, \gamma(b^k)$, respectively). Given with the initial configuration will be a starting point for the process B_t (either $B_0 = 0$; or B_0 starts on $\mathcal{S}_s$ for some $s \leq 0$.) If we assume $B_0 = 0$, then we will assume that the initial configuration gives probability one to h-sets. If the starting point is on $\mathcal{S}_s$, then we will assume the initial configuration gives probability one to $\bar{\gamma}_0$ satisfying

$$\mathbf{P}_1\{B[0, T_0] \cap \bar{\gamma}_0 = \emptyset\} > 0.$$

When an initial configuration is given we will let $\mathcal{F}_r$ denote the σ-algebra generated by $\bar{\gamma}$ and the Brownian motions $B^1, \ldots, B^k$ up through times $T_r^1, \ldots, T_r^k$. Let

$$Z_r = \mathbf{P}_1 \Big\{ B(0, T_r] \cap \bar{\gamma}_r = \emptyset \mid B(0, T_0] \cap \bar{\gamma}_0 = \emptyset \Big\},$$

where the conditioning is in terms of h-processes if necessary. For $s < r$, let

$$\tilde{Z}(s, r) = \mathbf{P}_1 \Big\{ B(0, T_r] \cap \bar{\gamma}_r = \emptyset; \tilde{U}(s, r) \mid B(0, T_0] \cap \bar{\gamma}_0 = \emptyset \Big\}.$$

Lemma 4.1 *There exist constants c_1, β such that if the initial configuration satisfies*

$$\mathbf{P}\{D_\epsilon \geq \epsilon\} = 1,$$

then

$$\mathbf{E}[\tilde{Z}(\frac{1}{4}, 1)^\lambda; U(\frac{1}{4}, 1)] \geq c_1 \epsilon^\beta.$$

Proof. Without loss of generality, we may assume that the initial configuration is a point mass at $\bar{\gamma}_0$ satisfying $D_\epsilon \geq \epsilon$. Cover the sphere of radius 1 by $O(\epsilon^{-(d-1)})$ balls of radius $\epsilon/16$ centered on the sphere. Then we can see that there must by an x such that

$$\mathbf{P}_1\{\delta_0 \geq \epsilon; B(T_0) \in \mathcal{B}(x, \epsilon/16) \mid B(0, T_0] \cap \bar{\gamma}_0 = \emptyset\} \geq c\epsilon^{d-1}\epsilon.$$

Without loss of generality we may assume $x = u$. The lemma now boils down to an estimate on Brownian motions: see Lemmas 3.6 and 3.7. $\square$

Lemma 4.2 *There exists a constant c such that for any initial configuration*

$$\mathbf{E}[\tilde{Z}(\frac{1}{2}, 1)^\lambda; U(\frac{1}{2}, 1)] \geq c\mathbf{E}[Z_1^\lambda].$$

Proof. Choose N sufficiently large so that

$$\sum_{n=N}^{\infty} n^2 2^{-n} \leq \frac{1}{8}.$$

Let $h_n = 1/4$ for $n \leq N$ and for $n > N$,

$$h_n = h_{n-1} + n^2 2^{-n},$$

so that $h_n \leq 3/8$ for all n. Let $D^n = D_{2^{-n}}$ and

$$r(n) = \inf \frac{\mathbf{E}[\tilde{Z}(h_n, 1)^\lambda; U(h_n, 1)]}{\mathbf{E}[Z_{h_n}^\lambda]},$$

where the infimum is over all initial configurations satisfying

$$D^n \geq \frac{1}{2}.$$

Note that the numerator in the infimum is increasing in h_n while the denominator is decreasing in h_n. Hence

$$\mathbf{E}[\tilde{Z}(\frac{1}{2},1)^\lambda; U(\frac{1}{2},1)] \geq r\mathbf{E}[Z_1^\lambda],$$

where $r = \inf_n r(n)$. By Lemma 4.1,

$$\mathbf{E}[\tilde{Z}(h_n;1)^\lambda; U(h_n,1)] \geq c_1 2^{-\beta n}, \tag{18}$$

for any such configuration, and hence $r(n) > 0$ for each n. We will show that for $n > N$,

$$r(n) \geq (1 - \frac{c}{n^2}) r(n-1), \tag{19}$$

and hence $r > 0$. This will prove the lemma.

Assume $n > N$ and we have an initial configuration with

$$D^n \geq \frac{1}{2}.$$

Let $D(j) = D(j,n)$ be the $\mathcal{F}_{j2^{-n}}$-measurable random variable

$$\mathbf{P}_1\{\delta_{j2^{-n}} \geq 2^{-n-1} \mid B(0, T_{j2^{-n}}] \cap \bar{\gamma}_{j2^{-n}} = \emptyset\},$$

and let $\sigma = \sigma_n$ be the smallest positive integer j with $D(j) \geq 1/2$. Then if $j < n^2$, on the event $\{\sigma = j\}$,

$$\mathbf{E}\Big[\tilde{Z}(h_n,1)^\lambda; U(h_n,1) \mid \mathcal{F}_{j2^{-n}}\Big] = Z_{j2^{-n}}^\lambda \mathbf{E}\Big[(\frac{\tilde{Z}(h_n,1)}{Z_{j2^{-n}}})^\lambda; U(h_n,1) \mid \mathcal{F}_{j2^{-n}}\Big],$$

$$\mathbf{E}\Big[Z_{h_n}^\lambda \mid \mathcal{F}_{j2^{-n}}\Big] = Z_{j2^{-n}}^\lambda \mathbf{E}\Big[(\frac{Z_{h_n}}{Z_{j2^{-n}}})^\lambda \mid \mathcal{F}_{j2^{-n}}\Big].$$

The definition of $r(n)$ and Brownian scaling can be used to see that on $\{\sigma = j\}$,

$$\mathbf{E}\Big[(\frac{\tilde{Z}(h_n,1)}{Z_{j2^{-n}}})^\lambda; U(h_n,1) \mid \mathcal{F}_{j2^{-n}}\Big] \geq r(n-1)\mathbf{E}\Big[(\frac{Z_{h_n}}{Z_{j2^{-n}}})^\lambda \mid \mathcal{F}_{j2^{-n}}\Big],$$

and hence

$$\mathbf{E}[\tilde{Z}(h_n,1)^\lambda; U(h_n,1); \sigma = j] \geq r(n-1)\mathbf{E}[Z(h_n)^\lambda; \sigma = j].$$

Therefore,

$$\mathbf{E}[\tilde{Z}(h_n,1)^\lambda; U(h_n,1); \sigma < n^2] \geq r(n-1)\mathbf{E}[Z(h_n)^\lambda; \sigma < n^2]. \tag{20}$$

However, it is not difficult to see that there is a positive c such that if $D^{n-1} \leq 1/2$,

$$\mathbf{E}[Z(2^{-n})^\lambda; D(1) < \frac{1}{2}] \leq 1 - c.$$

By continuing we see that this implies

$$\mathbf{P}\{\sigma \geq n^2\} \leq (1-c)^{n^2},$$

and hence

$$\mathbf{E}[Z(h_n)^\lambda; \sigma \geq n^2] \leq ce^{-\beta_1 n^2}.$$

This and (18) imply that

$$\mathbf{E}[Z(h_n)^\lambda; \sigma < n^2] \geq (1 - \frac{c}{n^2})\mathbf{E}[Z(h_n)^\lambda],$$

which combined with (20) gives (19) and hence the lemma. □

We return to the random variable

$$Z_{r,u} = \mathbf{P}_1^u\{B[0, T_r] \cap \Lambda_r = \emptyset\}.$$

As above, let $A = A_{1/10}$, and

$$Z'_{r,u} = \mathbf{P}_1^u\{B[0, T_r] \cap \Lambda_r = \emptyset; B[T_{r-(1/2)}, T_r] \subset A\},$$

$$Z_u(s, r) = \frac{Z_{r,u}}{Z_{s,u}}, \quad Z'_u(s, r) = \frac{Z'_{r,u}}{Z_{s,u}}.$$

By applying Lemma 4.2 to the configuration $\bar{\gamma}_{r-1}$, we see that for $r \geq 2$, $\mathbf{x} \in \mathcal{S}^k$

$$\mathbf{E}^{\mathbf{x}}[Z'_u(r-1, r)^\lambda; U(r) \mid \mathcal{F}_{r-1}] \geq c\mathbf{E}^{\mathbf{x}}[Z_u(r-1, r)^\lambda \mid \mathcal{F}_{r-1}], \tag{21}$$

where $U(r) = U(r - \frac{1}{2}, r)$ is as defined above. In particular,

$$\mathbf{E}^{\mathbf{x}}[(Z'_r)^\lambda; U(r)] \geq c\mathbf{E}^{\mathbf{x}}[Z^\lambda_{r,u}]. \tag{22}$$

Let q be the infimum over all $x \in \mathcal{S} \cap A$ of the probability that a Brownian motion starting at x reaches the sphere of radius e without hitting the sphere of radius $e^{-1/2}$ or hitting the hyperplane $\{y = (y^1, \cdots) : y^1 = 0\}$. Then it is easy to see from the strong Markov property that

$$\mathbf{E}^{\mathbf{x}}[Z^\lambda_{r+1,u}] \geq q^k q^\lambda \mathbf{E}^{\mathbf{x}}[(Z'_{r,u})^\lambda; U(r)].$$

Therefore,

$$\mathbf{E}^{\mathbf{x}}[Z^\lambda_{r+1,u}] \geq c\mathbf{E}^{\mathbf{x}}[Z^\lambda_{r,u}],$$

and hence

$$\phi(r+1) \geq c\phi(r). \tag{23}$$

The same argument holds with different initial configurations so we can conclude

$$\tilde{\phi}(r+1) \geq c\tilde{\phi}(r).$$

Also, using (16),

$$\bar{\phi}(r+1) \geq \tilde{\phi}(r+1) \geq c\tilde{\phi}(r-1) \geq c\bar{\phi}(r),$$

which gives us (15).

One of the keys to proving (10) is (22). The other is the following lemma.

Lemma 4.3 *Let $A = A_{1/5}$ as in (7); let*

$$\tilde{Z}_{r,u} = \mathbf{P}_1^u\Big\{B[0,T_r]\cap\Lambda_r = \emptyset; B[0,T_r]\cap\mathcal{B}(0,2)\subset A\setminus\mathcal{B}(0,\frac{4}{5})\Big\}.$$

and let J_r the event

$$J_r = \Big\{B^j[0,T_r^j]\cap\mathcal{B}(0,2)\subset -A\setminus\mathcal{B}(0,\frac{4}{5}), j=1,\ldots,k\Big\}.$$

There exists a c such that if $\mathbf{x}=(x_1,\ldots,x_k)\in\mathcal{S}^k$ with $|x_j+u|\le 1/10, j=1,\ldots,k$, $r\ge 2$

$$\mathbf{E}^{\mathbf{x}}[\tilde{Z}_{r,u}^\lambda; J_r]\ge c\mathbf{E}^{\mathbf{x}}[Z_{r,u}^\lambda].$$

Before proving this lemma, let us derive a simple consequence. Let

$$\hat{Z}_r = \inf \mathbf{P}_1^y\Big\{B[0,T_r]\cap\Lambda_r = \emptyset; B[0,T_r]\cap\mathcal{B}(0,2)\subset A\setminus\mathcal{B}(0,\frac{4}{5})\Big\},$$

where the infimum is over all $y\in\mathcal{S}_0$ with $|y-u|\le 1/10$. By the Harnack inequality, on the event J_r, $\hat{Z}_r\ge c\tilde{Z}_r$. We can therefore replace the conclusion in Lemma 4.3 with

$$\mathbf{E}^{\mathbf{x}}[\hat{Z}_r^\lambda; J_r]\ge c\mathbf{E}^{\mathbf{x}}[Z_{r,u}^\lambda].$$

It is then an easy application of the strong Markov property to see that (22) and Lemma 4.3 imply (13) and hence Theorem 2.1. Therefore we need only prove Lemma 4.3 to finish the proof of Theorem 2.1. We will use a series of lemmas to prove this lemma. The proof of the next lemma can be found in [16, Lemma 3.4]. We state two versions of the result, but they can easily be seen to be equivalent by Brownian scaling. We note that for $d=2$ the lemma is immediate from the Beurling estimates; however, for $d=3$ some work is needed.

Lemma 4.4 *Let Y_ϵ, Y_r^* be the random variables on $(\Omega,\mathbf{P})$:*

$$Y_\epsilon = \sup_{|B^1(0)-y|\le\epsilon}\mathbf{P}_1^y\{B[0,T_1]\cap B^1[0,T_1^1]=\emptyset\},$$

$$Y_r^* = \sup_{y\in\mathcal{S}_0}\mathbf{P}_1^y\{B[0,T_r]\cap B^1[0,T_r^1]=\emptyset\},$$

For every $M<\infty$ there exist $b>0$ and $C<\infty$ such that for all $\epsilon>0, r>1$,

$$\mathbf{P}\{Y_\epsilon\ge\epsilon^b\}\le C\epsilon^M,$$

$$\mathbf{P}\{Y_r^*\ge e^{-rb}\}\le Ce^{-rM}.$$

Lemma 4.5 *There exists a β such that if $|x-u|\le\epsilon/2$,*

$$\mathbf{P}^{u,x}\Big\{B^1[0,T_1^1]\cap B^2[0,T_1^2]=\emptyset \mid B^1[0,T_1^1]\cap\mathcal{B}(0,1-\epsilon)=\emptyset\Big\}\le\epsilon^\beta,$$

$$\mathbf{P}^{u,x}\Big\{B^1[0,T_1^1]\cap B^2[0,T_1^2]=\emptyset \mid (B^1[0,T_1^1]\cup B^2[0,T_1^2])\cap\mathcal{B}(0,1-\epsilon)=\emptyset,\Big\}\le\epsilon^\beta.$$

Proof. (sketch) We will sketch the idea in the second inequality; the first is similar. Fix ϵ and assume that B^1, B^2 are h-processes conditioned so that

$$B^i[0,T_i^i] \cap \mathcal{B}(0,1-\epsilon) = \emptyset.$$

Let

$$\tau_r^i = \inf\{t : |B_t^i - u| = r\}.$$

Then if $\epsilon < \delta < 1/4$, routine estimates for this h-process can be used to show that

$$\mathbf{P}^{u,x}\{B^1[0,\tau_{2\delta}^1] \cap B^2[0,\tau_{2\delta}^2] \neq \emptyset \mid B^1[0,\tau_\delta^1] \cap B^2[0,\tau_\delta^2] = \emptyset\} \geq c.$$

The lemma is obtained by iterating this estimate. $\square$

Lemma 4.6 *For any $\epsilon > 0$, let J_ϵ be the event*

$$J_\epsilon = \{B^1[0,T_1^1] \cap \mathcal{B}(0,1-\epsilon) = \emptyset\},$$

and let X, X_ϵ be the random variables on $(\Omega, \mathbf{P})$,

$$X = \mathbf{P}_1^u\{B[0,T_1] \cap B^1[0,T_1^1] = \emptyset\},$$

$$X_\epsilon = \mathbf{P}_1^u\{B[0,T_1] \cap B^1[0,T_1^1] = \emptyset; B[0,T_1] \cap \mathcal{B}(0,1-\epsilon) = \emptyset\}.$$

Then there exist constants c_1, β such that for all $|x-u| \leq \epsilon/2$,

$$\mathbf{E}^x(X^\lambda) \leq c_1 \epsilon^\beta,$$

$$\mathbf{E}^x(X_\epsilon^\lambda) \leq c_1 \epsilon^{\lambda+\beta},$$

$$\mathbf{E}^x(X^\lambda; J_\epsilon) \leq c_1 \epsilon^{1+\beta},$$

$$\mathbf{E}^x(X_\epsilon^\lambda; J_\epsilon) \leq c_1 \epsilon^{1+\lambda+\beta}.$$

Proof. Let Y_ϵ be as in Lemmas 4.4, and let $M = \max\{3\lambda_2, 2\}$. Choose b, C so that

$$\mathbf{P}^x\{Y_\epsilon \geq \epsilon^b\} \leq C\epsilon^M.$$

Then

$$\mathbf{E}^x(X^\lambda) \leq \mathbf{E}^x(Y_\epsilon^\lambda) \leq \epsilon^{b\lambda} + C\epsilon^M.$$

This gives the first inequality. For the third inequality note that the gambler's ruin estimate (Lemma 3.1) gives $\mathbf{P}^x(J_\epsilon) \asymp \epsilon$. Hence,

$$\mathbf{E}^x(X^\lambda; J_\epsilon) \leq \mathbf{E}^x(Y_\epsilon^\lambda; J_\epsilon) \leq \epsilon^{b\lambda+1} + C\epsilon^M.$$

For the second inequality , write

$$X_\epsilon = \mathbf{P}_1^u\{B[0,T_1] \cap \mathcal{B}(0,1-\epsilon) = \emptyset\}\tilde{Y}_\epsilon,$$

where

$$Y_\epsilon = \mathbf{P}_1^u\{B[0,T_1] \cap B^1[0,T_1^1] = \emptyset \mid B[0,T_1] \cap \mathcal{B}(0,1-\epsilon) = \emptyset\}.$$

Then the gambler's ruin estimate gives

$$\mathbf{E}^x(X_\epsilon^\lambda) \leq c\epsilon^\lambda \mathbf{E}^x(\tilde{Y}_\epsilon^\lambda),$$

$$\mathbf{E}^x[X_\epsilon^\lambda; J_\epsilon] \leq c\epsilon^\lambda \mathbf{E}^x[\tilde{Y}_\epsilon^\lambda; J_\epsilon].$$

We then must prove

$$\mathbf{E}^x(\tilde{Y}_\epsilon^\lambda) \leq c\epsilon^\beta,$$

$$\mathbf{E}^x[\tilde{Y}_\epsilon^\lambda \mid J_\epsilon] \leq c\epsilon^\beta.$$

But this follows from Lemma 4.5 and the fact that if $0 \leq Y \leq 1$, then

$$\mathbf{E}[Y^\lambda] \leq \mathbf{E}[Y]^{1\wedge\lambda}. \quad \Box$$

Lemma 4.7 *There exist c, β such that for all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$,*

$$E^{\mathbf{x}}(Z_{r,u}^\lambda) \leq c\phi(r)|x_1 - u|^\beta.$$

Proof. Note that $Z_{r,u} \leq XY_r$ where

$$X = \mathbf{P}_1^u\{B[0,T_1] \cap B^1[0,T_1^1] = \emptyset\},$$

$$Y_r = Y_{r,u} = \mathbf{P}_1^u\Big\{B[T_1,T_r] \cap (B^1[T_1^1,T_r^1] \cup \cdots \cup B^k[T_1^k,T_r^k]) = \emptyset \mid B[0,T_1] \cap \Lambda_1 = \emptyset\Big\}.$$

By Brownian scaling, (14), and (23),

$$\mathbf{E}^{\mathbf{x}}(Y_r^\lambda \mid \mathcal{F}_1) \leq \bar{\phi}(r-1) \leq c\phi(r).$$

By Lemma 4.6,

$$\mathbf{E}^{\mathbf{x}}(X^\lambda) \leq c|x_1 - u|^\beta.$$

Hence

$$\mathbf{E}^x(Z_{r,u}^\lambda) \leq \mathbf{E}^{\mathbf{x}}(X^\lambda Y_r^\lambda) = \mathbf{E}^{\mathbf{x}}[X^\lambda \mathbf{E}(Y_r^\lambda \mid \mathcal{F}_1)] \leq c|x_1 - u|^\beta \phi(r). \quad \Box$$

Lemma 4.8 *For any $\epsilon > 0$, let $V_r^j = V_r^j(\epsilon)$ be the event*

$$V_r^j = \{B^j[0,T_r^j] \cap \mathcal{B}(0,1) \subset \mathcal{B}(B^j(0),\epsilon)\}.$$

There exists a constant c such that for all $\epsilon > 0$, and $j = 1, \ldots, k$,

$$\sup_{\mathbf{x}\in\mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[Z_{r,u}^\lambda; V_r^1 \cap \cdots \cap V_r^j] \geq c\epsilon^j \phi(r).$$

Proof. Note that the gambler's ruin estimate can be used to conclude that

$$\mathbf{P}^{\mathbf{x}}(V_1^1 \cap \cdots \cap V_1^j) \asymp \epsilon^j.$$

By an argument similar to those in Lemmas 4.6 and 4.7, we can see that if $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$,

$$\mathbf{E}^{\mathbf{x}}[Z_{r,u}^{\lambda}; V_r^1 \cap \cdots \cap V_r^j] \leq c_1 |x_1 - u|^{\beta} \phi(r) \epsilon^j. \tag{24}$$

Without loss of generality we may assume $r > 1, \epsilon < 1/2$. We will write Z for $Z_{r,u}$. For any j, let

$$\rho^j = \rho_\epsilon^j = \inf\{t : B_t^j \in \mathcal{B}(0,1) \setminus \mathcal{B}(B^j(0), \epsilon)\},$$
$$\sigma^j = \sigma_\epsilon^j = \inf\{t : |B_t^j| = 1 - \epsilon\},$$
$$\tau^j = \tau_\epsilon^j = \inf\{t \geq \rho^j : |B_t^j| = 1\}.$$

It is not difficult to show that

$$\mathbf{P}^{\mathbf{x}}\{\rho^j \leq \sigma^j \leq \tau^j \leq T_r^j\} \geq c, \tag{25}$$

where we emphasize that c does not depend on r or ϵ. Choose $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$, which may depend on r, such that

$$\mathbf{E}^{\mathbf{x}}[Z^{\lambda}] = \phi(r).$$

By Lemma 4.7, there is a δ_1 such that if $|y_1 - u| \leq \delta_1$,

$$\mathbf{E}^{y_1, x_2, \ldots, x_k}[Z^{\lambda}] \leq \frac{1}{2}\phi(r).$$

Fix such a δ_1. Using (25), we claim that there exists a c (which may depend on δ_1, but we have fixed δ_1) such that

$$\mathbf{P}^{x_1}\{|B^1(\tau^1) - u| \leq \delta_1 \mid \rho^1 \leq \sigma^1 \leq \tau^1 \leq T_r^1\} \geq c\epsilon.$$

(The argument to establish the claim goes approximately as follows: with probability at least c we have $\sigma^1 < \tau^1 < T_r^1$; given $\sigma^1 < T_r^1$, gambler's ruin implies that with probability at least $c\epsilon$, $T_{-1}^1 < T_r^1$; and given $T_{-1}^1 < T_r^1$, there is at least a $c\delta_1^{d-1}$ probability that $|B^1(\tau^1) - u| \leq \delta_1$.) Hence, using the strong Markov property on B^1,

$$\mathbf{E}^{\mathbf{x}}[Z^{\lambda}; \rho^j \leq T_r^j] \leq (1 - c\epsilon)\phi(r),$$

and hence

$$\mathbf{E}^{\mathbf{x}}[Z^{\lambda}; V_r^1] = \mathbf{E}^{\mathbf{x}}[Z^{\lambda}; \rho^1 > T_r^j] \geq c\epsilon\phi(r).$$

This proves the lemma for $j = 1$.

To prove the result for general j, let

$$\phi_j(r) = \phi_{j,\epsilon}(r) = \sup_{\mathbf{x} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{x}}[Z^{\lambda}; V_r^1 \cap \cdots \cap V_r^j].$$

Suppose we have shown that

$$\phi_{j-1}(r) \geq c\epsilon^{j-1}\phi(r).$$

Choose $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$, which may depend on r, such that

$$\phi_{j-1}(r) = \mathbf{E}^{\mathbf{x}}[Z^{\lambda}; V_r^1 \cap \cdots \cap V_r^{j-1}].$$

By (24), there exists a δ_j such that for $|y-u| \leq \delta_j$,

$$\mathbf{E}^{y_1,x_2,\ldots,x_k}[Z^\lambda; V_r^1 \cap \cdots \cap V_r^{j-1}] \leq \frac{1}{2}\phi_{j-1}(r).$$

We now define times ρ^j, σ^j, τ^j and proceed in the same way as the $j=1$ case. $\square$

We note that from (24) we can conclude that, for all ϵ small, the $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$ for which the supremum in the conclusion of Lemma 4.8 is attained satisfies

$$|x_j - u| \geq 20\epsilon. \tag{26}$$

Likewise (for small ϵ), if $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$ with $|x_j - u| < 5\epsilon$ for some j,

$$\mathbf{E}^{\mathbf{x}}[Z_{r,u}^\lambda; V_r^1 \cap \cdots \cap V_r^k] \leq (\frac{1}{10}) \sup_{\mathbf{y} \in \mathcal{S}^k} \mathbf{E}^{\mathbf{y}}[Z_{r,u}^\lambda; V_r^1 \cap \cdots \cap V_r^k]. \tag{27}$$

We now fix an $\epsilon_0 > 0$ sufficiently small such that (26) and (27) hold, and such that for each $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$,

$$\mathcal{S} \not\subset \mathcal{B}(x^1, 30\epsilon_0) \cup \cdots \cup \mathcal{B}(x^k, 30\epsilon_0). \tag{28}$$

Since we have fixed ϵ_0, constants $c, c_1, c_2, \ldots$ may now depend on ϵ_0.

Lemma 4.9 *Let $V_r^j = V_r^j(\epsilon_0)$ as in Lemma 4.8. Let*

$$\tilde{Z}_r = \tilde{Z}_{r,u} = \mathbf{P}_1^u\Big\{B[0,T_r] \cap \Lambda_r = \emptyset; B[0,T_r] \cap \mathcal{B}(0,1) \subset \mathcal{B}(u, 1-\epsilon_0)\Big\}.$$

There exists a $c > 0$ such that for all r,

$$\sup \mathbf{E}^{\mathbf{x}}[\tilde{Z}_r^\lambda; V_r^1 \cap \cdots \cap V_r^k] \geq c\phi(r),$$

where the supremum is over all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$ with $|x_j - u| \geq 20\epsilon_0$.

Proof. We will write Z_r for $Z_{r,u}$. Choose $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$, which may depend on r, such that

$$\mathbf{E}^{\mathbf{x}}[Z_r; V_r^1 \cap \cdots \cap V_r^k] = q(r),$$

where $q(r)$ is the supremum in the conclusion of Lemma 4.8 with $j = k$ and $\epsilon = \epsilon_0$. By (26), $|x_j - u| \geq 20\epsilon_0$. Let

$$\rho = \inf\{t : B(t) \in \mathcal{B}(0,1) \setminus \mathcal{B}(u, 1-\epsilon_0)\},$$

$$\tau = \inf\{t > \rho : |B(t)| = 1\},$$

$$\tilde{T}_r = \inf\{t > \tau : B(t) \in \mathcal{S}_r\}.$$

Let

$$X_r = \mathbf{P}_1^u\{\tau < T_r; B[\tau, \tilde{T}_r] \cap \Lambda_r = \emptyset\}.$$

Note that

$$Z_r - \tilde{Z}_r \leq X_r.$$

But an argument as in Lemma 4.8 can be used to show that

$$\mathbf{E}^{\mathbf{x}}[X_r^\lambda; V_r^1 \cap \cdots \cap V_r^k] \leq (1-c)q(r).$$

Hence

$$\mathbf{E}^{\mathbf{x}}[Z_r^\lambda - (Z_r - \tilde{Z}_r)^\lambda; V_r^1 \cap \cdots \cap V_r^k] \geq (1-c)q(r).$$

The lemma now follows, using Lemma 3.10. $\square$

For any $a > 0$, let

$$\hat{V}_r^j = \hat{V}_r^j(\epsilon_0, a) = \{B^j[0, T_r^j] \cap \mathcal{B}(0,1) \subset \mathcal{B}(B^j(0), \epsilon_0); B^j[0, T_r^j] \cap \mathcal{B}(u, a) = \emptyset\},$$

$$\hat{Z}_r = \hat{Z}_r(a) = \mathbf{P}_1^u\{B[0,T_r] \cap \Lambda_r = \emptyset; B[0,T_r] \cap \mathcal{B}(0,1) \subset \mathcal{B}(u, \epsilon_0);$$
$$B[0,T_r] \cap \mathcal{B}(B^j(0), a) = \emptyset, j = 1, \ldots, k\}.$$

By using Lemma 4.6, we can see that by choosing $a = a_0$ sufficiently small we have

$$\sup \mathbf{E}^{\mathbf{x}}[\hat{Z}_r^\lambda; \hat{V}_r^1 \cap \cdots \cap \hat{V}_r^k] \geq c\phi(r),$$

where the supremum is over all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$ with $|x_j - u| \geq 20\epsilon_0$. With this fixed $a_0 > 0$, we can then use the Harnack inequality to conclude the following.

Corollary 4.10 *Let $\hat{V}_r^j = \hat{V}_r^j(\epsilon_0, a_0)$ be as above and for $z \in \mathcal{S}_0$, let*

$$\hat{Z}_{r,z} = \hat{Z}_{r,z}(a_0) = \mathbf{P}_1^z\Big\{B[0,T_r] \cap \Lambda_r = \emptyset; B[0,T_r] \cap \mathcal{B}(0,1) \subset \mathcal{B}(u,\epsilon_0);$$
$$B[0,T_r] \cap \mathcal{B}(B^j(0), a_0) = \emptyset, j = 1, \ldots, k\Big\}.$$

Then

$$\sup_{\mathbf{x}} \inf_{\mathbf{y}} \inf_{z} \mathbf{E}^y[\hat{Z}_{r,z}^\lambda; \hat{V}_r^1 \cap \cdots \cap \hat{V}_r^k] \geq c\phi(r),$$

where the supremum is over all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{S}^k$ with $|x_j - u| \geq 20\epsilon_0$; the first infimum is over all $\mathbf{y} \in \mathcal{S}^k$ with $|y - x| \leq a_0/2$; and the second infimum is over all $z \in \mathcal{S}_0$ with $|z - u| \leq a_0/2$.

Let $A = A_{1/10}$ as before, and let $W = W(x_1, \ldots, x_k, \epsilon_0, a_0)$ be the event:

$$(i) \;\; B[0, T_1] \cap \Lambda_1 = \emptyset;$$

$$(ii) \;\; |e^{-1}B^j(T_1^j) - x_j| \leq a_0/2, j = 1, \ldots, k;$$

$$(iii) \;\; |e^{-1}B(T_1) - u| \leq a_0/2;$$

$$(iv) \;\; B^j[0, T_1^j] \cap \{|z| \geq e(1-\epsilon_0)\} \subset \mathcal{B}(ex_j, 3e\epsilon_0), j = 1, \ldots, k;$$

and

$$(v) \;\; B[0, T_1] \cap \{|z| \geq e(1-\epsilon_0)\} \subset \mathcal{B}(eu, 3e\epsilon_0).$$

Let $\mathbf{y} = (y_1, \ldots, y_k) \in \mathcal{S}$ and let $B^1, \ldots, B^k, B$ start at $y_1, \ldots, y_k, u$ respectively. Then it is straightforward to show that there is a $c > 0$ such that for all $|x^j - u| \geq 20\epsilon_0$,

$$\bar{\mathbf{P}}[W] \geq c.$$

It is then not difficult to use this estimate with Corollary 4.10, to conclude Lemma 4.3 and hence we have (13). In particular, we may replace $\phi(r)$ with $ce^{-\xi r}$ in all the results above.

We now summarize some of the main results of this section in a way that we will use them. Assume we have a random initial configuration $\bar{\gamma}_0$ and assume we have a Brownian motion B starting at the origin. We assume that $\bar{\gamma}_0$ is an h-set with probability one so that conditioning given

$$B(0, T_0] \cap \bar{\gamma}_0 = \emptyset,$$

makes sense. Let

$$Z_r = \mathbf{P}_1 \Big\{ B(0, T_r] \cap (\bar{\gamma}_0 \cup \Lambda_r) = \emptyset \mid B(0, T_0] \cap \bar{\gamma}_0 = \emptyset \Big\},$$

and for $s < r$,

$$Z(s, r) = \mathbf{P}_1 \Big\{ B(0, T_r] \cap (\bar{\gamma}_0 \cup \Lambda_r) = \emptyset \mid B(0, T_s] \cap (\bar{\gamma}_0 \cup \Lambda_s) = \emptyset \Big\}.$$

Note that $Z_r = Z_s Z(s, r)$ and $Z(s, r) \leq \bar{Z}(s, r)$ where

$$\bar{Z}(s, r) = \sup_{y \in \mathcal{S}_s} \mathbf{P}^y \{ B[0, T_r] \cap \Lambda(s, r) = \emptyset \}.$$

Let c_4 be a constant such that

$$\bar{\phi}(r) \leq c_4 e^{-\xi r}.$$

Then for all $s < r$,

$$\mathbf{E}[Z(s, r)^\lambda \mid \mathcal{F}_s] \leq \mathbf{E}[\bar{Z}(s, r)^\lambda \mid \mathcal{F}_s] \leq c_4 e^{-\xi r}.$$

Let $A = A_{1/10}$ as before. Let

$$U_r = \{ \Lambda(r - \frac{1}{2}, r) \subset -A \},$$

$$\tilde{Z}(r - 1, r) = \mathbf{P}_1 \Big\{ B(0, T_r] \cap (\bar{\gamma}_0 \cup \Lambda_r) = \emptyset; B[T_{r-(1/2)}, r] \subset A \mid B(0, T_{r-1}] \cap (\bar{\gamma}_0 \cup \Lambda_{r-1}) = \emptyset \Big\}.$$

Then we have proved the following.

Corollary 4.11 *There exists a constant c_3 such that for any initial configuration and every $r \geq 1$,*

$$\mathbf{E}[\tilde{Z}(r - 1, r)^\lambda; U_r \mid \mathcal{F}_{r-1}] \geq c_3 \mathbf{E}[Z(r - 1, r)^\lambda \mid \mathcal{F}_{r-1}].$$

Moreover for all $s > r$,

$$\mathbf{E}[\tilde{Z}(r - 1, s)^\lambda \mid \mathcal{F}_{r-1}] \geq c_3 e^{-\xi(s-r)} \mathbf{E}[\tilde{Z}(r - 1, r)^\lambda \mid \mathcal{F}_{r-1}].$$

We note that

$$\begin{aligned}\mathbf{E}[\tilde{Z}(r-1,s)^\lambda \mid \mathcal{F}_{r-1}] &\leq \mathbf{E}[\tilde{Z}(r-1,r)^\lambda \bar{Z}(r,s)^\lambda \mid \mathcal{F}_{r-1}] \\ &= \mathbf{E}[\mathbf{E}[\tilde{Z}(r-1,r)^\lambda \bar{Z}(r,s)^\lambda \mid \mathcal{F}_r] \mid \mathcal{F}_{r-1}] \\ &\leq c_4 e^{-\xi(s-r)} \mathbf{E}[\tilde{Z}(r-1,r)^\lambda \mid \mathcal{F}_{r-1}],\end{aligned}$$

so the inequality goes in both directions. We will often restrict ourselves to initial configurations such that

$$c_3 e^{-\xi r} \leq \mathbf{E}[Z_r^\lambda] \leq c_4 e^{-\xi r}. \tag{29}$$

This will not be a big restriction. If a given configuration $\bar{\gamma}_0$ does not satisfy the lower bound (all configurations satisfy the upper bound), we consider the configuration generated by $\bar{\gamma}_0 \cup \Lambda_1$.

Assume that our initial configuration satisfies (29). Assume $1 \leq s \leq 8$, $0 \leq m \leq n-s$, and consider the measure $\mathbf{Q}_n$ on Ω whose density is

$$\mathbf{E}[Z_n^\lambda]^{-1} Z_n^\lambda.$$

Note that $\mathbf{Q}_n$ depends on λ and the initial configuration. By (29) we know that

$$\mathbf{E}[Z_n^\lambda]^{-1} Z_n^\lambda \asymp e^{n\xi} Z_n^\lambda.$$

Assume V is an event that is measurable with respect to $\Lambda(m, m+s)$ and let

$$\tilde{p}(V) = \sup \mathbf{P}(V \mid \mathcal{F}_m).$$

Then we have shown that

$$\mathbf{E}[Z_n^\lambda; V \mid \mathcal{F}_m] \leq c Z_m^\lambda \tilde{p}(V) e^{-\xi(n-m)}.$$

In particular,

$$\mathbf{Q}_n[V \mid \mathcal{F}_m] \leq c\tilde{p}(V). \tag{30}$$

Conversely, suppose that the event V is measurable with respect to $\Lambda(m+1, m+s)$. Suppose also that every $\Lambda(m+1, m+s)$ in V satisfies

$$\Lambda(m+1, m+s) \subset -A_{1/10}.$$

Let

$$\hat{p}(V) = \inf \mathbf{P}(V \mid \mathcal{F}_m).$$

Then by Corollary 4.11 we can see that

$$\mathbf{E}[Z_n^\lambda; V \mid \mathcal{F}_m] \geq c\mathbf{E}[Z_{m+1}^\lambda \mid \mathcal{F}_m] \hat{p}(V) e^{-(n-m)\xi}.$$

In particular,

$$\mathbf{Q}_n[V \mid \mathcal{F}_m] \geq c\hat{p}(V). \tag{31}$$

We finish this section with some estimates that will be needed in future sections. It is well known and easy to verify that there is a β such that for $a > 0$, $m > 0$,

$$\mathbf{P}[T_{m+1}^j - T_m^j \geq a e^{2m}\} \leq e^{-a\beta}.$$

Hence, if $0 \le m \le n-1$, $a > 0$,

$$\begin{aligned}\mathbf{E}\Big[Z_n^\lambda; T_{m+1}^j - T_m^j \ge ae^{2m} \mid \mathcal{F}_m\Big] &\le \mathbf{E}\Big[Z_m^\lambda; \bar{Z}(m+1,n)^\lambda; T_{m+1}^j - T_m^j \ge ae^{2m} \mid \mathcal{F}_m\Big] \\ &\le ce^{-\xi(n-m)} e^{-a\beta} Z_m^\lambda. \end{aligned} \tag{32}$$

Similarly, if we let

$$Z_n^* = Z_n^*(m,a) = \mathbf{P}^1\{B(0,T_n) \cap (\bar{\gamma}_0 \cup \Lambda) = \emptyset; T_{m+1} - T_m \ge ae^{2m} \mid B(0,T_0) \cap \bar{\gamma}_0 = \emptyset\},$$

then

$$\mathbf{E}[(Z_n^*)^\lambda \mid \mathcal{F}_m] \le ce^{-\xi(n-m)} e^{-a\beta} Z_m^\lambda. \tag{33}$$

The next lemma quantifies the idea that Brownian motions conditioned to not intersect are transient.

Lemma 4.12 *There exist c_1, β such that the following holds. Assume $s < m \le r$, and let $J^j = J^j(s,m,r)$ be the event*

$$J^j = \{B^j[T_m^j, T_r^k] \cap \mathcal{B}(0, e^s) \ne \emptyset\}.$$

Then if $\mathbf{x} \in \mathcal{S}^k$

$$\mathbf{E}^x[\bar{Z}(m,r)^\lambda; J^1 \cup \cdots \cup J^k \mid \mathcal{F}_m] \le c_1 e^{-(m-s)\beta} e^{-(r-m)\xi}.$$

Proof. It suffices to show that

$$\mathbf{E}^x[\bar{Z}(m,r)^\lambda; J^1 \mid \mathcal{F}_m] \le ce^{-(m-s)\beta} e^{-(r-m)\xi}.$$

Let

$$\begin{aligned} \rho = \rho(m,s) &= \inf\{t \ge T_m^1 : B^1(t) \in \mathcal{S}_s\}, \\ \tau = \tau(m,s) &= \inf\{t \ge \rho : B^1(t) \in \mathcal{S}_m\}, \\ \Lambda' = \Lambda'(m,s,r) &= B^1[\tau, T_r^1] \cup B^2[T_m^2, T_r^2] \cup \cdots \cup B^k[T_m^k, T_r^k], \\ Z' = Z'(m,s,r) &= \sup_{x \in \mathcal{S}_m} \mathbf{P}_1^x\{B[0,T_r] \cap \Lambda' = \emptyset\}. \end{aligned}$$

Assume $d = 3$ and write $J = J^1$ for both the event and the indicator function of the event. Then

$$\bar{Z}(m,r)^\lambda J \le Z' J.$$

By transience of three dimensional Brownian motion (see Lemma 3.3),

$$\mathbf{P}^{\mathbf{x}}(J \mid \mathcal{F}_m) \le e^{-(m-s)}.$$

Hence, by the strong Markov property

$$\mathbf{E}^x[J(Z')^\lambda \mid \mathcal{F}_m] \le ce^{-(m-s)} e^{-\xi(r-m)}.$$

Now assume $d = 2$. Let

$$Y = Y(s,m) = \mathbf{P}_1^u\{B[0,T_m] \cap B[T_m^1, \rho] = \emptyset \mid B[0,T_m] \cap \Lambda_m = \emptyset\}.$$

By using the Beurling projection estimate, Lemma 3.2 (using $B[0,T_m]$ as the fixed curve and $B^1[T^1_m,\rho]$ as the Brownian motion), we can see that

$$\mathbf{E}[Y^\lambda \mid \mathcal{F}_m] \leq ce^{-\beta(m-s)}.$$

But,

$$\bar{Z}(m,r)^\lambda J \leq YZ',$$

so again we get the lemma. □

Lemma 4.12 concerns the transience of $B^1,\ldots,B^k$. The transience of B can be proved similarly (proof omitted):

Lemma 4.13 *There exist c_1,β such that the following holds. Let $s \leq m \leq r$ and*

$$Z'' = Z''(s,m,r) = \mathbf{P}_1\Big\{B(0,T_r]\cap(\bar{\gamma}_0\cup\Lambda_r) = \emptyset; B[T_m,T_n]\cap\mathcal{B}(0,e^s) \neq \emptyset \mid B(0,T_m]\cap(\bar{\gamma}_0\cup\Lambda_m) = \emptyset\Big\}.$$

Then for all $\mathbf{x} \in \mathcal{S}^k$,

$$\mathbf{E}^{\mathbf{x}}[(Z-Z'')^\lambda \mid \mathcal{F}_m] \leq c_1 e^{-\beta(m-s)}e^{-\xi(r-m)}.$$

The following is an easy corollary.

Corollary 4.14 *There exist c_1,β such that the following holds. Let $s \leq r$ and*

$$Z' = Z'(s,r) =$$

$$\mathbf{P}_1\Big\{B(0,T_r]\cap(\bar{\gamma}_0\cup\Lambda_r) = \emptyset; B[T_m,T_r]\cap\mathcal{B}(0,e^{m-s}) = \emptyset, s \leq m \leq n\} \mid B(0,T_0]\cap\bar{\gamma}_0 = \emptyset\Big\}.$$

Let $K = K_s$ be the event

$$K = \{\Lambda(m,r)\cap\mathcal{B}(0,e^{m-s}) = \emptyset,\ s \leq m \leq r\}.$$

Then for $r \leq s^5$,

$$\mathbf{E}[Z^\lambda] - \mathbf{E}[(Z')^\lambda;K] \leq c_1 e^{-\beta s}e^{-\xi r}.$$

The following lemma will be needed in Section 6.

Lemma 4.15 *For every $\epsilon > 0$, there exist s and a such that the following is true. For $j = 1,\ldots,k$, let*

$$J_m = J_{m,n,s,j} = \{B^j[T^j_{ms},T^j_{ns}]\cap\mathcal{B}(0,e^{(m-1)s}) \neq \emptyset\},$$

and let

$$X = X_{s,n,j} = \sum_{m=1}^n J_m,$$

where we let J_m denote either the event or its indicator function. Then for every $\mathbf{x} \in \mathcal{S}^k$,

$$\mathbf{E}^{\mathbf{x}}[Z^\lambda_{sn}; X \geq \epsilon n] \leq e^{-an}e^{-\xi sn}.$$

Proof. Without loss of generality we will assume $j = 1$. Suppose

$$B^1[T^1_{ms}, T^1_{ns}] \cap \mathcal{B}(0, e^{(m-1)s}) \neq \emptyset.$$

Then there exists a smallest integer $i \geq 0$ such that

$$B^1[T^1_{ms}, T^1_{(m+i+1)s}] \cap \mathcal{B}(0, e^{(m-1)s}) \neq \emptyset,$$

i.e.,

$$B^1[T^1_{(m+i)s}, T^1_{(m+i+1)s}] \cap \mathcal{B}(0, e^{(m-1)s}) \neq \emptyset.$$

Let $\rho_m = \rho_{m,s}$ be the largest integer $i \geq 0$ such that

$$B^1[T^1_{sm}, T^1_{s(m+1)}] \cap \mathcal{B}(0, e^{(m-i)s}) \neq \emptyset.$$

Then we can see that

$$X \leq \sum_{m=0}^{n-1} \rho_m.$$

and that ρ_m is $\mathcal{F}_{s(m+1)}$-measurable. Let $\mathbf{Q} = \mathbf{Q}_{sn}$ denote the probability given by $(\mathbf{E}[Z_{sn}])^{-1} Z_{sn}$, Then by (30), we see that there exist c, β such that

$$\mathbf{Q}[\rho_{m+1} \geq i \mid \mathcal{F}_{sm}] \leq c e^{-\beta s i}.$$

The result can then be obtained by large deviation estimates for geometric random variables (see Lemma 3.8(b)), handling separately

$$\sum_{m \text{ even}} \rho_m,$$

and

$$\sum_{m \text{ odd}} \rho_m. \quad \Box$$

5 Convergence to Stationarity

In this section we will describe the results on convergence to a stationary measure. We start by setting up some notation. For $r < s$, let $\mathcal{G}_{r,s}$ be the set of continuous functions

$$\gamma : [0, b] \to \mathbb{R}^d,$$

with $\gamma(0) \in \mathcal{S}_r, \gamma(b) \in \mathcal{S}_s$, and $|\gamma(t)| < e^s,\ t < b$. The number b can be any number, and given a particular $\gamma \in \mathcal{G}_{r,s}$ we write $b(\gamma)$ or b_γ for the number b such that the domain of γ is $[0, b]$. This is a metric space under the metric ρ in which $\rho(\gamma, \tilde{\gamma}) < \epsilon$ if there exists a continuous bijective time change $h : [0, b(\gamma)] \to [0, b(\tilde{\gamma})]$ with $|t - h(t)| < \epsilon$ and $|\gamma(t) - \tilde{\gamma}(h(t))| < \epsilon$ for all $t \in [0, b(\gamma)]$. There is a natural one-to-one correspondence between $\mathcal{G}_{r,s}$ and $\mathcal{G}_{0,s-r}$ given by Brownian scaling. Specifically, if $\gamma \in \mathcal{G}_{r,s}$ we associate $\tilde{\gamma} \in \mathcal{G}_{0,s-r}$ by

$$b(\tilde{\gamma}) = e^{-2r} b(\gamma),$$

$$\tilde{\gamma}(t) = e^{-r} \gamma(e^{2r} t).$$

We will write $\mathcal{G}_s$ for $\mathcal{G}_{0,s}$, and elements of $\mathcal{G}_{r,s}$ will be considered equally well as elements of $\mathcal{G}_{s-r}$. We let $\mathcal{C}_s = \mathcal{G}_{-\infty,s}$ be the set of functions $\gamma : [0, b] \to \mathbb{R}^d$ with $\gamma(0) = 0; \gamma(b) \in \mathcal{S}_s$ and $|\gamma(t)| < e^s,\ t < b$. There is a natural one-to-one correspondence between $\mathcal{C}_s$ and $\mathcal{C}_0$ given by Brownian scaling. We let $\tilde{\mathcal{G}}_{r,s}$ be the set of excursions from $\mathcal{S}_r$ to $\mathcal{S}_s$, i.e., the set of $\gamma \in \mathcal{G}_{r,s}$ with $|\gamma(t)| > e^r,\ t > 0$. Again we set $\tilde{\mathcal{G}}_s = \tilde{\mathcal{G}}_{0,s}$ and note that there is a natural one-to-one correspondence between $\tilde{\mathcal{G}}_{r,s}$ and $\tilde{\mathcal{G}}_{s-r}$.

If $0 < r < s < n$, we define

$$\Phi_s : \mathcal{G}_{r,n} \to \mathcal{G}_{s,n},$$
$$\tilde{\Phi}_s : \mathcal{G}_{r,n} \to \tilde{\mathcal{G}}_{s,n},$$

as follows. Let $\gamma \in \mathcal{G}_{r,n}$ and let $b = b(\gamma)$. Let

$$\tau = \tau_s = \inf\{t : \gamma(t) \in \mathcal{S}_s\},$$

$$\sigma = \sigma_s = \sup\{t \le b : \gamma(t) \in \mathcal{S}_s\}.$$

Then set

$$b(\Phi_s\gamma) = b - \tau, \quad b(\tilde{\Phi}_s\gamma) = b - \sigma,$$
$$(\Phi_s\gamma)(t) = \gamma(t + \tau), \quad 0 \le t \le b - \tau,$$
$$(\tilde{\Phi}_s\gamma)(t) = \gamma(t + \sigma), \quad 0 \le t \le b - \sigma.$$

Alternatively, by abuse of notation, we will write

$$\Phi_s\gamma = \gamma[\tau, b],$$

$$\tilde{\Phi}_s\gamma = \gamma[\sigma, b].$$

(It will be convenient for us to use this notation below; it should not present any confusion.) In a similar fashion we can define

$$\Phi_s : \mathcal{C}_n \to \mathcal{G}_{s,n},$$
$$\tilde{\Phi}_s : \mathcal{C}_n \to \tilde{\mathcal{G}}_{s,n}.$$

Let $\{\mathcal{O}_x; x \in \mathcal{S}\}$ be a fixed collection of rotations with $\mathcal{O}_x u = x$. (For $d = 2$ there is an obvious choice; for $d = 3$ we will make the following arbitrary choice. We let $\mathcal{O}_u$ be the identity and let $\mathcal{O}_{-u}$ be a rotation by π along the great circle parallel to the xy-plane. For other x, we let $\mathcal{O}_x$ be the rotation along the great circle containing u and x.) If $\gamma_1 \in \mathcal{G}_r, \gamma_2 \in \mathcal{G}_s$, we define $\gamma_1 \oplus \gamma_2$ to be the element of $\mathcal{G}_{r+s}$ obtained by attaching γ_2 to the end of γ_1. More specifically, let $\gamma_2' \in \mathcal{G}_{r,r+s}$ be obtained by scaling γ_2 and let $\tilde{\gamma}_2 = \mathcal{O}\gamma_2'$ where the rotation $\mathcal{O}$ is chosen so that $\gamma_1(b(\gamma_1)) = \tilde{\gamma}_2(0)$. Then

$$b(\gamma_1 \oplus \gamma_2) = b(\gamma_1) + b(\gamma_2'),$$
$$(\gamma_1 \oplus \gamma_2)(t) = \begin{cases} \gamma_1(t), & 0 \le t \le b(\gamma_1), \\ \tilde{\gamma}_2(t - b(\gamma_1)), & b(\gamma_1) \le t \le b(\gamma_1) + b(\gamma_2'). \end{cases}$$

Similarly, if $\gamma_1 \in \mathcal{C}_0, \gamma_2 \in \mathcal{G}_r$ we define $\gamma_1 \oplus \gamma_2 \in \mathcal{C}_r$; we can, of course, consider $\gamma_1 \oplus \gamma_2$ as an element of $\mathcal{C}_0$. We will write n-fold "additions" with the understanding that

$$\gamma_1 \oplus \cdots \oplus \gamma_n = (\gamma_1 \oplus \cdots \oplus \gamma_{n-1}) \oplus \gamma_n.$$

Let $\mathcal{H}$ be the set of $\gamma \in \mathcal{G}_1$ with $\gamma(0) = u$, and let

$$\mathcal{Y} = \mathcal{C}_0^k \times \mathcal{H}^k \times \mathcal{H}^k \times \cdots .$$

We write elements of $\mathcal{Y}$ as

$$(\bar\gamma_0, \bar\eta_1, \bar\eta_2, \ldots),$$

where

$$\bar\gamma_0 = (\gamma_0^1, \ldots, \gamma_0^k), \quad \gamma_0^k \in \mathcal{C}_0^k,$$
$$\bar\eta_n = (\eta_n^1, \ldots, \eta_n^k), \quad \eta_n^j \in \mathcal{H}.$$

We define the shift

$$(\bar\gamma_0, \bar\eta_1, \bar\eta_2, \ldots) \longrightarrow (\bar\gamma_1, \bar\eta_2, \bar\eta_3, \ldots),$$

where

$$\bar\gamma_1 = \bar\gamma_0 \oplus \eta_1$$

(the operation $\oplus$ is done separately on each of the k components). It is easy to see that this transformation is invertible, and the inverse operation can be described easily (we omit it). We let

$$\bar\gamma_n = \bar\gamma_0 \oplus \bar\eta_1 \oplus \cdots \oplus \bar\eta_n,$$

which we will consider either as an element of $\mathcal{C}_0^k$ or $\mathcal{C}_n^k$. Assume that we have a probability measure ν on $\mathcal{C}_0^k$. We will assume that ν is an h-measure. By this we mean that ν is supported on curves $\bar\gamma_0$ that are h-sets as described in Section 4 (Here and below we will write $\bar\gamma_0$ as shorthand for the set

$$\gamma_0^1[0, b(\gamma_0^1)] \cup \cdots \cup \gamma_0^k[0, b(\gamma_0^k)]. \quad)$$

To say that $\bar\gamma_0$ is an h-set is to say that there is a well-defined measure on $\mathcal{C}_0$ that corresponds to a Brownian motion B starting at the origin conditioned so that $B(0, T_0] \cap \bar\gamma_0 = \emptyset$. We extend ν to be a measure on $\mathcal{Y}$ by specifying that $\bar\eta_1, \bar\eta_2, \ldots$ are independent, identically distributed random variables, independent of $\bar\gamma_0$, each having the Wiener distribution on $\mathcal{H}^k$. The Wiener distribution on $\mathcal{H}^k$ is the measure obtained by starting independent Brownian motions $B^1, \ldots, B^k$ at u and considering

$$(B^1[0, T_1^1], \ldots, B^k[0, T_1^k]).$$

Let $\mu_0 = \nu$. Let $\mathcal{F}_n$ denote the σ-algebra generated by $\bar\gamma_n$, and let Z_n be the $\mathcal{F}_n$-measurable random variable

$$Z_n = \mathbf{P}_1\{B(0, T_n] \cap \bar\gamma_n = \emptyset \mid B(0, T_0] \cap \bar\gamma_0 = \emptyset\}$$

(here we are considering $\bar\gamma_n$ as an element of $\mathcal{C}_n^k$). Let

$$Y_n = -\log(Z_n/Z_{n-1}) = -\log \mathbf{P}_1\{B(0, T_n] \cap \bar\gamma_n = \emptyset \mid B(0, T_{n-1}] \cap \bar\gamma_{n-1} = \emptyset\}.$$

Let $\mu_n = \mu_n(\lambda, \nu)$ be the measure on $\mathcal{C}_n^k$ given by $\bar\gamma_n$ with the measure whose density with respect to ν is

$$\frac{Z_n^\lambda}{\mathbf{E}_\nu[Z_n^\lambda]} \frac{\exp(-\lambda \sum_{j=1}^n Y_j)}{\mathbf{E}_\nu[\exp(-\lambda \sum_{j=1}^n Y_j)]}.$$

We can consider μ_n as a measure on $\mathcal{C}_0^k$ (an h-measure, in fact) and hence we can also consider it as a measure on $\mathcal{Y}$. Our goal is to find an invariant measure $\mu = \mu(\lambda)$ such that

$$\mu_n(\lambda, \mu) = \mu. \tag{34}$$

Of course, it suffices to show this for $n = 1$.

If $m < n$ and ν is any measure on $\mathcal{C}_0^k$ or $\mathcal{G}_{-n,0}^k$, we write $\Pi_m \nu$ for the measure on $\mathcal{G}_m^k$ induced by the projection $\Phi_{-m,0}$. The projections are done separately on each component. In Section 7, we will show that for every m, there exists a measure $\mu^{(m)} = \mu^{(m)}(\lambda)$ on $\mathcal{G}_{-m,0}^k$ and positive constants c, β such that for any initial h-measure ν and all $n \geq m^2$,

$$\|\Pi_m \mu_n - \mu^{(m)}\| \leq c e^{-\beta m}. \tag{35}$$

Here $\|\cdot\|$ denotes variation distance, and we emphasize that the constants c, β can be chosen uniformly for $\lambda \in [\lambda_1, \lambda_2]$ (and may change from line to line). This implies for all $s < m$,

$$\|\Pi_s \mu_n - \Pi_s \mu^{(m)}\| \leq c e^{-\beta m}, \quad n \geq m^2.$$

If we fix s and take $s < m < r$, we can see (by considering $\Pi_s \mu_n$ for large n),

$$\|\Pi_s \mu^{(m)} - \Pi_s \mu^{(r)}\| \leq c e^{-\beta m}.$$

We therefore have a Cauchy sequence and hence can find a limit measure $\tilde{\mu}^s$ such that

$$\|\Pi_s \mu_n - \tilde{\mu}^s\| \leq c e^{-\beta \sqrt{n}}, \quad s^2 \leq n.$$

We emphasize that this bound is independent of the initial distribution ν. It is not difficult to see that $\{\tilde{\mu}^s\}$ give a consistent family of measures. It is not difficult, using (32), to see that there is a constant c such that the expected value of $b(\gamma^j)$ under μ^s is less than c. Hence, this will be true in the limit, and hence we get a measure μ on $\mathcal{C}_0^k$ such that

$$\mathbf{E}_\mu[b(\gamma^j)] \leq c, \quad j = 1, \ldots, k.$$

With this and the technical lemma of the next section, one can show that μ is an h-measure. Given this, it is clear that it satisfies (34).

The exponent $\xi = \xi(\lambda)$ satisfies

$$\mathbf{E}_\mu(e^{-\lambda Y_1}) = e^{-\xi}.$$

Since μ is invariant,

$$\mathbf{E}_\mu[\exp(-\lambda \sum_{j=1}^n Y_j)] = e^{-n\xi}.$$

Suppose $\tilde{\mu}$ is another measure on $\mathcal{C}_0^k$ such that

$$\|\Pi_{\sqrt{n}} \mu - \Pi_{\sqrt{n}} \tilde{\mu}\| \leq c e^{-\beta \sqrt{n}}.$$

Then it is not difficult, using the ideas in Lemma 4.12 and 4.13 to see that

$$|\mathbf{E}_\mu[e^{-\lambda Y_1}] - \mathbf{E}_{\tilde{\mu}}[e^{-\lambda Y_1}]| \leq c e^{-\beta \sqrt{n}}. \tag{36}$$

Let

$$R_{0,n} = e^{n\xi}\mathbf{E}[\exp(-\lambda\sum_{j=1}^{n} Y_j) \mid \mathcal{F}_0].$$

Note that

$$R_{0,n+1} = R_{0,n}\frac{\mathbf{E}[e^{\xi}e^{-\lambda Y_{n+1}}\exp(-\lambda\sum_{j=1}^{n} Y_j) \mid \mathcal{F}_0]}{\mathbf{E}[\exp(-\lambda\sum_{j=1}^{n} Y_j) \mid \mathcal{F}_0]}.$$

Since the convergence rate to μ is independent of the initial distribution, we get by (36)

$$R_{0,n+1} = R_{0,n}[1 + O(e^{-\beta\sqrt{n}})].$$

and hence the limit

$$R_0 = \lim_{n\to\infty} R_{0,n}$$

exists and

$$R_0 = R_{0,n}[1 + O(e^{-\beta\sqrt{n}})]. \tag{37}$$

Note also that $\mathbf{E}_\mu[R_0] = 1$. Similarly, we define

$$R_{n,m} = e^{m\xi}\mathbf{E}[\exp(-\lambda\sum_{j=n+1}^{n+m} Y_j) \mid \mathcal{F}_n],$$

$$R_n = \lim_{m\to\infty} R_{n,m}.$$

We now define another measure $\bar{\mu}$. Let the invariant measure μ be the initial distribution and extend μ to $\mathcal{Y}$ as above. Define $\bar{\mu}$ by saying that on $\mathcal{F}_n$ the density of $\bar{\mu}$ with respect to μ is

$$e^{n\xi}\exp(-\lambda\sum_{j=1}^{n} Y_j)R_n.$$

We will need a slight generalization of (36) in Section 8. Suppose $0 \le a_1, a_2, a_3 \le 3$ are integers and $1 \le m_1 < m_2 < m_3 \le m$. Suppose again that $\tilde{\mu}$ is a measure on $\mathcal{C}_0^k$ such that

$$\|\Pi_{\sqrt{n}}\mu - \Pi_{\sqrt{n}}\tilde{\mu}\| \le ce^{-\beta\sqrt{n}}.$$

Then,

$$|\mathbf{E}_\mu[Y_{m_1}^{a_1}Y_{m_2}^{a_2}Y_{m_3}^{a_3}e^{-\lambda(Y_1+\cdots+Y_m)}] - \mathbf{E}_{\tilde{\mu}}[Y_{m_1}^{a_1}Y_{m_2}^{a_2}Y_{m_3}^{a_3}e^{-\lambda(Y_1+\cdots+Y_m)}]| \le ce^{-\xi m}e^{-\beta\sqrt{n}}. \tag{38}$$

This can be done in a straightforward manner. The only reason we restrict ourselves to third moments is that all that we will need and we want the constant c to be uniform. Higher moments could be bounded similar, but the constant would depend on the particular moment.

6 A Harnack Type Inequality

Let $0 < m \leq n/2 < s/2 < \infty$, and assume we have an initial configuration $\bar{\gamma}_0$ satisfying

$$\mathbf{E}[Z_n^\lambda] \geq c_3 e^{-\xi n}.$$

(By Corollary 4.11, this is not a very restrictive assumption on $\bar{\gamma}_0$.) Consider the random variable

$$Z(n,s) = \mathbf{P}_1\Big\{ B(0,T_s] \cap (\bar{\gamma}_0 \cup \Lambda_s) = \emptyset \mid B(0,T_n] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset \Big\}.$$

The value of this random variable depends on the entire path

$$\bar{\gamma}_0 \cup \Lambda_s.$$

The purpose of this section is to approximate $Z(n,s)$ by a random variable $Z^{**}(n,s)$ that depends only on $\tilde{\Lambda}(n-2m,n) \cup \Lambda(n,s)$. We start by defining

$$Z^*(n,s) = Z^*(n,s;m) = \mathbf{P}_1\Big\{ B(0,T_s] \cap (\bar{\gamma}_0 \cup \Lambda_s) = \emptyset;$$

$$B[T_n,T_s] \cap \mathcal{B}(0,e^{n-m}) = \emptyset \mid B(0,T_n] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset \Big\}.$$

It is not difficult to use Lemma 4.13 to show that

$$\mathbf{E}[|Z^*(n,s) - Z(n,s)|^\lambda] \leq c e^{-\beta m} e^{-\xi(s-n)}.$$

Note that we can also write

$$Z^*(n,s) = \mathbf{P}_1\Big\{ B[T_{n-m},T_s] \cap \Lambda(n-m,s) = \emptyset;$$

$$B[T_n,T_s] \cap \mathcal{B}(0,e^{n-m}) = \emptyset \mid B(0,T_n] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset \Big\}.$$

Also if $W = W(n,m)$ is the event

$$W = \{\ [\Lambda(0,n) \cap \{z : |z| \geq e^{n-m}\}] = [\tilde{\Lambda}(n-2m,n) \cap \{z : |z| \geq e^{n-m}\}]\ \},$$

then Lemma 4.12 can be used to show that

$$\mathbf{E}[Z_n^\lambda; W^c] \leq c e^{-\beta m} e^{-\xi(s-n)}.$$

On the event W,

$$Z^*(n,s) = \mathbf{P}_1\Big\{ B[T_{n-m},T_s] \cap [\tilde{\Lambda}(n-2m,s) \cup \Lambda(n,s)] = \emptyset;$$

$$B[T_n,T_s] \cap \mathcal{B}(0,e^{n-m}) = \emptyset \mid B(0,T_n] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset \Big\}.$$

However, even on the event W, $Z^*(n,s)$ still depends on more than just $\tilde{\Lambda}(n-2m,n)\cup\Lambda(n,s)$ since the conditioning includes $\bar{\gamma}_0 \cup \Lambda_n$. We will replace $Z^*(n,s)$ with

$$Z^{**}(n,s) = \mathbf{P}_1\Big\{B[T_{n-m},T_s]\cap[\tilde{\Lambda}(n-2m,s)\cup\Lambda(n,s)] = \emptyset;$$

$$B[T_n,T_s]\cap\mathcal{B}(0,e^{n-m}) = \emptyset \mid B(0,T_n]\cap\tilde{\Lambda}(n-2m,n) = \emptyset\Big\}.$$

We need to have a similar estimate for, $Z^{**}(n,s)$,

$$\mathbf{E}[|Z^{**}(n,s) - Z(n,s)|^\lambda] \leq ce^{-\beta m}e^{-\xi(s-n)}. \tag{39}$$

The purpose of this section is to prove a lemma that will establish (39).

Let $Q_1 = Q_1(m,n)$ be the (random) measure on $\mathcal{G}_{n-m,n}$ obtained by considering

$$\Phi_{n-m,n}B(0,T_n] = B[T_{n-m},T_m],$$

where B_t is an h-process starting at the origin conditioned so that

$$B(0,T_n]\cap(\bar{\gamma}_0\cup\Lambda_n) = \emptyset.$$

This is a random measure in the sense that the measure Q_1 depends on $\bar{\gamma}_0\cup\Lambda_n$. Let $Q_2 = Q_2(m,n)$ be the measure on $\mathcal{G}_{n-m,n}$ obtained by considering

$$\Phi_{n-m,n}\tilde{B}(0,T_n],$$

where $\tilde{B}_t$ is an h-process starting at the origin conditioned so that

$$\tilde{B}(0,T_n]\cap\tilde{\Lambda}(n-2m,n) = \emptyset.$$

We will prove the following lemma. The inequality (39) is an immediate corollary.

Lemma 6.1 *There exist c,β such that for all $m \leq n/2$ and all initial configurations with*

$$\mathbf{E}[Z_n^\lambda] \geq c_3 e^{-\xi n},$$

$$\mathbf{E}[\bar{Z}_n^\lambda; \|Q_1 - Q_2\| \geq ce^{-\beta m}] \leq ce^{-\beta m}e^{-n\xi},$$

where $\|\cdot\|$ denotes variation distance.

The proof of the lemma is somewhat complicated so we will start by making some reductions. Without loss of generality, we will assume $m = n/2$, for otherwise we could let the Brownian motions run until they reach $\mathcal{S}_{n-2m}$ and consider that as the initial configuration. We will also write $8n$ and $4n$ for n and $n/2$; this will only affect the constants c,β. Hence we will show

$$\mathbf{E}[\bar{Z}_{8n}^\lambda; \|Q_1 - Q_2\| \geq ce^{-\beta n}] \leq ce^{-\beta n}e^{-8\xi n}, \tag{40}$$

where $Q_i = Q_i(4n,8n)$. In this change of variables, Q_i has become the measure on $\mathcal{G}_{4n,8n}$ obtained by considering $\Phi_{4n,8n}B(0,T_n]$ under the conditionings

$$B(0,T_{8n}]\cap(\bar{\gamma}_0\cup\Lambda_{8n}) = \emptyset, \quad \text{if } i = 1;$$

$$B(0,T_{8n}] \cap \tilde{\Lambda}(0,8n) = \emptyset, \quad \text{if } i = 2.$$

Let $V_1 = V_1(n)$ be the $\mathcal{F}_{8n}$-measurable event

$$V_1 = \{(\Lambda_{8n} \cap \{|x| \geq e^{jn}\}) = (\tilde{\Lambda}[(j-1)n, 8n] \cap \{|x| \geq e^{jn}\}), \;\; j = 1, \ldots, 7\}.$$

In order for the event $(V_1)^c$ to occur, one of the Brownian motions $B^1, \ldots, B^k$ must hit $\mathcal{S}_{(j-1)n}$ at some time after hitting $\mathcal{S}_{jn}$ but before reaching $\mathcal{S}_{8n}$ for the first time. By Lemma 4.12,

$$\mathbf{E}[\bar{Z}_{8n}^{\lambda}; (V_1)^c] \leq ce^{-\beta n} e^{-8\xi n}.$$

Hence to prove (40) it suffices to prove that

$$\mathbf{E}[\bar{Z}_{8n}^{\lambda}; \|Q_1 - Q_2\| \geq ce^{-\beta n}; V_1] \leq ce^{-\beta n} e^{-8\xi n}.$$

Note that on the event V_1,

$$\tilde{\Lambda}(0,8n) \cap \{z : |z| \geq e^n\} = (\bar{\gamma}_0 \cup \Lambda_{8n}) \cap \{z : |z| \geq e^n\}.$$

Let

$$Z' = Z'_{8n} = \mathbf{P}_1\{B(0,T_{8n}] \cap (\bar{\gamma}_0 \cup \Lambda_n) \neq \emptyset; B[T_{2n}, T_{8n}] \cap \mathcal{B}(0,e^n) = \emptyset \mid$$
$$B(0,T_0] \cap \bar{\gamma}_0 = \emptyset\}.$$

By Lemma 4.13,

$$\mathbf{E}[(Z_{8n} - Z')^{\lambda}] \leq e^{-\beta n} e^{-8\xi n}.$$

Hence, if we let $Q'_1 = Q'_1(4n, 8n)$ be the measure on $\mathcal{G}_{4n,8n}$ obtained by considering

$$\Phi_{4n,8n} B(0, T_{8n}],$$

where $B(0,T_{8n}]$ is conditioned so that

$$B(0,T_{8n}] \cap (\bar{\gamma}_0 \cup \Lambda_n) = \emptyset \text{ and } B[T_{2n}, T_{8n}] \cap \mathcal{B}(0,e^n) = \emptyset,$$

then

$$\mathbf{E}[Z_{8n}^{\lambda}; \|Q_1 - Q'_1\| \geq ce^{-\beta n}] \leq ce^{-\beta n} e^{-8\xi n},$$

for appropriately chosen c, β. Similarly, if we let $Q'_2 = Q'_2(4n, 8n)$ be the measure on $\mathcal{G}_{4n,8n}$ obtained by considering

$$\Phi_{4n,8n} B(0, T_{8n}],$$

where $B(0,T_{8n}]$ is conditioned so that

$$B(0,T_{8n}] \cap \tilde{\Lambda}(0,8n) = \emptyset \text{ and } B[T_{2n}, T_{8n}] \cap \mathcal{B}(0,e^n) = \emptyset,$$

then

$$\mathbf{E}[Z_{8n}^{\lambda}; \|Q_2 - Q'_2\| \geq ce^{-\beta n}] \leq ce^{-\beta n} e^{-8\xi n}.$$

Hence to prove (40), it suffices to prove that

$$\mathbf{E}[Z_{8n}^{\lambda}; \|Q'_1 - Q'_2\| \geq ce^{-\beta n}; V_1] \leq ce^{-\beta n} e^{-8\xi n}.$$

We will assume we are on the event V_1. We now let B_t be an h-process conditioned to hit $\mathcal{S}_{8n}$ before hitting

$$\Lambda(0,8n) \cup \mathcal{B}(0,e^n) = \tilde{\Lambda}(0,8n) \cup \mathcal{B}(0,e^n).$$

Under the conditionings used to derive the measures Q_1', Q_2', the process

$$B_t, \quad T_{2n} \le t \le T_{8n},$$

has the distribution of such an h-process. The only possible difference between Q_1' and Q_2' comes in the "initial" distribution of $B(T_{2n})$. For $i = 1, 2$, $2n \le r \le 8n$, let

$$h_i(x,r) = h_i(x,r;\Lambda(0,8n) \cup \mathcal{B}(0,e^n)),$$

be the density with respect to normalized surface measure on $\mathcal{S}_r$ of the distribution of $B(T_r)$ given these two possible conditionings. For $2n \le |y| \le r$, let

$$g(y,x,r) = g(y,x,r;\Lambda(0,8n) \cup \mathcal{B}(0,e^n)),$$

be the density (in x) with respect to normalized surface measure on $\mathcal{S}_r$ of the first hitting time of an h-process as described above, starting at y. Then for $2n \le s \le r \le 8n$,

$$h_i(x,r) = \int_{\mathcal{S}_s} h_i(y,s) g_r(y,x,r) \, d\sigma_s(y),$$

where, as before, σ_s denotes normalized surface measure on $\mathcal{S}_s$. Let $Y = Y_{4n}$ be the $\mathcal{F}_{8n}$-measurable random variable

$$Y = \int_{\mathcal{S}_{4n}} |h_1(x,4n) - h_2(x,4n)| \, d\sigma_{4n}(x).$$

Note that on the event V_1,

$$\|Q_1' - Q_2'\| = Y.$$

Hence it suffices to prove that

$$\mathbf{E}[Z_{8n}^{\lambda}; Y \ge ce^{-\beta n}; V_1] \le ce^{-\beta n}. \tag{41}$$

The technique to show that Y is small is coupling. In Section 3, we discussed coupling of h-processes. The basic idea of the proof of (41) is simple. We give exponential estimates to show that the number of times m between $2n$ and $4n$ that

$$\Lambda_{8n} \cap \{e^{m-1} \le |z| \le e^{m+3}\}$$

is "thin" is at least some small constant times n, except for a set of exponentially small probability. Then given that there are of order n such thin spots, two h-processes in the complement of Λ_{8n} starting at different points on $\mathcal{S}_{2n}$ will have order n chances to couple (using an appropriate coupling) and hence the probability of no coupling is exponentially small in n. The rest of this section is devoted to making this idea precise. Readers who are willing to accept that this idea can be made precise are invited to skip to the next section!

Recall the definition of R_ϵ from Section 3 and let $R_\epsilon(m) = R_{e^m\epsilon}$ be a scaling of R_ϵ. Suppose there exist $\epsilon, \delta, c, \beta$ such that the following holds, except for an $\mathcal{F}_{8n}$-measurable set of probability at most $ce^{-\beta n}$. There exists a set

$$K = K(\Lambda_{8n}) = \{l_1, \ldots, l_r\}, \quad r = [3\delta n],$$

of integers between $2n$ and $4n$ with $l_i \leq l_{i+1} - 5$ satisfying the following. First, for all $l \in K$,

$$\Lambda_{8n} \cap \{z; e^{l-1} \leq |z| \leq e^{l+2}\} \subset R_\epsilon(l).$$

Moreover, let $B_t, \tilde{B}_t$ denote two h-process conditioned to reach $\mathcal{S}_{8n}$ before hitting $\Lambda_{8n} \cup \mathcal{S}_n$, starting on $\mathcal{S}_{2n}$ with initial distributions given by $h_1(x, 2n)$ and $h_2(x, 2n)$ respectively, defined on the probability space $(\Omega_1, \mathbf{P}_1)$, with hitting times $T_r, \tilde{T}_r$, respectively. Let

$$X = X_n = \#\{l \in K : B[T_{l-1}, T_{l+2}] \cap R_\epsilon(l) \cap \{e^{l-1} \leq |z| \leq e^{l+1}\} \neq \emptyset\},$$

$$\tilde{X} = \tilde{X}_n = \#\{l \in K : \tilde{B}[\tilde{T}_{l-1}, \tilde{T}_{l+2}] \cap R_\epsilon(l) \cap \{e^{l-1} \leq |z| \leq e^{l+1}\} \neq \emptyset\}.$$

Then

$$\mathbf{P}_1\{X \geq \delta n\} + \mathbf{P}_1\{\tilde{X} \geq \delta n\} \leq ce^{-\beta n}.$$

By the coupling argument for h-processes in Section 3 (see Lemma 3.15), we can see that this will imply that $Y \leq ce^{-\beta n}$ for any Λ_{8n} not in the exceptional set. This will give the lemma.

Let ϵ, δ be numbers to be determined later, and also let $m \geq 6$ be a positive integer to be determined later. Let $U^j = U^j(\epsilon)$ be the event,

$$(i)\ |B^j(T_0) - u| \leq \epsilon/2;$$

$$(ii)\ B^j[T_0^j, T_5^j] \subset R_\epsilon \cap \{z : |z| \geq e^{-\epsilon}\};$$

(iii) For every $x \in R_\epsilon \cap \{e \leq |z| \leq e^3\}$,

$$\mathbf{P}_1^x\{B[0, \eta] \cap B^j[T_0^j, T_5^j] \neq \emptyset\} \geq \frac{1}{50},$$

where $\eta = \eta_\epsilon = \inf\{t : |B_t - B_0| = \epsilon\}$;

$$(iv)\ |e^{-5}B^j(T_5^j) - u| \leq \epsilon/2.$$

Let $U = U_\epsilon = U^1(\epsilon) \cap \cdots \cap U^k(\epsilon)$ and

$$p(\epsilon) = \epsilon^{k(d-1)}\mathbf{P}^{\mathbf{u}}[U(\epsilon)].$$

At least for ϵ sufficiently small, $p(\epsilon) > 0$ and $p(\epsilon) \to 0$ as $\epsilon \to 0$. If $\mathbf{x} \in \mathcal{S}^k$, $x = (x^1, \ldots, x^k)$, $|x^j - u| \leq \epsilon/2$, $j = 1, \ldots, k$, then by the Harnack inequality (for ϵ small),

$$c_1 p(\epsilon) \leq \mathbf{P}^{\mathbf{x}}[U(\epsilon)] \leq c_2 p(\epsilon).$$

We now let $U_i = U_i(\epsilon, m)$ be the event that the Brownian motions

$$B^1[T^1_{im+1}, T^1_{im+6}], \ldots, B^k[T^k_{im+1}, T^k_{im+6}],$$

appropriately scaled, satisfy U. Note that

$$\mathbf{P}[U_i(\epsilon, m) \mid \mathcal{F}_{im}] \asymp p(\epsilon)$$

(the factor $\epsilon^{k(d-1)}$ gives, up to a constant multiple, the probability that each of the k Brownian motions is within distance $\epsilon/2$ of u at time T^j_{im+1}).

For $n_1 < n_2 < n_3$, let $W(n_1, n_2, n_3)$ be the event

$$\Lambda(n_2, n_3) \cap \{|z| \leq e^{n_1}\} = \emptyset,$$

and let

$$\tilde{U}_i = \tilde{U}_i(\epsilon, m) = U_i \cap W(im + 4, im + 5, (i+1)m).$$

It follows from the work in Section 4, that for any $N \geq (i+1)m$,

$$\mathbf{E}[Z^\lambda_N; \tilde{U}_i \mid \mathcal{F}_{im}] \geq c e^{-\xi(N-im)} \mathbf{E}[Z^\lambda_{im+1} \mid \mathcal{F}_{im}] p(\epsilon).$$

If we let $\mathbf{Q} = \mathbf{Q}_N$ denote the measure on Ω with density $E[Z^\lambda_N]^{-1} Z^\lambda_N$,

$$\mathbf{Q}_{8n}[\tilde{U}_i \mid \mathcal{F}_{im}] \geq c p(\epsilon)$$

(see (31)). Let $S = S(m, n)$ be the collection of integers i such that

$$2n \leq im \leq 3n.$$

Let r be the cardinality of S and note that $r = (n/m) + O(1)$. Let

$$G = G(\epsilon, n, m) = \sum_{i \in S} I(\tilde{U}_i).$$

By standard large deviation estimates for binomial random variables (see Lemma 3.8(a)), we can find a c_5 (independent of ϵ, m), and positive constant $\alpha = \alpha(\epsilon)$ such that

$$\mathbf{Q}_{8n}[G \leq c_5 p(\epsilon) r] \leq e^{-\alpha r}. \tag{42}$$

We emphasize that c_5 holds for all ϵ sufficiently small and all m sufficiently large.

Let $c_6 = c_5/8$ and let $\mathcal{V} = \mathcal{V}(\epsilon, n, m)$ be the collection of all subsets $\hat{S}$ of S of cardinality $[c_6 p(\epsilon) r]$. Note that by Lemma 3.9, the cardinality of $\mathcal{V}$ is

$$\binom{r}{[c_6 p(\epsilon) r]} \leq c[c_6 p(\epsilon)]^{-c_6 p(\epsilon) r} e^{2 c_6 p(\epsilon) r}. \tag{43}$$

Consider a particular $\hat{S} \in \mathcal{V}$ and let $J_i = J_i(\epsilon, m)$ be the event (in $\bar{\Omega} = \Omega \times \Omega_1$),

$$J_i = \{B[T_{im-1}, T_{im+2}] \cap R_\epsilon(im) \neq \emptyset\}.$$

We can estimate

$$\bar{\mathbf{E}}[Z^\lambda_{8n}; \cap_{i \in \hat{S}} (J_i \cap \tilde{U}_i)]$$

from above as follows. Let $\bar{\mathcal{F}}_s$ be the σ-algebra generated by γ_0, Λ_s, and $\{B_t : 0 \leq t \leq T_s\}$. Note that if $s_1 < s_2$,

$$\bar{E}[Z^\lambda_{s_2} \mid \bar{\mathcal{F}}_{s_1}] \leq c\bar{\phi}(s_2 - s_1) Z^\lambda_{s_1} \leq c e^{-\xi(s_2 - s_1)} Z^\lambda_{s_1}.$$

Also if we let

$$v(\epsilon) = \mathbf{P}_1^u\{B[0,T_6] \cap R_\epsilon \neq \emptyset\},$$

then $v(\epsilon) \to 0$. If $i \in \hat{S}$, we can see that

$$\mathbf{E}[\bar{Z}^\lambda_{im+6}; J_i \cap \tilde{U}_i \mid \bar{\mathcal{F}}_{im}] \leq cp(\epsilon)v(\epsilon).$$

By combining these estimates (doing each one essentially $[c_6 p(\epsilon) r]$ times), we can see that there is a c_7 such that

$$\bar{\mathbf{E}}[Z^\lambda_{8n}; \cap_{i\in\hat{S}}(J_i \cap \tilde{U}_i)] \leq c_7^{c_6 p(\epsilon) r} e^{-\xi(8n - 6c_6 p(\epsilon) r)}[p(\epsilon)v(\epsilon)]^{c_6 p(\epsilon) r}.$$

This implies that

$$\mathbf{Q}_{8n}[\cap_{i\in\hat{S}}(J_i \cap \tilde{U}_i)] \leq c[c_7 e^{6\xi} p(\epsilon)v(\epsilon)]^{c_6 p(\epsilon) r}.$$

(Here we have extended $\mathbf{Q}_{8n}$ naturally to a measure on $\bar{\Omega}$.)

For each $\hat{S} \in \mathcal{V}$, let

$$J(\hat{S}) = J(\hat{S}, \epsilon, m) = \cap_i (J_i \cap \tilde{U}_i).$$

Then by combining the last estimate with (43), we see that there is a c_8 such that

$$\mathbf{Q}_{8n}[\cap_{\hat{S}\in\mathcal{V}}(J_i \cap \tilde{U}_i)] \leq c[c_8 v(\epsilon)]^{c_6 p(\epsilon) r}.$$

We now fix an ϵ sufficiently small so that $c_8 v(\epsilon) < 1/2$; since we have fixed the ϵ, constants may now depend on ϵ. We can therefore write the last inequality

$$\mathbf{Q}_{8n}[\cap_{\hat{S}\in\mathcal{V}}(J_i \cap \tilde{U}_i)] \leq ce^{-\beta r}.$$

We emphasize that this holds for all m (sufficiently large). We will now choose m.

Let

$$F = F_{n,m} = \sum_{i=0}^{n-1} I([W(im, (i+1)m, 8n)]^c).$$

By Lemma 4.15, we can find m sufficiently large so that

$$\mathbf{Q}_{8n}[F \geq \frac{1}{2} c_5 p(\epsilon) r] \leq ce^{-\beta r}.$$

We now fix such an m. Hence, if we let

$$\tilde{G} = \sum_{i\in S} I(U_i \cap W(im+4, im+5, 8n),$$

we have

$$\mathbf{Q}_{8n}[\tilde{G} \leq \frac{1}{2} c_5 p(\epsilon)] \leq ce^{-\beta r}.$$

Note that since we have fixed m, we can write this as

$$\mathbf{Q}_{8n}[\tilde{G} \leq \frac{1}{2} c_5 p(\epsilon)] \leq ce^{-\beta n}.$$

This will allow us to finish the lemma, choosing the δ as mentioned above to be $c_6 p(\epsilon) m^{-1}$.

7 Convergence Continued

In this section we continue the discussion started in Section 5 of the convergence to an equilibrium distribution. As was discussed in that section, the derivation consists broadly of two parts: approximating the distribution generated by Z_n^λ by the distribution of a Markov chain (more precisely, a subMarkov chain) and determining the rate of convergence to equilibrium of that Markov chain. We have already done most of the work necessary for the approximation. Here we will summarize those results in a convenient way and then discuss the rate of convergence of the Markov chain.

Assume that an initial configuration $\bar{\gamma}_0 = (\bar{\gamma}_0^1, \ldots, \bar{\gamma}_0^k) \in \mathcal{C}_0^k$ is given. We assume that (with probability one) $\bar{\gamma}_0$ is an h-set so that there is a well-defined process B defined on $(\Omega_1, \mathbf{P}_1)$ that corresponds to Brownian motion starting at the origin conditioned so that

$$B(0, T_0] \cap \bar{\gamma}_0 = \emptyset.$$

The conditioning on B goes only through time T_0; however, we assume that B is defined for all $t \in [0, \infty)$. As before, we have independent Brownian motions $B^1, \ldots, B^k$ starting at the endpoints of $\bar{\gamma}_0$, defined on the probability space $(\Omega, \mathbf{P})$. For convenience, we will assume we have another Brownian motion $\tilde{B}$ starting at the origin, defined on the probability space $(\Omega_2, \mathbf{P}_2)$, with stopping times $\tilde{T}_r$. We let $(\bar{\Omega}, \bar{\mathbf{P}}) = (\Omega \times \Omega_1 \times \Omega_2, \mathbf{P} \times \mathbf{P}_1 \times \mathbf{P}_2)$, so that all the Brownian motions on $\bar{\Omega}$ are independent.

As before set

$$\Lambda_n = B^1[0, T_n^1] \cup \cdots \cup B^k[0, T_n^k],$$

$$\bar{\gamma}_n = \bar{\gamma}_0 \cup \Lambda_n,$$

and for $m < n$,

$$\Lambda(m, n) = B^1[T_m^1, T_n^1] \cup \cdots \cup B^k[T_m^k, T_n^k],$$

$$\tilde{\Lambda}(m, n) = B^1[\sigma^1, T_n^1] \cup \cdots \cup B^k[\sigma^k, T_n^k],$$

where

$$\sigma^j = \sigma^j(m, n) = \sup\{t \le T_n^j : B_t^j \in \mathcal{S}_m\}.$$

For $0 \le r < m < n$, define the random variables

$$Z_n = \mathbf{P}_1 \Big\{ B(0, T_n] \cap \bar{\gamma}_n = \emptyset \mid B[0, T_0] \cap \bar{\gamma}_0 = \emptyset \Big\},$$

$$Z(m, n) = \mathbf{P}_1 \Big\{ B(0, T_n] \cap \bar{\gamma}_n = \emptyset \mid B[0, T_m] \cap \bar{\gamma}_m = \emptyset \Big\},$$

$$Z(r, m, n) = \mathbf{P}_1 \Big\{ B(0, T_n] \cap [\tilde{\Lambda}(r, m) \cup \Lambda(m, n)] = \emptyset \mid B(0, T_m] \cap \bar{\gamma}_m = \emptyset \Big\},$$

$$\tilde{Z}(r, m, n) = \mathbf{P}_2 \Big\{ \tilde{B}(0, T_n] \cap [\tilde{\Lambda}(r, m) \cup \Lambda(m, n)] = \emptyset \mid \tilde{B}(0, T_m] \cap \tilde{\Lambda}(r, m) = \emptyset \Big\}.$$

Note that $Z_n = Z(0, n)$ and $Z(r, n) = Z(r, m) Z(m, n)$. By Corollary 4.11, there exist constants c_3, c_4 such that for all $n > 1$,

$$c_3 e^{-(n-1)\xi} Z_1^\lambda \le \mathbf{E}[Z_n^\lambda \mid \mathcal{F}_1] \le c_4 e^{-(n-1)\xi} Z_1^\lambda.$$

In particular, if we let the initial distribution on $\mathcal{C}_0^k$ be given by $\bar{\gamma}_1$ (rather than $\bar{\gamma}_0$) we would have

$$c_3 e^{-n\xi} \le \mathbf{E}[Z_n^\lambda] \le c_4 e^{-n\xi}. \tag{44}$$

Without loss of generality, we will assume that our initial distribution on $\bar{\gamma}_0$ satisfies (44), realizing that if it does not satisfy it we can consider $\bar{\gamma}_1$ instead.

Fix an integer $m > 6$, and consider Z_{2mn} for integers m. We know that

$$Z_{2mn} = Z(0,2m)Z(2m,4m)\cdots Z(2(n-1)m,2nm).$$

Let

$$Z_{2mn}^{(n)} = Z(0,2m)Z(2m,4m)\cdots Z(2(n-2)m,2(n-1)m)Z((2n-3)m,2(n-1)m,2nm).$$

It follows from Lemmas 4.12 and 4.13 that

$$\mathbf{E}[|Z_{2mn} - Z_{2mn}^{(n)}|^\lambda] \le ce^{-\beta m}e^{-2mn\xi}.$$

Similarly, if for $i = 2,\ldots,n-1$, we set

$$Z_{2mn}^{(i)} = Z(0,2m)Z(2m,4m)\cdots Z(2(i-2)m,2(i-1)m)Z((2i-3)m,2(i-1)m,2im)$$
$$Z((2i-1)m,2im,2(i+1)m)\cdots Z((2n-3)m,2(n-1)m,2nm),$$

we can show for each i,

$$\mathbf{E}[|Z_{2mn}^{(i)} - Z_{2mn}^{(i+1)}|^\lambda] \le ce^{-\beta n}e^{-2mn\xi}.$$

Hence we can see by repeated application of Lemma 3.10, that for $n \le m^5$,

$$\mathbf{E}[|Z_{2mn} - Z_{2mn}^{(2)}|^\lambda] \le ce^{-\beta n}e^{-2mn\xi}.$$

We note that we have changed the values of both c and β in this last step. If we define $\tilde{Z}_{2mn}^{(i)}$ in the same way as $Z_{2mn}^{(i)}$ except that every $Z((2j-3)m,2(j-1)m,2jm))$ is replaced with $\tilde{Z}((2j-3)m,2(j-1)m,2jm))$, then using Lemma 6.1 we can show that

$$\mathbf{E}[|Z_{2mn} - \tilde{Z}_{2mn}^{(2)}|^\lambda] \le ce^{-\beta m}e^{-2mn\xi}, \quad n \le m^5.$$

Let $\mu(m,n)$ be the measure induced on $\mathcal{C}_{2nm}^k$ by the weighting $[\mathbf{E}(Z_{2mn}^\lambda)]^{-1}Z_{2mn}$, and let $\tilde{\mu}(m,n)$ be the measure induced by the weighting $(\mathbf{E}[(\tilde{Z}_{2mn}^{(1)})^\lambda)])^{-1}(\tilde{Z}_{2mn}^{(1)})^\lambda$. Then, using (44), we see that for $n \le m^4$,

$$\|\mu(m,n) - \tilde{\mu}(m,n)\| \le ce^{-\beta m}.$$

This also will be true if we project these measures by $\tilde{\Phi}_{(n-2)m,nm}$ or $\tilde{\Phi}_{(n-1)m,nm}$. The advantage of the measure $\tilde{\mu}(m,n)$ is that it is produced by a Markov chain on $\tilde{\mathcal{G}}_{0,m}^k$. To be specific, let ν be any measure on $\tilde{\mathcal{G}}_{0,m}^k$ which we consider as a measure on $\tilde{\mathcal{G}}_{-m,0}^k$. Let

$$\hat{Z} = \hat{Z}_m = \mathbf{P}_2\{\tilde{B}[0,T_m] \cap \bar{\gamma}_m = \emptyset \mid \tilde{B}[0,T_0] \cap \bar{\gamma}_0 = \emptyset\}.$$

Then we let ν_1 be the measure induced by projecting the measure $(\mathbf{E}[\hat{Z}^\lambda])^{-1}\hat{Z}^\lambda$ by $\tilde{\Phi}_{m,2m}$. To obtain ν_2 we consider ν_1 as a measure on $\tilde{\mathcal{G}}_{0,m}^k$ and do the same procedure. This defines the Markov chain. Now suppose we start with any initial configuration γ_0 (which need not be in $\tilde{\mathcal{G}}^k_{0,m}$). Let $\nu_i, i=0,\ldots,n-2$ be the measure obtained by projecting the measure $\tilde{\mu}(m,i+2)$ by $\tilde{\Phi}_{m(i+1),m(i+2)}$. Then it is easy to check that ν_i is obtained from ν_{i-1} by the Markov chain just described (in particular, the distribution ν_i depends only on ν_{i-1}).

It is for this Markov chain that we will show that there is an invariant measure $\nu^{(m)}$ on $\tilde{\mathcal{G}}^k_m$ such that for any initial distribution ν_0 on $\tilde{\mathcal{G}}^k_m$,

$$\|\nu_n - \nu^{(m)}\| \leq ce^{-\beta n}. \tag{45}$$

We emphasize that β, c are independent of m. In particular, if $\mu(m,n)$ is defined as in the previous paragraph, then for any initial distribution $\bar{\gamma}_0$,

$$\|\nu^{(m)} - \tilde{\Phi}_{(2n-1)m,2nm}\mu(m,n)\| \leq ce^{-\beta m}, \quad m/2 \leq n \leq m^4.$$

Another application of Lemmas 4.12 and 4.13 shows that

$$\|\Phi_{(2n-(1/2))m,2nm}\mu(m,n) - \Phi_{(2n-(1/2))m,2nm}\tilde{\Phi}_{(2n-1)m,2nm}\mu(m,n)\| \leq ce^{-\beta m}.$$

Hence

$$\|\Phi_{m/2,m}\nu^{(m)} - \Phi_{(2n-(1/2))m,2nm}\| \leq ce^{-\beta m}, \quad m/2 \leq n \leq m^4.$$

This establishes (35). It remains to derive (45).

For the remainder of this section we will investigate the Markov chain on $\tilde{\mathcal{G}}^k_m$. Let $\mathcal{W} = \mathcal{W}_m$ denote Wiener measure on $\tilde{\mathcal{G}}^k_m$, i.e., the measure induced by

$$(\tilde{\Phi}_{0,m}B^1[0,T^1_m], \ldots \tilde{\Phi}_{0,m}B^k[0,T^k_m]),$$

where $B^1,\ldots,B^k$ start at the origin (and have no other conditioning). We let $Y_0, Y_1, \ldots$ denote the values of the chain (so that $Y_i \in \tilde{\mathcal{G}}^k_m$). As before, if $\bar{\gamma}_0 \in \tilde{\mathcal{G}}^k_{-m,0}$, let

$$Z_{2m} = Z_{2m}(\bar{\gamma}_0) = \mathbf{P}^1\Big\{B[0,T_{2m}] \cap (\bar{\gamma}_0 \cup \Lambda(0,2m)) = \emptyset \mid B[0,T_0] \cap \bar{\gamma}_0 = \emptyset\Big\}.$$

We will assume that the conditioning is on an event of positive probability, i.e., that

$$\mathbf{P}_1\{B[0,T_0] \cap \bar{\gamma}_0 = \emptyset\} > 0. \tag{46}$$

For the remainder of this section we will write just $\tilde{\mathcal{G}}$ for the set of $\bar{\gamma}_0 \in \tilde{\mathcal{G}}^k_m$ that satisfy (46). It is easy to check that the Markov chain on $\bar{\gamma}_0 \in \tilde{\mathcal{G}}^k_m$ is actually a Markov chain on $\tilde{\mathcal{G}}$ (assuming that $Y_0 \in \tilde{\mathcal{G}}$). Let $K(\bar{\gamma}_0, \bar{\gamma}_1) = K_m(\bar{\gamma}_0, \bar{\gamma}_1)$ be the kernel of the chain with respect to Wiener measure, i.e., if $Y_0 = \bar{\gamma}_0$,

$$\mathbf{E}[Z^\lambda_{2m}; Y_1 \in V] = \int_V K(\bar{\gamma}_0, \bar{\gamma}_1)\, d\mathcal{W}(\bar{\gamma}_1).$$

We will show that there exist constants c_1, c_2 (independent of m) and bounded functions f_1, f_2 (depending on m) such that for all $\bar{\gamma}_0, \bar{\gamma}_1 \in \tilde{\mathcal{G}}$,

$$c_1 f_1(\bar{\gamma}_0) f_2(\bar{\gamma}_1) \leq K(\bar{\gamma}_0, \bar{\gamma}_1) \leq c_2 f_1(\bar{\gamma}_0) f_2(\bar{\gamma}_1).$$

Let $\nu_n = \nu_{n,m}$ denote the distribution of Y_n, normalized so that μ_n is a probability measure. Then it follows [15, Lemma 4.4] that there exists a limiting measure $\nu^{(m)}$ and c, β (depending only on c_1, c_2 and hence independent of m) such that for any initial distribution on Y_0,

$$\|\nu_m - \nu^{(m)}\| \le ce^{-\beta n}.$$

In particular, for $n \ge m/2$,

$$\|\nu_m - \nu^{(m)}\| \le ce^{-\beta m}.$$

Hence to derive (45) it suffices to prove (46).

Let

$$f_1(\bar\gamma_0) = \mathbf{E}[Z_m^\lambda] = \mathbf{E}[Z_m(\bar\gamma_0)^\lambda].$$

By the work in Section 4 (see Corollary 4.11), we know that

$$f_1(\bar\gamma_0) \asymp \mathbf{E}[Z_{m-2}^\lambda] \asymp e^{-\xi m}\mathbf{E}[Z_1^\lambda].$$

In fact, if we let $A = A_{1/10}$ as in Corollary 4.11 and let

$$U_m = \{\Lambda(m - \frac{1}{2}, m) \subset -A\},$$

and

$$\hat Z_m = \mathbf{P}_1\Big\{B[0, T_m] \cap (\bar\gamma_0 \cup \Lambda(0, m)) = \emptyset; B[T_{m-(1/2)}, T_0] \subset A; \\ \mid B[0, T_0] \cap \bar\gamma_0 = \emptyset\Big\},$$

then the corollary states that

$$f_1(\bar\gamma_0) \asymp \mathbf{E}[\hat Z_{m-2}; U_{m-2}].$$

Note that f_1 measures how easily the configuration $\bar\gamma_0$ can be extended to configurations that have a reasonable chance of being avoided. In this case the "end" of the configuration is more important than the "beginning". The function f_2 will be similar except that it will emphasize how easy it is for $\bar\gamma_1$ to appear at the end of the configuration and hence will focus more on the beginning of $\bar\gamma_1$. Let

$$Z'_{m-1,2m} = \mathbf{P}_1\{B[\sigma(m-1, 2m), T_{2m}] \cap \tilde\Lambda(m-1, 2m) = \emptyset\},$$

where, as before,

$$\sigma(m-1, 2m) = \sup\{t \le T_{2m} : B_t \in \mathcal{S}_{m-1}\}.$$

We define f_2 by

$$f_2(\bar\gamma_1) = \mathbf{E}[(Z'_{m-1,2m})^\lambda] \mid \tilde\Lambda(m, 2m) = \bar\gamma_1].$$

This definition is a little imprecise. By $\tilde\Lambda(m, 2m)$ we mean

$$(\tilde\Phi_{m,2m} B^1[T_m^1, T_{2m}^1], \ldots, \tilde\Phi_{m,2m} B^k[T_m^k, T_{2m}^k]),$$

and we have to make precise the conditioning, which as written is conditioning on an event of probability zero. Making sense of this conditioning is not difficult. In fact, we can describe how to get the distribution of $\tilde\Lambda(m-1, 2m)$ given $\tilde\Lambda(m, 2m) = \bar\gamma_1 = (\gamma_1^1, \ldots, \gamma_1^k)$. We will give the description now, leaving the verification to the reader. Start independent Brownian motions $B^1, \ldots, B^k$ at $\gamma_1^1(0), \ldots, \gamma_1^k(0) \in \mathcal{S}_m$, i.e., at the beginning points of $\bar{\gamma}_1$. Let $V = V_m$ be the event

$$V = \{T^j_{m-1} < T^j_{2m},\ j = 1, \ldots, k\}.$$

Note that $\mathbf{P}(V) > c$, where c is independent of m (recalling that we have chosen $m \geq 6$). Choose $B^1, \ldots, B^k$ conditioned on the event V. Then the set $\tilde{\Lambda}(m-1, 2m)$ is the set

$$\bar{\gamma}_1 \cup B^1[0, T^1_{m-1}] \cup \cdots \cup B^k[0, T^k_{m-1}].$$

We will need the analogue of Lemma 4.2. The proof is essentially the same, so we will only state the result. Let $A = A_{1/10}$ as above and let

$$\hat{Z}'_{m-1,2m} = \mathbf{P}_1\Big\{ B[\sigma(m-1, 2m), T_{2m}] \cap \tilde{\Lambda}(m-1, 2m) = \emptyset;$$

$$B[\sigma(m-1, 2m), T_{2m} \cap \{|z| \leq e^{m-(1/2)}\} \subset A \Big\},$$

$$U'_m = \Big\{ \tilde{\Lambda}(m-1, 2m) \cap \{|z| \leq e^{m-(1/2)}\Big\} \subset -A\}.$$

Then there exists a constant c such that for every $\bar{\gamma}_1 \in \tilde{\mathcal{G}}$,

$$\mathbf{E}[(\hat{Z}'_{m-1,2m})^\lambda; U'_m \mid \tilde{\Lambda}(m, 2m) = \bar{\gamma}_1] \geq$$

$$\geq c\mathbf{E}[(Z'_{m-1,2m})^\lambda] \mid \tilde{\Lambda}(m, 2m) = \bar{\gamma}_1].$$

Once we have the result, it is not hard to obtain (35). Basically we tie the ends of good extensions of $\bar{\gamma}_0$ to the good extensions of $\bar{\gamma}_1$. We omit the details.

8 Moment Calculations

In this section we do the moment calculations necessary to compute the derivatives of the intersection exponent as a function of λ. There is nothing deep about these calculations. Suppose we had a stationary sequence of mean zero random variables

$$\ldots, Y_{-1}, Y_0, Y_1, \ldots,$$

with

$$\mathcal{G}_\backslash = \sigma\{Y_m : m \leq n\},$$

such that for $n > 0$,

$$\mathbf{E}(Y_n \mid \mathcal{G}_0) = \mathbf{E}(Y_n)[1 + O(\epsilon_n)],$$

$$\mathbf{E}(Y_n^2 \mid \mathcal{G}_0) = \mathbf{E}(Y_n^2)[1 + O(\epsilon_n)],$$

$$\mathbf{E}(Y_n^3 \mid \mathcal{G}_0) = \mathbf{E}(Y_n^3)[1 + O(\epsilon_n)],$$

where $\epsilon \to 0$ sufficiently quickly. Then straightforward estimates given

$$\mathbf{E}(Y_1 + \ldots + Y_n \mid \mathcal{G}_0) = O(1),$$

$$\mathrm{Var}(Y_1 + \ldots + Y_n \mid \mathcal{G}_0) = \sigma^2 n + O(1),$$
$$\mathbf{E}[(Y_1 + \cdots + Y_n)^3 \mid \mathcal{G}_0] = O(n).$$

This is the idea behind the calculations of this section.

We will let μ_0 be the probability measure on $\mathcal{C}_0^k$ which gives probability one to $\bar{\gamma}_0 = (\bar{\gamma}, \ldots, \bar{\gamma})$ where

$$\gamma(t) = tu, \quad 0 \leq t \leq 1.$$

This is an arbitrary choice, but we want some initial distribution that does not depend on λ. This is a nice h-set, so it makes sense to speak of Brownian motions starting at the origin conditioned to avoid $\bar{\gamma}_0$. We let $B^1, \ldots, B^k$ be independent Brownian motions starting at u and define Λ_n as before. As before, we define

$$Z_n = \mathbf{P}_1\{B(0, T_n] \cap (\Lambda_n \cup \bar{\gamma}_0) = \emptyset \mid B(0, T_0] \cap \bar{\gamma}_0 = \emptyset\},$$
$$Z(m, n) = \mathbf{P}_1\{B(0, T_n] \cap (\Lambda_n \cup \bar{\gamma}_0) = \emptyset \mid B(0, T_m] \cap (\Lambda_m \cup \bar{\gamma}_0) = \emptyset\},$$
$$\Psi_n = -\log Z_n,$$
$$\Psi(m, n) = -\log Z(m, n).$$

As in Section 5, let

$$Y_n = \Psi(n-1, n),$$

so that $\Psi_n = Y_1 + \cdots + Y_n$. We define μ_n (which does depend on λ) as the probability measure on $\mathcal{C}_n^k$ induced by

$$(\gamma \oplus B^1[0, T_n^1], \cdots, \gamma \oplus B^k[0, T_n^k]),$$

with density

$$[\mathbf{E}(Z_n^\lambda)]^{-1} Z_n^\lambda.$$

We will actually think of μ_n as a measure on $\mathcal{C}_0^k$. We will write $\mathbf{E}_n$ for expectations with respect to μ_n. We emphasize that for $n = 0$ this measure does not depend on λ, but for $n > 0$ it does. We have seen that the measures μ_n converge to a measure $\mu = \mu(\lambda)$. We will write $\mathbf{E}_\mu$ for expectations with respect to μ.

Let

$$\phi_n(\lambda) = \mathbf{E}_0[Z_n^\lambda] = \mathbf{E}_0[e^{-\lambda \Psi_n}].$$

Let

$$\kappa_n = \kappa_n(\lambda) = e^{n\xi} \phi_n(\lambda),$$

and

$$\kappa = \kappa(\lambda) = \lim_{n \to \infty} \kappa_n.$$

From (37) we know that

$$\kappa_n = \kappa[1 + O(e^{-\beta \sqrt{n}})].$$

We also set as before,

$$R_{m,n} = e^{(n-m)\xi} \mathbf{E}[e^{-\lambda \Psi(m,n)} \mid \mathcal{F}_m],$$
$$R_m = \lim_{n \to \infty} R_{m,n}.$$

Note that

$$R_m = R_{m,n}[1 + O(e^{-\beta\sqrt{n-m}})].$$

Let

$$f_n(\lambda) = -\frac{1}{n}\log \phi_n(\lambda).$$

Then from the results of Section 4,

$$f_n(\lambda) = \xi(\lambda) + O(\frac{1}{n}).$$

By differentiating we can see that

$$f_n'(\lambda) = \frac{1}{n}\mathbf{Q}_n[\Psi_n],$$

$$f_n''(\lambda) = -\frac{1}{n}\left[\mathbf{Q}_n[\Psi_n^2] - (\mathbf{Q}_n[\Psi_n])^2\right],$$

$$f_n'''(\lambda) = \frac{1}{n}\left[\mathbf{Q}_n[\Psi_n^3] - 3(\mathbf{Q}_n[\Psi_n^2])(\mathbf{Q}_n[\Psi_n]) + 2(\mathbf{Q}_n[\Psi_n])^3\right].$$

Here $\mathbf{Q}_n = \mathbf{Q}_{n,\lambda}$ represents expectations

$$\mathbf{Q}_n[X] = \frac{\mathbf{E}_0[Xe^{-\lambda\Psi_n}]}{\mathbf{E}_0[e^{-\lambda\Psi_n}]} = \kappa_n^{-1}e^{n\xi}\mathbf{E}_0[Xe^{-\lambda\Psi_n}].$$

We start with the first moment. Note that

$$\mathbf{Q}_n[\Psi_n] = \sum_{i=1}^{n}\mathbf{Q}_n[Y_i] = \kappa_n^{-1}e^{n\xi}\sum_{i=1}^{n}\mathbf{E}_0[Y_i e^{-\lambda(Y_1+\cdots+Y_n)}],$$

and

$$\mathbf{E}_0[Y_i e^{-\lambda(Y_1+\cdots+Y_n)}] = \kappa_{i-1}e^{-(i-1)\xi}\mathbf{E}_{i-1}[Y_1 e^{-\lambda(Y_1+\cdots+Y_{n-i+1})}].$$

Also, note that

$$\mathbf{E}_{i-1}[Y_1 e^{-\lambda(Y_1+\cdots+Y_{n-i+1})} \mid \mathcal{F}_1] = Y_1 e^{-\lambda Y_1}e^{-\xi(n-i)}R_{1,n-i},$$

and hence

$$\mathbf{E}_{i-1}[Y_1 e^{-\lambda(Y_1+\cdots+Y_{n-i+1})}] = \mathbf{E}_{i-1}[Y_1 e^{-\lambda Y_1}e^{-\xi(n-i)}R_{1,n-i}].$$

Hence,

$$\mathbf{Q}_n[Y_i] = \frac{\kappa_{i-1}}{\kappa_n}\mathbf{E}_{i-1}[\Psi_1 e^{\xi-\lambda\Psi_1}R_{1,n-i+1}]. \qquad (47)$$

By the convergence results, we know that

$$\kappa_{i-1} = \kappa_n[1 + O(e^{-\beta\sqrt{i}})],$$

$$\mathbf{E}_{i-1}[\Psi_1 e^{\xi-\lambda\Psi_1}R_{1,n-i+1}] = \mathbf{E}_\mu[\Psi_1 e^{\xi-\lambda\Psi_1}R_{1,n-i+1}][1 + O(e^{-\beta\sqrt{i}})],$$

$$R_{1,n-i+1} = R_1[1 + O(e^{-\beta\sqrt{n-i}})].$$

Hence,

$$\mathbf{Q}_n[Y_i] = \mathbf{E}_\mu[\Psi_1 e^{\xi-\lambda\Psi_1}R_1][1 + O(e^{-\beta\sqrt{i}})][1 + O(e^{-\beta\sqrt{n-i}})].$$

In particular we see that

$$\mathbf{Q}_n[\Psi_n] = an + O(1),$$

where

$$a = a(\lambda) = \mathbf{E}_\mu[\Psi_1 e^{\xi - \lambda\Psi_1} R_1].$$

We can also write

$$a = \mathbf{E}_{\bar{\mu}}[\Psi_1],$$

where $\bar{\mu}$ is as defined in Section 5.

Before tackling the higher moments, it will be useful to set up some notation. Assume $1 \leq i \leq j \leq n$. Define the following (which depend on λ):

$$a(n) = \mathbf{E}_\mu[Y_1 e^{\xi - \lambda Y_1} R_{1,n-1}],$$

$$a(n;i) = \mathbf{E}_\mu[Y_1 Y_i e^{i\xi - \lambda(Y_1 + \cdots + Y_i)} R_{i,n-i}],$$

$$a(n;i,j) = \mathbf{E}_\mu[Y_1 Y_i Y_j e^{j\xi - \lambda(Y_1 + \cdots + Y_j)} R_{j,n-j}].$$

We define $a(\infty), a(\infty;i), a(\infty;i,j)$ to the be corresponding quantities where $R_{1,n-1}, R_{i,n-i}, R_{j,n-j}$ is replaced by R_1, R_i, R_j, respectively. We also define $a_l(n), a_l(n;i), a_l(n;i,j), a_l(\infty)$, $a_l(\infty;i)$, $a_l(\infty;i,j)$ similarly by replacing μ with μ_l. Note that $a = a(\infty)$ and

$$a(\infty) = \mathbf{E}_{\bar{\mu}}[Y_1 e^{\xi - \lambda Y_1}],$$

$$a(\infty;i) = \mathbf{E}_{\bar{\mu}}[Y_1 Y_i e^{i\xi - \lambda(Y_1 + \cdots + Y_i)}],$$

$$a(\infty;i,j) = \mathbf{E}_{\bar{\mu}}[Y_1 Y_i Y_j e^{j\xi - \lambda(Y_1 + \cdots + Y_j)}].$$

Suppose $1 \leq i \leq j \leq m \leq n$. Then by an argument similar to that used to derive (47),

$$\mathbf{Q}_n[Y_i Y_j] = \frac{\kappa_{i-1}}{\kappa_n} a_{i-1}(n-j+1; j-i+1),$$

$$\mathbf{Q}_n[Y_i Y_j Y_m] = \frac{\kappa_{i-1}}{\kappa_n} a_{i-1}(n-m+1; j-i+1, m-i+1).$$

Assume $i > 1$ and consider

$$a_l(n;i) = \mathbf{E}_l[Y_1 Y_i e^{i\xi - \lambda(Y_1 + \cdots + Y_i)} R_{i,n-i}].$$

By first conditioning with respect to $\mathcal{F}_{i-1}$ we can see that

$$a_l(n;i) = \mathbf{E}_l[Y_1 e^{(i-1)\xi - \lambda(Y_1 + \cdots + Y_{i-1})}] \tilde{\mathbf{E}}_l[Y_i e^{\xi - \lambda Y_i} R_{i,n-i}],$$

where $\tilde{\mathbf{E}}_l$ represents the measure whose density with respect to $\mathbf{E}_l$ is

$$\frac{Y_1 e^{(i-1)\xi - \lambda(Y_1 + \cdots + Y_{i-1})}}{\mathbf{E}_l[Y_1 e^{(i-1)\xi - \lambda(Y_1 + \cdots + Y_{i-1})}]}.$$

By the convergence result, we can see that

$$\tilde{\mathbf{E}}_l[Y_i e^{\xi - \lambda Y_i} R_{i,n-i}] = \mathbf{E}_\mu[Y_1 e^{\xi - \lambda Y_1} R_{1,n-i}][1 + O(e^{-\beta\sqrt{i}})],$$

hence

$$a_l(n;i) = a_l(i-1)a(n-i+1)[1+O(e^{-\beta\sqrt{i}})].$$

Similarly we can show

$$a_l(n;i,j) = a_l(i-1)a(n-i+1;j-i+1)[1+O(e^{\beta\sqrt{i}})],$$

$$a_l(n;i,j) = a_l(j-1;i)a(n-j+1)[1+O(e^{\beta\sqrt{j-1}})].$$

Also,

$$a_l(n;i) = a_l(\infty;i)[1+O(e^{-\beta\sqrt{n-i}})],$$

$$a_l(n;i,j) = a_l(\infty;i,j)[1+O(e^{-\beta\sqrt{n-j}})].$$

These results also hold if a_l is replaced with a.

We now tackle the second moment,

$$\begin{aligned}\mathbf{Q}_n[\Psi_n^2] - (\mathbf{Q}_n[\Psi_n])^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{Q}_n[Y_iY_j] - \mathbf{Q}_n(Y_i)\mathbf{Q}_n(Y_j) \\ &= \sum_{i=1}^{n}[\mathbf{Q}_n[Y_i^2] - (\mathbf{Q}_n[Y_i])^2] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}[\mathbf{Q}_n[Y_iY_j] - \mathbf{Q}_n(Y_i)\mathbf{Q}_n(Y_j)].\end{aligned}$$

The convergence results tell us that

$$\mathbf{Q}_n[Y_i^2] = \mathbf{E}_\mu[\Psi_1^2 e^{\xi-\lambda\Psi_1}R_1] + O(e^{-\beta\sqrt{i}}) + O(e^{-\beta\sqrt{n-i}}),$$

and hence

$$\sum_{i=1}^{n}\mathbf{Q}_n[Y_i^2] = n\mathbf{E}_\mu[\Psi_1^2 e^{\xi-\lambda\Psi_1}R_1] + O(1),$$

and

$$\sum_{i=1}^{n}(\mathbf{Q}_n[Y_i^2] - \mathbf{Q}_n[Y_i]^2) = n[a(\infty;1) - a(\infty)^2] + O(1).$$

Let

$$r_{i,n} = \sum_{j=i+1}^{n}(\mathbf{Q}_n[Y_iY_j] - \mathbf{Q}_n[Y_i]\mathbf{Q}_n[Y_j]),$$

and

$$r = \sum_{j=2}^{\infty}[a(\infty;j) - a(\infty)^2].$$

The convergence results can be used to show that

$$r_{i,n} \le c, \tag{48}$$

and

$$r_{i,n} = r + O(e^{-\beta\sqrt{i}}) + O(e^{-\beta\sqrt{n-i}}). \tag{49}$$

This then implies

$$\mathbf{Q}_n[\Psi_n^2] - (\mathbf{Q}_n[\Psi_n])^2 = n\sigma^2 + O(1),$$

where

$$\sigma^2 = a(\infty;1) - a(\infty)^2 + 2r.$$

For the third moment, we have,

$$nf'''(\lambda) = \sum_{1 \le i,j,k \le m} \Delta(i,j,m)$$

where

$$\Delta(i,j,m) = \mathbf{Q}_n(Y_iY_jY_m) - \mathbf{Q}_n(Y_i)\mathbf{Q}_n(Y_jY_m) - \mathbf{Q}_n(Y_j)\mathbf{Q}_n(Y_iY_m) - \mathbf{Q}_n(Y_m)\mathbf{Q}_n(Y_iY_j)$$
$$+2\mathbf{Q}_n(Y_i)\mathbf{Q}_n(Y_j)\mathbf{Q}_n(Y_m).$$

Assume that $i \le j \le m$. If $m - j \ge \delta$, then the convergence results tell us that

$$\mathbf{Q}_n(Y_iY_jY_m) = \mathbf{Q}_n(Y_iY_j)\mathbf{Q}_n(Y_m) + O(e^{-\beta\sqrt{\delta}}),$$

$$\mathbf{Q}_n(Y_iY_m) = \mathbf{Q}_n(Y_i)\mathbf{Q}_n(Y_m) + O(e^{-\beta\sqrt{\delta}}),$$

$$\mathbf{Q}_n(Y_jY_m) = \mathbf{Q}_n(Y_j)\mathbf{Q}_n(Y_m) + O(e^{-\beta\sqrt{\delta}}).$$

Hence,

$$\Delta(i,j,m) \le ce^{-\beta\sqrt{m-j}}.$$

Similarly,

$$\Delta(i,j,m) \le ce^{-\beta\sqrt{j-i}},$$

and hence

$$\Delta(i,j,m) \le e^{-\beta\sqrt{\max(m-j,j-i)}},$$

from which it follows by a straightforward estimate that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} \Delta(i,j,m) \le cn.$$

We finish this section by sketching the argument to show that $\sigma^2 = \sigma^2(\lambda) > 0$ for every $\lambda > 0$. Let

$$g_n(\lambda) = -\log \phi_n(\lambda),$$

where

$$\phi_n = \mathbf{E}[Z_n^\lambda].$$

We assume the same initial condition as the previous section. Note that $g_n = nf_n$ where f_n is as defined in the previous section, and let

$$\sigma_n^2 = \sigma_n^2(\lambda) = g_n''(\lambda).$$

In the previous section we showed that

$$\sigma_n^2 = \sigma^2 n + O(1).$$

Hence in order to prove that $\sigma^2 > 0$, it suffices to show that

$$\lim_{n\to\infty} \sigma_n^2 = \infty. \tag{50}$$

Define as in Section 7,

$$\tilde{Z}(r,m,n) = \mathbf{P}_1\Big\{B[0,T_n] \cap [\tilde{\Lambda}(r,m) \cup \Lambda(m,n)] = \emptyset \mid B[0,T_m] \cap \tilde{\Lambda}(r,m) = \emptyset\Big\}.$$

For a given $m > 8$, let

$$\tilde{Z}_{2nm,m} = \tilde{Z}(-m,0,2m)\tilde{Z}(m,2m,4m)\tilde{Z}(3m,4m,6m)\cdots\tilde{Z}((2n-3)m,(2n-2)m,2nm).$$

It follows from the work in that section and previous sections that for $n \leq m^2$,

$$\mathbf{E}[Z_{nm}^\lambda; |Z_{2nm} - \tilde{Z}_{2nm,m}| \geq e^{-\beta m} Z_{2nm}] \leq c e^{-\beta m} e^{-2mn\xi}.$$

In particular, if we let $\tilde{\sigma}_{n,m}^2$ be the variance of $-\log \tilde{Z}_{2n^2}$ with respect to the probability measure

$$\tilde{Z}_{2nm,m}[\mathbf{E}[\tilde{Z}_{2nm,m}]]^{-1},$$

then as $n \to \infty$,

$$\sigma_{2n^2}^2 \sim \tilde{\sigma}_{2nn,n}^2.$$

Hence to prove (50), it suffices to show that

$$\tilde{\sigma}_{2n^2,n}^2 \to \infty.$$

We will, in fact, show that there is a constant $c > 0$ such that for all $m > 4$,

$$\tilde{\sigma}_{2nm,m}^2 \geq cn.$$

Fix $m > 4$ and let

$$Y_n = Y_{n,m} = -\log \tilde{Z}((2n-3)m,(2n-2)m,2nm).$$

Let $\mathcal{H} = \mathcal{H}_m$ be the σ-algebra generated by sets

$$\tilde{\Lambda}((2n-1)m,2nm), \quad n = 0,1,2,\ldots.$$

Note that $Y_1, Y_2, \ldots$ are conditionally independent, given $\mathcal{H}$. Now if $X_1,\ldots,X_n$ are independent nonnegative random variables, then it is easy to check that they are also independent with respect to the probability measure

$$\frac{e^{-\lambda(X_1+\cdots+X_n)}}{\mathbf{E}[e^{-\lambda(X_1+\cdots+X_n)}]}, \quad \lambda > 0.$$

Hence

$$\mathrm{Var}[Y_1 + \cdots + Y_n \mid \mathcal{H}] = \mathrm{Var}[Y_1 \mid \mathcal{H}] + \cdots + \mathrm{Var}[Y_n \mid \mathcal{H}],$$

where the variance is with respect to the probability measure

$$\frac{\tilde{Z}^{\lambda}_{2mn,m}}{\mathbf{E}[\tilde{Z}^{\lambda}_{2mn,m}]}.$$

If we can show that there is a $c > 0$ such that

$$\mathrm{Var}[Y_i \mid \mathcal{H}] \geq c,$$

then we will have

$$\mathrm{Var}[Y_1 + \cdots + Y_n \mid \mathcal{H}] \geq cn.$$

Since conditioning can only reduce the variance, this implies

$$\mathrm{Var}[Y_1 + \cdots + Y_n] \geq cn,$$

which gives us the result.

To prove that $\mathrm{Var}[Y_i \mid \mathcal{H}| \geq c$, is actually quite easy and we omit the proof. The basic idea is to note that

$$\mathrm{Var}[Y_i \mid \mathcal{H}] = \mathrm{Var}[Y_i \mid \tilde{\Lambda}((2i-3)m, (2i-2)m), \tilde{\Lambda}((2i-1)m, 2im)],$$

and to show directly using ideas similar to those in Section 4 to show that there is some positive variance.

9 Continuity at $\lambda = 0$

In this section we show that the intersection exponent $\xi(\lambda)$ is continuous at 0. More precisely we show that

$$\lim_{\lambda \to 0} \xi(\lambda) = 0, \quad d = 3, \tag{51}$$

$$\lim_{\lambda \to 0} \xi(\lambda) = \alpha, \quad d = 2, \tag{52}$$

where $\alpha = \alpha_k$ is the k disconnection exponent defined by

$$\mathbf{P}\{Z_n > 0\} \asymp e^{-n\alpha}.$$

We will also prove that for $d = 2$

$$a_0 < \infty, \tag{53}$$

where a_0 is as in Section 2.

The proof of (51) is straightforward. Suppose $d = 3$. Let $\delta > 0$ and let A_δ be as in Section 3. It is not difficult (see Lemma 3.5) to show that there exist $r = r(\delta), C_1 = C_1(\delta)$ with $r \to 0$ as $\delta \to 0$ such that

$$\mathbf{P}^{-\mathbf{u}}\{\Lambda_n \cap A_\delta = \emptyset\} \geq C_1 e^{-rn}.$$

However (see Lemma 3.4) there exists an $s = s(\delta), C_2 = C_2(\delta)$ such that

$$\mathbf{P}^u_1\{B[0, T_n] \subset A_\delta\} \geq C_2 e^{-sn}.$$

In other words, with probability at least $C_1 e^{-rn}$,

$$Z_n \geq C_2 e^{-sn}.$$

Hence,

$$\mathbf{E}^{-\mathbf{u}}[Z_n^\lambda] \geq C_1 C_2^\lambda e^{-n(r+\lambda s)},$$

and

$$\xi(\lambda) \leq r + \lambda s,$$

$$\lim_{\lambda \to 0} \xi(\lambda) \leq r.$$

By letting $r \to 0$, we obtain the result.

Now let $d = 2$. Since $\mathbf{E}[Z_n^\lambda] \leq \mathbf{P}\{Z_n > 0\}$ we immediately see that

$$\lim_{\lambda \to 0} \xi(\lambda) \geq \alpha.$$

If we try to adapt the $d = 3$ argument for $d = 2$, we will only be able to prove that

$$\lim_{\lambda \to 0} \xi(\lambda) \leq k/2,$$

but it is known [24, 25] that $\alpha < k/2$.

For ease we will use the same initial configuration as in Section 8, i.e., we start with $\bar{\gamma}_0 = (\gamma, \ldots, \gamma)$, with

$$\gamma(t) = tu, \quad 0 \leq t \leq 1.$$

In this section, we will let $\mathbf{Q}_n$ denote the conditional measure on Ω given no disconnection,

$$\mathbf{Q}_n(V) = \frac{\mathbf{P}\{V; Z_n > 0\}}{\mathbf{P}\{Z_n > 0\}}.$$

We will assume $n \geq 8$, and let

$$X_n = \bar{Z}(4, n-4) = \sup_{x \in \mathcal{S}_4} \mathbf{P}_1^x\{B[0, T_{n-4}] \cap [\bar{\gamma}_0 \cup \Lambda_{n-4}] = \emptyset\}.$$

We have seen, see (11), (12), (15), that

$$\mathbf{E}[X_n^\lambda] \asymp e^{-n\xi},$$

where the multiplicative constants may depend on λ (and, in particular, have not been shown to be uniform for all $\lambda \geq 0$). We will show that

$$\mathbf{Q}_n[-\log X_n] \leq cn, \tag{54}$$

where we use $\mathbf{Q}_n[Y]$ to denote the expectation of Y with respect to $\mathbf{Q}_n$. It follows immediately that there exists a β_1 such that

$$\mathbf{P}\{X_n \geq e^{-n\beta_1}\} \geq \frac{1}{2}\mathbf{P}\{Z_n > 0\} \geq ce^{-n\alpha}. \tag{55}$$

Hence for any $\lambda > 0$,

$$\mathbf{E}[X_n^\lambda] \geq \mathbf{E}[X_n^\lambda; X_n \geq e^{-n\beta_1}] \geq ce^{-n\beta_1\lambda}e^{-n\alpha}.$$

Therefore,

$$\xi(\lambda) \leq \alpha + \beta_1\lambda,$$

and letting λ go to zero we get (52). If $b(a)$ is defined as in Section 2, then (55) can be rephrased as

$$b(a) = \alpha, \quad a \geq \beta_1.$$

It also follows from (55) that for every $\lambda > 0$ and every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\mathbf{E}[X_n^\lambda; X_n \leq e^{-n(\beta_1+\epsilon)}]}{\mathbf{E}[X_n^\lambda]} = 0.$$

From this we get that $a(\lambda) \leq \beta_1$, and hence we get (53). So we need only prove (54).

It will be convenient to use the complex map $z \to \log z$ to convert this problem to a slightly different problem. We denote the real and imaginary parts of a complex number by $\Re(z)$ and $\Im(z)$, respectively. The logarithm takes the complex plane to an infinite cylinder, which can be thought of as the complex plane with two points identified if their difference is an integer multiple of $2\pi i$. Brownian motion is invariant (up to a random time change that will not be relevant) under this transformation (see, e.g., [3, Theorem V.1.1]). The initial configuration now is the negative real axis, combined with all translates of this axis by integer multiples of $2\pi i$. Brownian motions $B^1, \ldots, B^k$ are started at the origin and run until T_n^j, the first time B^j hits $\{\Re(z) = n\}$. The paths of the Brownian motions are augmented by the $2j\pi i$ translates of the paths (j integer), and we write Λ_n for the path along with all its translates. The random variables Z_n, X_n, and the measure $\mathbf{Q}_n$ are transformed in a natural way. Given the paths $B^1, ..., B^k$, if $Z_n > 0$, there is a unique open domain D bounded by $\{\Re(z) = 0\}$, $\{\Re(z) = n\}$ and Λ_n such that

$$\partial D \cap \{\Re(z) = 0, \Im(z) \in (0, 2\pi)\} \neq \emptyset.$$

Take this domain D (but not its translates) and let $\eta : [0,1] \to \mathbb{C}$ denote a continuous function with $\eta(0,1) \subset D$; $\Re(\eta(1)) = n$; $\Re(\eta(0)) = 0$; $\Im(\eta(0)) \in (0, 2\pi)$. Since $Z_n > 0$, such an η must exist. Note that $\eta(0,1)$ splits $\{0 < \Re(z) < n\}$ into two pieces. Let $\partial D_1, \partial D_2$ be the intersection of

$$\partial D \cap \{\frac{1}{2} \leq \Re(z) \leq n - \frac{1}{2}\}$$

with the "upper" and "lower" pieces, respectively. Note that $\partial D_1, \partial D_2$ are independent of the η chosen.

For any $z \in D$, let

$$d_1(z) = \mathrm{dist}(z, \partial D_1), \quad d_2(z) = \mathrm{dist}(z, \partial D_2).$$

Let

$$A = A_n = \{z \in D : 4 \leq \Re(z) \leq n - 4; d_1(z) = d_2(z)\}.$$

Note that $d_1(z) \leq \pi$ if $z \in A$, and hence $\mathrm{dist}(z, \partial D) = d_1(z)$. For $z \in A$, let $r(z) = d_1(z)/8 = d_2(z)/8$. The open balls of radius $r(z)$,

$$\{\ \mathcal{B}^o(z, r(z)), \quad z \in A\ \}$$

form an open cover of A; hence we can find a finite set $z_1, \ldots, z_K \in A$ such that

$$A \subset \bigcup_{j=1}^{K} \mathcal{B}^o(z_j, r(z_j)).$$

From this finite set we can find a sequence, which we also denote by $z_1, \ldots, z_K$, such that

$$\mathcal{B}^o(z_1, r(z_1)) \cap \{\Re(z) = 4\} \neq \emptyset,$$

$$\mathcal{B}^o(z_K, r(z_K)) \cap \{\Re(z) = n-4\} \neq \emptyset,$$

$$z_j \in \mathcal{B}^o(z_{j-1}, r(z_{j-1})), \quad j = 1, \ldots, K.$$

(To find this sequence we use the fact that the set of points equidistant from two disjoint compact sets is connected.) By "erasing loops" if necessary, we can find a subsequence of this sequence, which we still denote as $z_1, \ldots, z_K$, such that the above three conditions hold as well as

$$z_j \notin \mathcal{B}^o(z_m, r(z_m)), \quad j \geq m+2. \tag{56}$$

Take $y \in \{\Re(z) = 4\} \cap \mathcal{B}^o(z_1, r(z_1))$ and start a Brownian motion B at y. Let

$$\tau = \inf\{t : B_t \in \mathcal{B}^o(z_2, r(z_2)) \cup [\mathcal{B}^o(z_1, 4r(z_1))]^c\}.$$

It is easy to check that there is a $\rho > 0$ such that

$$\mathbf{P}_1^y\{B(\tau) \in \mathcal{B}^o(z_2, r(z_2))\} \geq \rho.$$

By repeating this estimate, we see that

$$\mathbf{P}_1^y\{B[0, T_{n-4}] \subset \cup_{j=1}^K \mathcal{B}^o(z_j, 4r(z_j))\} \geq \rho^K,$$

and hence

$$-\log X_n \leq cK.$$

For integer m, we say that (x, y) is a 2^{-m}-approach if $x \in \partial D_1, y \in \partial D_2$, $|x - y| = 2^{-m}$. Let $\mathcal{B}_{x,y}$ denote the closed disk with diameter the line segment connecting x and y. Let $U_m = U_m^{(n)}$ be the maximal number N of 2^{-m} approaches, $(x_1, y_1), \ldots (x_N, y_N)$ that can be chosen with $3 \leq \Re(x_j), \Re(y_j) \leq n - 3$ and

$$\mathcal{B}_{x_l, y_l} \cap \mathcal{B}_{x_j, y_j} = \emptyset, \quad l \neq j.$$

Let

$$U = U^{(n)} = \sum_{m=1}^{\infty} U_m.$$

By using (56) it is easy to show that

$$K \leq cU,$$

where K is the K of the last line of the last paragraph. We will show that there exists a constant c such that

$$\mathbf{Q}_n(U) \leq cn, \tag{57}$$

which will imply (54).

If l is an integer, we say that a 2^{-m} approach (x,y) is an $(l,2^{-m})$-approach if $x,y \in \Lambda_l$ but at least one of x,y is not in Λ_{l-1}. Let $U_m(l)$ denote the maximal number of $(l,2^{-m})$-approaches (x,y) that can be chosen so that the disks $\mathcal{B}_{x,y}$ are disjoint. Let

$$U(l) = \sum_{m=1}^{\infty} U_m(l).$$

Note that

$$U^{(n)} \leq \sum_{l=1}^{n} U(l).$$

We will show that there exists a constant c such that

$$\mathbf{Q}_n(U(l)) \leq c.$$

To do this, it suffices to show that there exist c,β such that for each m,

$$\mathbf{Q}_n(U_m(l)) \leq ce^{-\beta m}.$$

Fix m,l and write V for $U_m(l)$. Let τ_1^1 be the smallest $t \geq T_{l-1}^1$ such that there exists a $y \in \Lambda_{l-1} \cup B^1[T_{l-1}^1, t]$ such that $|B_t^1 - y| = 2^{-m}$ and

$$\Lambda_{l-1} \cup B^1[T_{l-1}^1, t] \cup L(B_t^1, y)$$

disconnects. Here $L(x,y)$ denotes the line segment with endpoints x,y, and we say that Γ disconnects if there does not exists a continuous curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) \in \{\Re(z) = 0, \Im(z) \in (0,2\pi)\}, \Re(\gamma(1)) = n, \gamma(0,1) \in \{0 < \Re(z) < n\}$, such that $\gamma(0,1) \cap \Gamma = \emptyset$. Let ρ_1^1 be the smallest t greater than τ_1^1 such that $|B_t^1 - B^1(\tau_1^1)| = 2^{-m+1}$. For integer $r > 1$, let τ_r^1 be the smallest $t \geq \rho_{j-1}^1$ such that there exists a $y \in \Lambda_{l-1} \cup B^1[T_{l-1}^1, t]$ such that $|B_t^1 - y| = 2^{-m+1}$ and

$$\Lambda_{l-1} \cup B^1[T_{l-1}^1, t] \cup L(B_t^1, y)$$

disconnects; and let ρ_r^1 be the smallest $t > \tau_r^1$ with $|B_t^1 - B^1(\tau_r^1)|$. Let $J = J^1$ be the largest r such that $\tau_J^1 \leq T_l^1$, and consider the event

$$\{J \geq r\}.$$

In order for this event to occur it is necessary (although not sufficient) that: Λ_{l-1} does not disconnect; For each $s \leq r$, no disconnection occurs between τ_1^s and ρ_1^s; no disconnection occurs between ρ_1^r and T_{l+1}^1; and finally $\Lambda(l+1,n)$ does not disconnect. The first probability is bounded by $ce^{-\alpha(l-1)}$; the last probability by $ce^{-\alpha(n-l)}$ (here we are really talking conditional probabilities). The Beurling estimates say that the second probability is bounded by $e^{-\beta r}$, and the third probability by $e^{-\beta m}$. Hence

$$\mathbf{Q}_n[J^1] \leq ce^{-\beta m}.$$

Similarly, we let τ_1^2 be the smallest $t > T_{l-1}^2$ such that there exists a y in

$$\Lambda_{l-1} \cup B^1[T_{l-1}^1, T_l^1] \cup B^2[T_{l-1}^2, t],$$

such that $|B_t^2 - y| = 2^{-m}$ and

$$\Lambda_{l-1} \cup B^1[0, T_l^1] \cup B^2[T_{l-1}^2, t] \cup L(B_t^2, y)$$

disconnects, and similarly as above. The only difference is that we have included $B^1[T_{l-1}^1, T_l^1]$. The random times ρ_r^2, τ_r^2 as well as J^2 and we show as above

$$\mathbf{Q}_n[J^2] \leq ce^{-\beta m}.$$

We define $J^3, \ldots, J^k$ similarly, and hence we get

$$\mathbf{Q}_n[J^1 + \cdots + J^k] \leq ce^{-\beta m}.$$

But it is not difficult to check that

$$U_m(l) \leq c[J^1 + \cdots + J^k],$$

so we have proved (57).

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