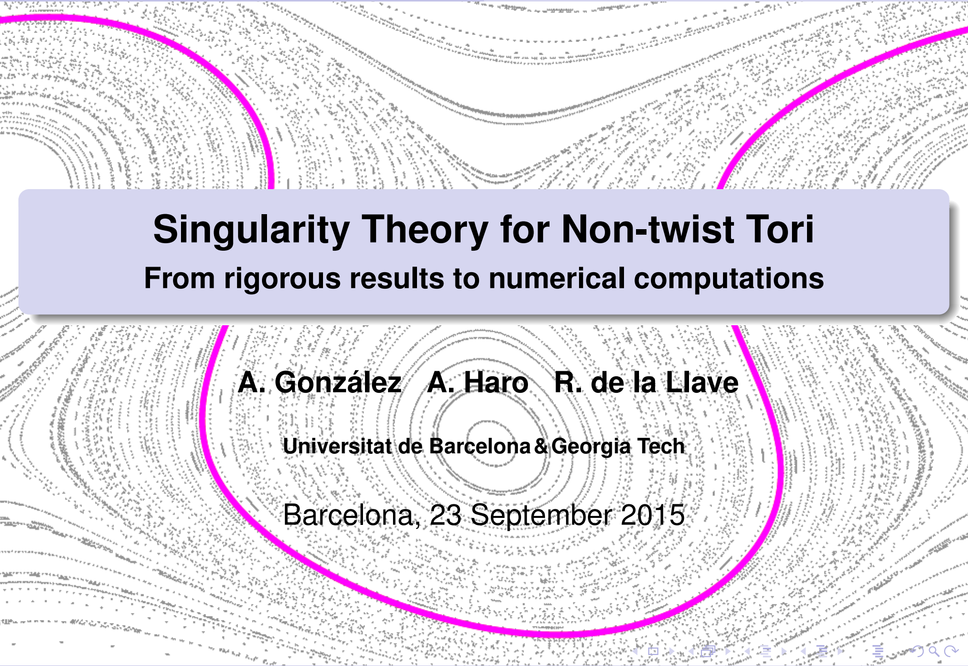

Singularity Theory for Non-twist Tori

From rigorous results to numerical computations

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KAM Theory for non-degenerate Hamiltonian systems

Existence and persistence of quasi-periodic motions in Hamiltonian systems

Classical KAM theorems guarantee persistence of Cantor families of invariant tori under perturbations of non-degenerate integrable systems.

Twist conditions ($\simeq$ diffeomorphic frequency map):

- Persistence of quasi-periodic motions

  Kolmogorov, Arnold, Moser, ...

- KAM results based on reducibility

  Celletti-Chierchia, Salamon-Zehnder, González-Jorba-de la Llave-Villanueva, ...

Rüssmann conditions ($\simeq$ weakly non-degenerate frequency map):

- Persistence of Cantor families of invariant tori

  Rüssmann, Broer, Huitema, Sevryuk, ...

KAM Theory for degenerate Hamiltonian systems

Existence and persistence of degenerate invariant tori with prescribed frequency

Non-twist tori or shearless tori are invariant tori for which the Birkhoof normal form is degenerate.
(The integrable approximation violates the twist condition)

There are many numerical and experimental evidences of existence of non-twist tori, in areas such as plasma physics, celestial mechanics, oceanography.

del Castillo-Negrete, Morrison, Meiss, Apte, Dullin, Beron-Vera, Haller, ...

Rigorous results:

- Existence and persistence of non-twist curves in 2-parameter families of apm. Delshams-de la Llave (2000)
- Existence of meandering twist curves in area preserving non-twist maps. Simó (1998)

KAM Theory for degenerate Hamiltonian systems

Existence and persistence of degenerate invariant tori with prescribed frequency

Our contribution:

- A methodology to study non-twist tori and bifurcations of KAM tori with fixed Diophantine frequency.
- The methodology works in any dimension, and for any kind of degeneracy.
- The methodology works in the far-from-integrable regime (including meandering non-twist tori and non-twist tori close to breakdown).
- The methodololgy gives rise to efficient numerical algorithms (and with time and effort, to computer assisted proofs.)

Motivating examples

Example: A collision of invariant tori

A family of non-twist apm

Consider the quadratic standard family of apm in $\mathbb{T} \times \mathbb{R}$:

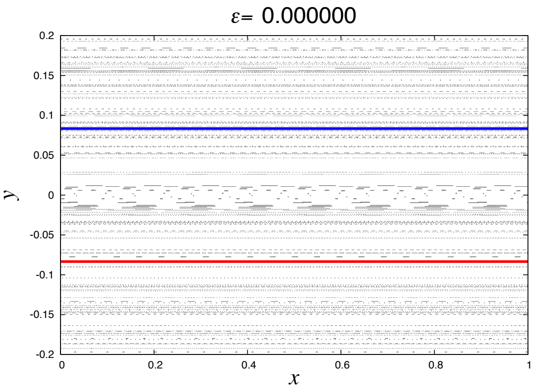
$$\begin{cases} \bar{x} = 0.375 + x + \bar{y}^2 \,, \\ \bar{y} = y - \dfrac{\varepsilon}{2\pi} \sin(2\pi x) \,. \end{cases}$$

Problem:

Study the bifurcations of invariant tori with frequency $\omega = \frac{3-\sqrt{5}}{2}$, with respect to ε.

Example: A collision of invariant tori

Numerical observations

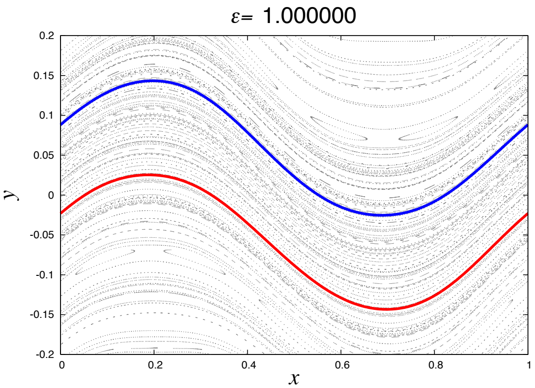


$T = \quad 0.1669253$

$T = -0.1669253$

Example: A collision of invariant tori

Numerical observations



$T = \quad 0.1123464$

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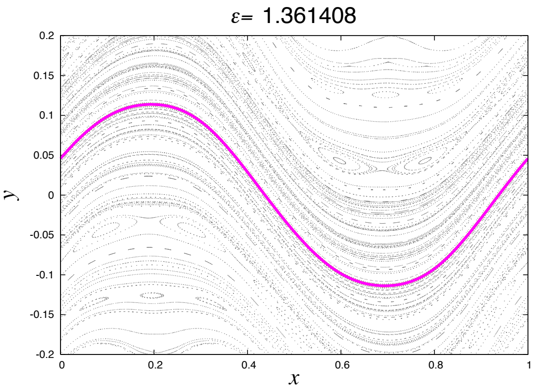
Example: A collision of invariant tori

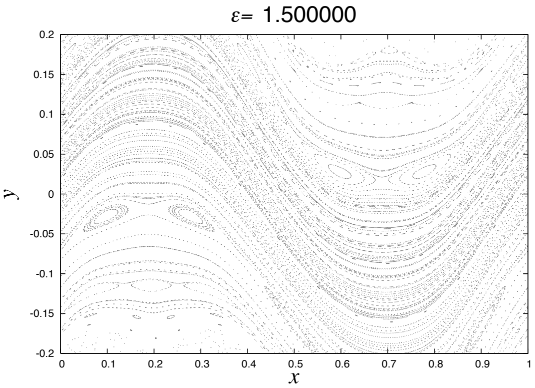
Numerical observations



$$T = 0.6558 \cdot 10^{-8}$$
$$T = -0.6558 \cdot 10^{-8}$$

Example: A collision of invariant tori

Numerical observations



Inspiration: Integrable systems

Frequency map of an integrable system in $\mathbb{T}^n \times \mathbb{R}^n$

Let

$$f_0(x,y) = \begin{pmatrix} x + \nabla W(y) \\ y \end{pmatrix}$$

be an integrable system, where $W : \mathbb{R}^n \to \mathbb{R}$.

- The frequency map is $\hat\omega = \nabla W : \mathbb{R}^n \to \mathbb{R}^n$.
- $\forall p \in \mathbb{R}^n$, $Z_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$ is f_0-invariant with frequency $\hat\omega(p)$:

$$f_0(Z_p(\theta)) = Z_p(\theta + \hat\omega(p)) \ .$$

- Z_p is twist iff the torsion

$$\bar T(p) = \mathrm{D}\,\hat\omega(p) = \mathrm{Hess}\, W(p)$$

is non-degenerate. Otherwise, Z_p is non-twist.

Inspiration: Integrable systems

Potential for frequency ω of an integrable system in $\mathbb{T}^n \times \mathbb{R}^n$

Consider the modified family (Moser, Herman):

$$f_{0,\lambda}(x,y) = \begin{pmatrix} x + \nabla W(y) \\ y \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \end{pmatrix} .$$

- $\forall p \in \mathbb{R}^n$, $Z_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$ is $f_{0,\lambda(p)}$-invariant with frequency ω:

  $$f_{0,\lambda(p)}(Z_p(\theta)) = Z_p(\theta + \omega) ,$$

  where $\lambda(p) = \omega - \nabla W(p)$.

- The potential $V : \mathbb{R}^n \to \mathbb{R}$, defined by

  $$V(p) = W(p) - p^\top \omega ,$$

  satisfies

  $$\lambda(p) = -\nabla_p V(p) , \; \bar{T}(p) = \mathrm{Hess}\, V(p).$$

Inspiration: Integrable systems

Invariant tori as critical points of the potential

Conclusions:

1. Z_{p_*} is f_0-invariant with frequency ω if and only if p_* is a critical point of the *potential* $V(p)$.
2. Z_{p_*} is a twist f_0-invariant torus with frequency ω if and only p_* is a non-degenerate critical point of $V(p)$.
3. Z_{p_*} is a non-twist f_0-invariant torus with frequency ω if and only p_* is a degenerate critical point of $V(p)$.

Inspiration: Integrable systems

A degenerate case in $\mathbb{T} \times \mathbb{R}$

Let f_μ be family of integrable symplectomorphisms in $\mathbb{T} \times \mathbb{R}$ with

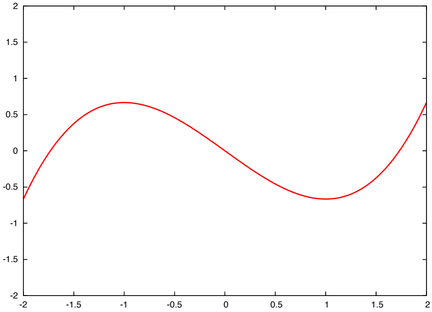
- frequency map $\hat{\omega}(y;\mu) = \omega + \mu + y^2$.

For fixed frequency ω:

- translation map: $\lambda(p;\mu) = -\mu - p^2$;
- potential: $V(p;\mu) = \mu p + \dfrac{p^3}{3}$.

Then, for fixed frequency ω:

- TWO invariant tori for $\mu < 0$.
- ONE NON-TWIST torus for $\mu = 0$.
- NO invariant tori for $\mu > 0$.



$\mu = -1.0$

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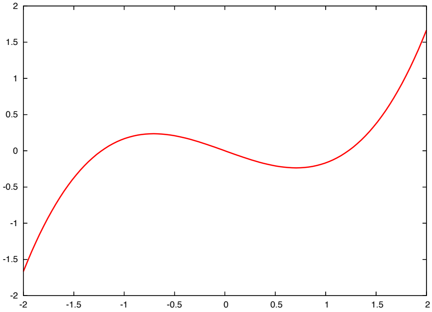

$\mu = -0.5$

Inspiration: Integrable systems

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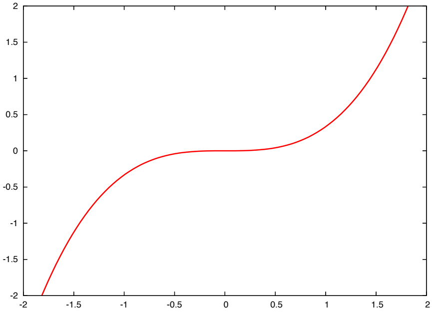

$\mu = 0$

Inspiration: Integrable systems

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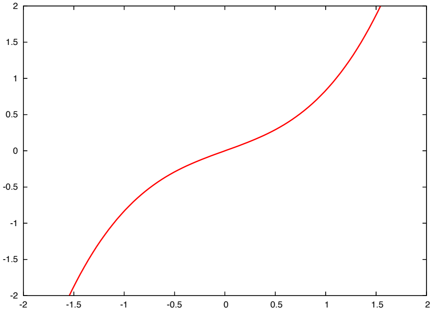

$\mu = 0.5$

Inspiration: Integrable systems

A degenerate case in $\mathbb{T} \times \mathbb{R}$

Let f_μ be family of integrable symplectomorphisms in $\mathbb{T} \times \mathbb{R}$ with

- frequency map $\hat{\omega}(y; \mu) = \omega + \mu + y^2$.

For fixed frequency ω:

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- potential: $V(p; \mu) = \mu p + \dfrac{p^3}{3}$.

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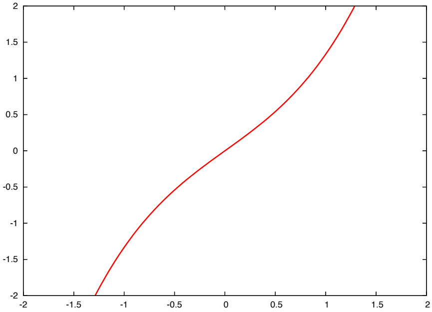


$\mu = +1.0$

Natural questions

(From the motivating examples)

For a fixed Diophantine frequency ω:

1. Is the potential and its properties persistent under perturbations of an integrable system?
2. Given a non-integrable system, are there sufficient conditions that guarantee the existence of a suitable *potential*?
3. Can the potential be used to develop a bifurcation theory of invariant tori?

Methodology

Methodology

Setting

- Let $\Omega = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}$ be the standard symplectic matrix in $\mathbb{T}^n \times \mathbb{R}^n$.
- Let $P \subset \mathbb{R}^m$ be an open subset of parameters ε.
- Let $f_\varepsilon : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ a smooth family of real-analytic exact symplectomorphisms, with primitive functions $S_\varepsilon : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$:

$$dS_\varepsilon = f_\varepsilon^*(y \, \mathrm{d}x) - y \, \mathrm{d}x.$$

- Let $\omega \in \mathbb{R}^n$ be fixed and Diophantine:

$$\left| \ell^\top \omega - m \right| \geq \gamma \, |\ell|_1^{-\tau}, \qquad \forall \ell \in \mathbb{Z}^n \setminus \{0\}, \; m \in \mathbb{Z}.$$

  where $\gamma > 0$, $\tau \geq n$.

Goal: Finding f_ε-invariant tori, with frequency ω.

Methodology

Step 1: KAM theory for families of invariant tori

Using **Newton-like methods** for families of symplectic deformations, find

$$(p;\varepsilon) \in U \times P \to \mathbf{K}(p;\varepsilon) = (\lambda(p;\varepsilon), K_{(p;\varepsilon)})$$

such that

$$\begin{aligned} f_\varepsilon \circ K_{(p;\varepsilon)}(\theta) + \begin{pmatrix} \lambda(p;\varepsilon) \\ 0 \end{pmatrix} &= K_{(p;\varepsilon)}(\theta+\omega)\,, \\ \left\langle K_{p;\varepsilon}(\theta) - \begin{pmatrix} \theta \\ p \end{pmatrix} \right\rangle &= 0\,. \end{aligned}$$

For each ε, $p \to K_{(p;\varepsilon)}$ is a **Lagrangian deformation.**

Non-degeneracy conditions for Lagrangian deformations are mild.

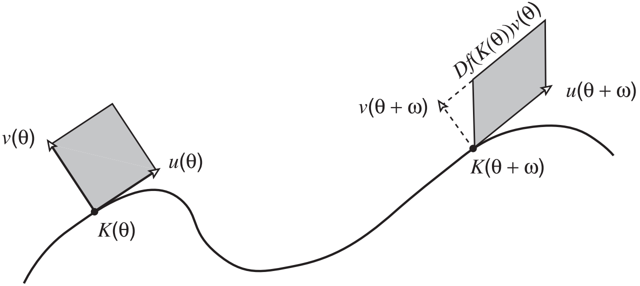
Methodology

Step 1: KAM theory for families of invariant tori

The **torsion** of $K_{(p,\varepsilon)}$ is the symmetric matrix $\bar{T}(p;\varepsilon) = \left\langle T_{(p;\varepsilon)}(\theta) \right\rangle$, where

$$T_{(p;\varepsilon)}(\theta) = N_{(p;\varepsilon)}(\theta+\omega)^{\top}\, \Omega\, \mathrm{D}_z f_\varepsilon(K_{(p;\varepsilon)}(\theta))\, N_{(p;\varepsilon)}(\theta),$$

$$N_{(p;\varepsilon)}(\theta) = J\, \mathrm{D}_\theta K_{(p;\varepsilon)}(\theta) \left(\mathrm{D}_\theta K_{(p;\varepsilon)}(\theta)^{\top} \mathrm{D}_\theta K_{(p;\varepsilon)}(\theta) \right)^{-1}.$$



The torus is twist if the torsion is non-degenerate.

Methodology

Step 2: Symplectic Geometry of families of invariant tori

Define the potential (depending on ε),

$$V(p;\varepsilon) = -p^\top \lambda(p;\varepsilon) - \left\langle S_\varepsilon \circ K_{(p;\varepsilon)}(\theta) \right\rangle \, .$$

Properties of the potential:

- $\lambda(p;\varepsilon) = -\nabla_p V(p;\varepsilon)$.
- $\bar{T}(p;\varepsilon)\, W_1(p;\varepsilon) = W_2(p;\varepsilon)\, \mathrm{Hess}\, V(p;\varepsilon)$ where
  $W_1(p;\varepsilon) = \mathrm{D}_p C(p;\varepsilon)$, with $C(p;\varepsilon) = \left\langle K^y_{(p;\varepsilon)}(\theta)^\top \mathrm{D}_\theta K^x_{(p;\varepsilon)}(\theta) \right\rangle^\top$,
  $W_2(p;\varepsilon) = \left\langle N^y_{(p;\varepsilon)}(\theta) - \mathrm{D}_\theta K^y_{(p;\varepsilon)}(\theta) \mathcal{R}_\omega T_{(p;\varepsilon)}(\theta) \right\rangle$.

Methodology

Step 3: Singularity Theory for critical points of the potential

For ε_* fixed:

- $K_{(p_*;\varepsilon_*)}$ is f_{ε_*}-invariant with frequency ω iff p_* is a critical point of the potential $V(p;\varepsilon_*)$.
- If $W_1(p_*;\varepsilon_*)$ and $W_2(p_*;\varepsilon_*)$ are invertible:

$$\dim \ker \bar{T}(p_*,\varepsilon_*) = \dim \ker \mathrm{Hess}_p V(p_*,\varepsilon_*).$$

KAM theory for translated tori

Translated tori

The funtional equation

Given:

- a homotopic to the identity real-analytic **symplectomorphism** $f : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$;
- a Diophantince vector $\omega \in \mathbb{R}^n$, and denote $R_\omega(\theta) = \theta + \omega$;

for each **height** $p \in \mathbb{R}^n$ we define the error function

$$E(p; K, \Lambda) \stackrel{\text{def}}{=} \begin{pmatrix} f \circ K - K \circ R_\omega + \Lambda \\ \langle K - Z_p \rangle \end{pmatrix},$$

for any

- homotopic to the zero-section **torus** $K : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$
- **translation** $\Lambda = \begin{pmatrix} \lambda \\ \sigma \end{pmatrix} \in \mathbb{R}^{2n}$

Our goal

To solve $E(p; K, \Lambda) = 0$ by a Newton-Nash-Moser-Zehnder method.

Newton method

The linearized equation

Let (K, Λ) be an approximate solution: $E(p; K, \Lambda) = (e, \nu)^\top$.

1. A Newton step involves solving a linear system for $(\Delta K, \Delta\Lambda)$:

$$\begin{aligned} \mathrm{D}f(K(\theta))\Delta K - \Delta K \circ R_\omega + \Delta\Lambda &= -e \\ \langle \Delta \rangle &= -\nu \end{aligned} \tag{1}$$

2. The approximately symplectic matrix

$$M(\theta) = \begin{pmatrix} \mathrm{D}K(\theta) & N(\theta) \end{pmatrix} ,$$

with $N(\theta) = J\mathrm{D}K(\theta)(\mathrm{D}K(\theta)^\top \mathrm{D}K(\theta))^{-1}$, almost reduces $\mathrm{D}f(K(\theta))$:

$$M(\theta+\omega)^{-1}\mathrm{D}f(K(\theta))M(\theta) \simeq (I_{2n} + \hat{T}(\theta)),$$

where

$$\hat{T}(\theta) = \begin{pmatrix} 0_n & T(\theta) \\ 0_n & 0_n \end{pmatrix} \text{ and } T(\theta) = N(\theta+\omega)^\top \Omega\, \mathrm{D}f(K(\theta))N(\theta).$$

Newton method

Homological equations

3. We write $\Delta K(\theta) = M(\theta)\xi(\theta)$, so the Newton equation (1) is approximately equivalent to

$$\begin{aligned} (I + \hat{T})\xi - \xi \circ R_\omega + (M \circ R_\omega)^{-1} \Delta\Lambda &= -(M \circ R_\omega)^{-1} e \\ \langle M\xi \rangle &= -\nu \end{aligned} \qquad (2)$$

4. $\mathcal{L}_\omega$ is the operator acting on functions $\xi : \mathbb{T}^n \to \mathbb{R}$ by

$$\mathcal{L}_\omega \, \xi \stackrel{\text{def}}{=} \xi - \xi \circ R_\omega,$$

We define its “inverse”, as the “small divisors" operator $\mathcal{R}_\omega$ acting on functions $\eta : \mathbb{T}^n \to \mathbb{R}$ by

$$\mathcal{R}_\omega \, \eta \stackrel{\text{def}}{=} \xi \text{ is the zero-average solution of } \mathcal{L}_\omega \, \xi = \eta - \langle \eta \rangle.$$

Notice that $\mathcal{R}_\omega \, \mathcal{L}_\omega \, \xi = \xi - \langle \xi \rangle$, $\mathcal{L}_\omega \mathcal{R}_\omega \eta = \eta - \langle \eta \rangle$.

Newton method

Approximate solution of Newton step

Define $M_+ = M \circ R_\omega$, i.e. $M_+(\theta) = M(\theta + \omega)$.

Let

$$\eta = -M_+^{-1} e$$

be the new r.h.s., and $\xi_0 = \langle \xi \rangle$.

6. $\xi = (I - \mathcal{R}_\omega \hat{T})\xi_0 - \mathcal{R}_\omega (M_+^{-1} - \hat{T}\mathcal{R}_\omega M_+^{-1})\Delta\Lambda + \mathcal{R}_\omega(\eta - \hat{T}\mathcal{R}_\omega \eta)$

   where ξ_0 and $\Delta\Lambda$ are computed from

$$\begin{pmatrix} \left\langle \hat{T} \right\rangle & \left\langle M_+^{-1} - \hat{T}\mathcal{R}_\omega M_+^{-1} \right\rangle \\ \left\langle M(I - \mathcal{R}_\omega \hat{T}) \right\rangle & -\left\langle M\mathcal{R}_\omega (M_+^{-1} - \hat{T}\mathcal{R}_\omega M_+^{-1}) \right\rangle \end{pmatrix} \begin{pmatrix} \xi_0 \\ \Delta\Lambda \end{pmatrix} = \begin{pmatrix} \tilde{\eta} \\ \tilde{\nu} \end{pmatrix}$$

where $\tilde{\eta} = \left\langle \eta - \hat{T}\mathcal{R}_\omega \eta \right\rangle$, $\tilde{\nu} = -\nu - \left\langle M\mathcal{R}_\omega(\eta - \hat{T}\mathcal{R}_\omega \eta) \right\rangle$.

Newton method

A mild non degeneracy condition

Non degeneracy condition

In order to perform the Newton step, it is sufficient that the $4n \times 4n$ matrix

$$Q(K) \stackrel{\text{def}}{=} \begin{pmatrix} \left\langle \hat{T} \right\rangle & \left\langle M_+^{-1} - \hat{T}\mathcal{R}_\omega M_+^{-1} \right\rangle \\ \left\langle M(I - \mathcal{R}_\omega \hat{T}) \right\rangle & -\left\langle M\mathcal{R}_\omega(M_+^{-1} - \hat{T}\mathcal{R}_\omega M_+^{-1}) \right\rangle \end{pmatrix}$$

be invertible.

Newton method

A mild non degeneracy condition

Integrable case

Non degeneracy condition is satisfied, since

$$Q(Z_p) = \begin{pmatrix} O_n & \langle T \rangle & I_n & O_n \\ O_n & O_n & O_n & I_n \\ I_n & O_n & O_n & O_n \\ O_n & I_n & O_n & O_n \end{pmatrix}$$

Translated tori theorem

Informal statement

Theorem (Translated tori theorem)

Let $(p_0; K_0, \Lambda_0)$ *be an* approximate solution *of the functional equation* $E(p; K, \Lambda) = 0$.

- **If** $Q(K_0)$ *is invertible and* $E(p_0; K_0, \Lambda_0)$ *is small enough,*
- **then**, *for each* p *close to* p_0 *there exist a unique* $\Lambda(p) \in \mathbb{R}^{2n}$ *and a unique* $K_p : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$ *s.t.*

$$\begin{aligned} f(K_p(\theta)) &= K_p(\theta + \omega) - \Lambda(p), \\ \left\langle K_p(\theta) - \begin{pmatrix} \theta \\ p \end{pmatrix} \right\rangle &= 0. \end{aligned}$$

Translated tori theorem

Informal statement (cont.)

Theorem (TTT cont.)

Moreover*:*

- $\Lambda(p)$ *and* K_p *are smooth with respect to* p*.*
- $\Lambda(p)$ *and* K_p *depend smoothly on parameters of* f*.*

If f **is exact** *w.r.t. to the Liouville form* $\alpha = y\,dx$*, that is* $f^*\alpha - \alpha = dS$ *for a* **primitive function** $S : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$*:*

- *The translation is horizontal:*

$$\Lambda(p) = \begin{pmatrix} \lambda(p) \\ 0 \end{pmatrix},$$

- $\lambda(p)$ *is a gradient:* $\lambda(p) = -\nabla_p V(p)$*.*

Applications

- Close-to-integrable systems:
  1. Persistence of invariant tori with fixed frequency under weak non-degeneracy conditions.
  2. Persistence of invariant tori for perturbations of isochronous systems.
  3. Robust bifurcation theory for non-twist tori.

- Numerical computations in far-to-integrable systems:
  1. A fold singularity.
  2. Birth of meandering.
  3. Continuation of non-twist tori.

Bifurcation theory for non-twist tori

A persistence result of non-twist tori

Hypotheses

Theorem (Persistence theorem of non-twist tori)

Given:

- $\omega \in \mathbb{R}^n$, *a Diophantine frequency vector.*
- f_0, *a real-analytic integrable symplectic map with frequency map* $\hat{\omega}(y) = \omega + \nabla V(y)$, *s.t.* $y^* = 0$ *is a finitely determined singularity of* $V(y)$.
- f_μ, *a* μ*-family of real-analytic integrable symplectic map with frequency map* $\hat{\omega}_\mu(y) = \omega + \nabla V_\mu(y)$, *s.t.* $V_\mu(y)$ *is a stable unfolding of the singularity at zero.*
- $f_{\mu,\varepsilon}$, *a perturbed family of exact symplectomorphims:*

$$f_{\mu,\varepsilon}(x, y) = \begin{pmatrix} x + \omega + \nabla V_\mu(y) \\ y \end{pmatrix} + \varepsilon g(x, y, \varepsilon).$$

A persistence result of non-twist tori

Theses

Theorem (Theses)

Then:

- *(KAM) For* (μ, ε) *sufficiently small, there exists a potential* $V_{\mu,\varepsilon}(p)$ *defined around* $0 \in \mathbb{R}^n$ *and an n-dimensional torus* $K_{p,\mu,\varepsilon} : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$*, depending smoothly on* (p, μ, ε)*, such that:*

$$f_{\mu,\varepsilon} \circ K_{p,\mu,\varepsilon}(\theta) = K_{p,\mu,\varepsilon}(\theta + \omega) + \begin{pmatrix} \nabla_p V_{\mu,\varepsilon}(p) \\ 0 \end{pmatrix} .$$

- *(ST) For* ε *sufficiently small, there exist* $\mu(\varepsilon)$ *and* $p(\varepsilon)$ *such that* $V_{\mu(\varepsilon),\varepsilon}$ *has a singularity at* $p(\varepsilon)$ *that is of the same type (in the sense of Singularity Theory) of the singularity of* V_0 *at zero.*

Numerical examples

Example 1: The collision of invariant tori

Applying the methodology

- Modified family:

$$\mathbf{f}_{(\varepsilon,\lambda)}(x,y) = \begin{pmatrix} 0.375 + x + \left(y - \frac{\varepsilon}{2\pi}\sin(2\pi x)\right)^2 + \lambda \\ y - \frac{\varepsilon}{2\pi}\sin(2\pi x) \end{pmatrix}.$$

- The primitive function of $f_{\varepsilon,\lambda}$ is

$$S_\varepsilon(x,y) = \frac{\varepsilon}{4\,\pi^2}\cos(2\pi x) + \frac{2}{3}\left(y - \frac{\varepsilon}{2\pi}\sin(2\pi x)\right)^3.$$

- The co-rank of the torsion of a torus is either 0 (twist) or 1 (non-twist).

Example 1: The collision of invariant tori

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Applying the methodology

- Modified family:

$$\mathbf{f}_{(\varepsilon,\lambda)}(x,y) = \begin{pmatrix} 0.375 + x + \left(y - \frac{\varepsilon}{2\pi}\sin(2\pi x)\right)^2 + \lambda \\ y - \frac{\varepsilon}{2\pi}\sin(2\pi x) \end{pmatrix} .$$

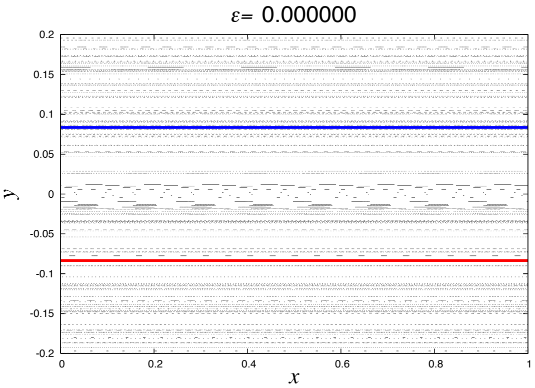
- The primitive function of $f_{\varepsilon,\lambda}$ is

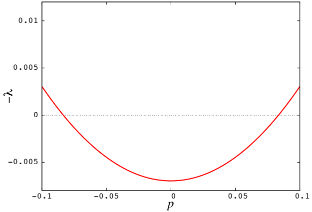
$$S_\varepsilon(x,y) = \frac{\varepsilon}{4\,\pi^2}\cos(2\pi x) + \frac{2}{3}\left(y - \frac{\varepsilon}{2\pi}\sin(2\pi x)\right)^3 .$$

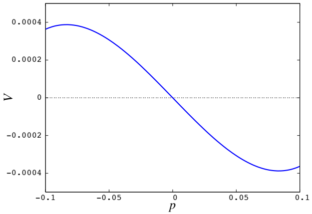
- The co-rank of the torsion of a torus is either 0 (twist) or 1 (non-twist).

Example 1: The collision of invariant tori

A fold singularity





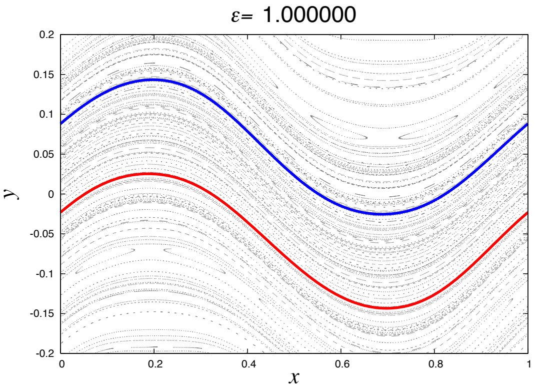


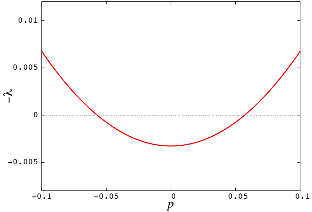
$T = \quad 0.1669253$

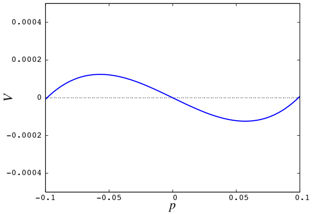
$T = -0.1669253$

Example 1: The collision of invariant tori

A fold singularity





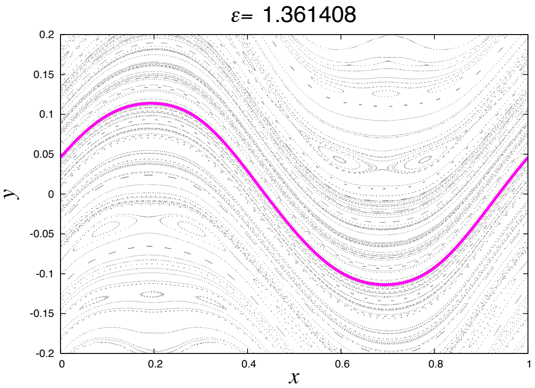


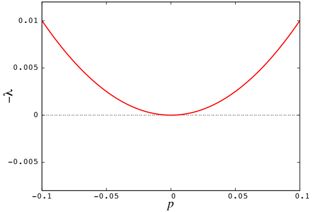
$T = \quad 0.1123464$

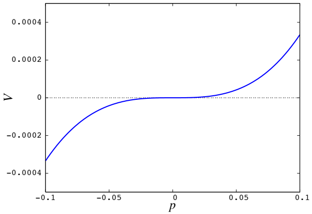
$T = -0.1123464$

Example 1: The collision of invariant tori

A fold singularity





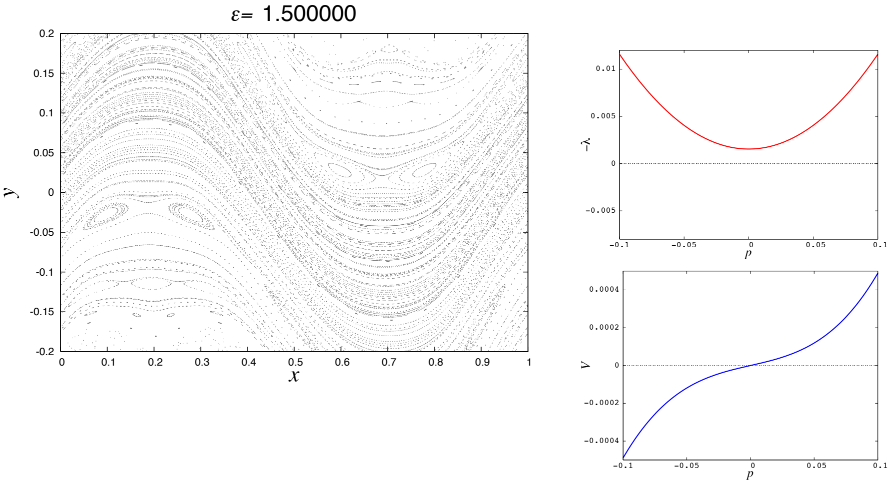


$T = \quad 0.6558 \cdot 10^{-8}$

$T = -0.6558 \cdot 10^{-8}$

Example 1: The collision of invariant tori

A fold singularity



Example 2: The birth of meandering tori

Another non-twist family

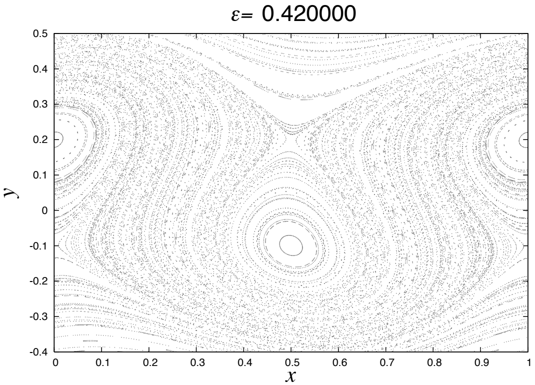
Consider a quadratic standard family of symplectomorphisms $f_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, defined by:

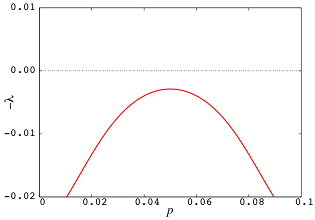
$$\begin{cases} \bar{x} &= x + (\bar{y} + 0.1)(\bar{y} - 0.2)\,, \\ \bar{y} &= y - \dfrac{\varepsilon}{2\pi} \sin(2\pi x)\,. \end{cases}$$

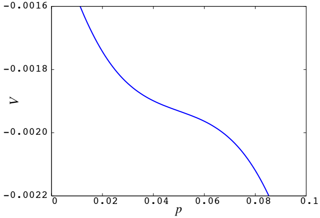
Problem: Look for invariant tori with frequency $\omega = \frac{\sqrt{5}-1}{32}$, with respect to parameter ε.

A fold singularity

The birth of meandering tori

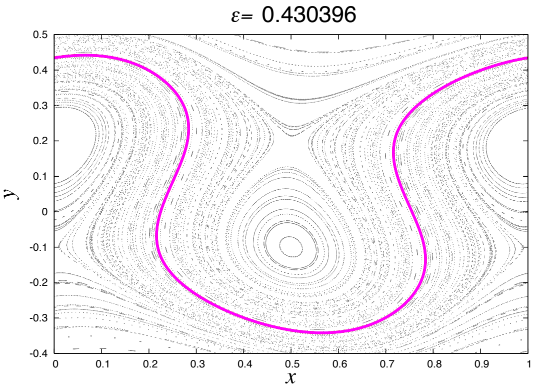


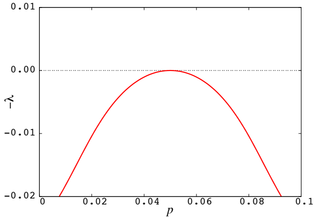


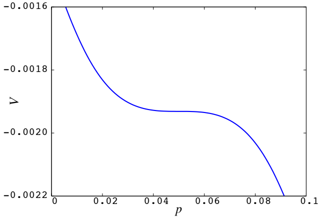


A fold singularity

The birth of meandering tori





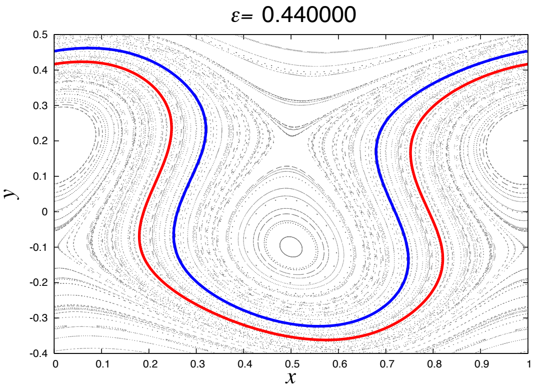


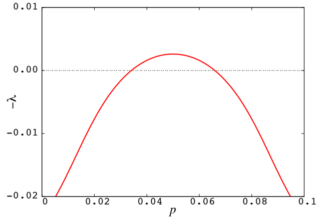
$T = -1.6261 \cdot 10^{-7}$

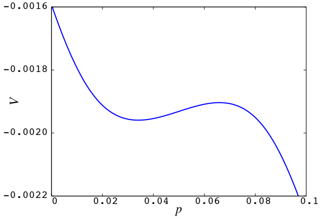
$T = \quad 1.6261 \cdot 10^{-7}$

A fold singularity

The birth of meandering tori





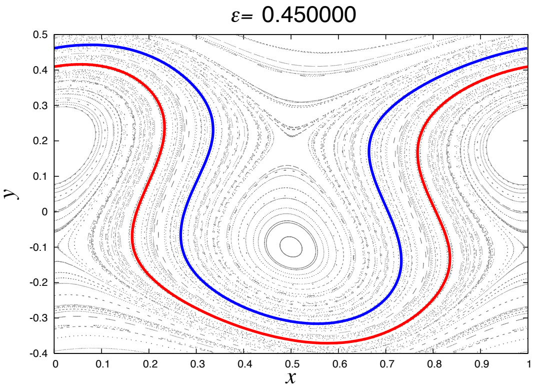


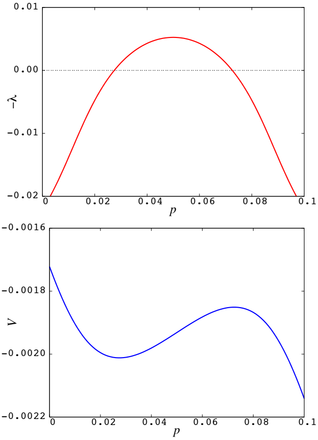
$T = -0.0211805$

$T = \quad 0.0211805$

A fold singularity

The birth of meandering tori





$T = -0.0309003$

$T = \quad 0.0309003$

Example 3: Continuation of a non-twist torus

The problem

Consider a quadratic standard family of symplectomorphisms $f_{\mu,\varepsilon} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, defined by:

$$\begin{cases} \bar{x} &= \omega + \mu + x - \bar{y}^2 \,, \\ \bar{y} &= y - \dfrac{\varepsilon_1}{2\pi} \sin(2\pi x) - \dfrac{\varepsilon_2}{4\pi} \cos(4\pi x) \,. \end{cases}$$

with $\omega = \frac{\sqrt{5}-1}{2}$.

μ is the **unfolding parameter.**

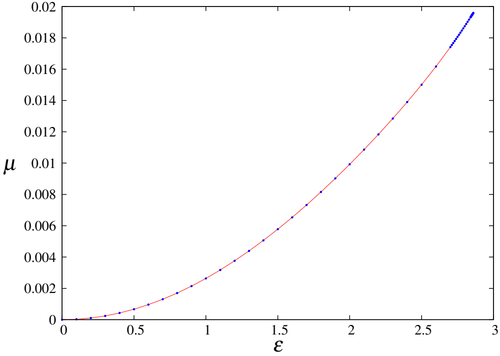
Fixed φ, take $\varepsilon_1 = \cos(2\pi\varphi)\varepsilon$, $\varepsilon_2 = \sin(2\pi\varphi)\varepsilon$, with ε as the **continuation parameter.**

Parameter φ **breaks reversibility.**

Problem: Look for **non-twist** invariant tori with frequency $\omega = \frac{\sqrt{5}-1}{2}$ (for suitable μ), with respect to parameter ε, from $\varepsilon = 0$ (and $\mu = 0$).

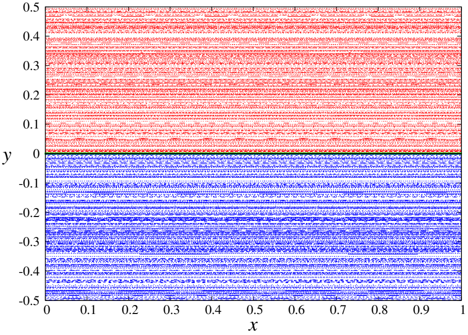
Example 3: Continuation of a non-twist torus

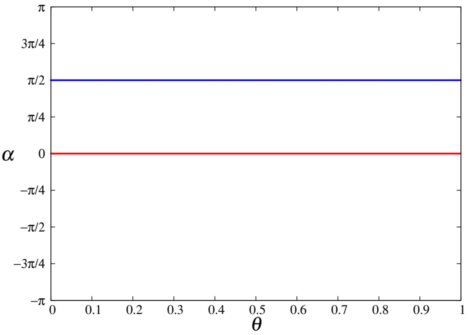
Case $\varphi = \frac{1}{8}$



Example 3: Continuation of a non-twist torus and its adapted symplectic frame

$\varepsilon = 0.000$, $\mu = 0.0000000000$

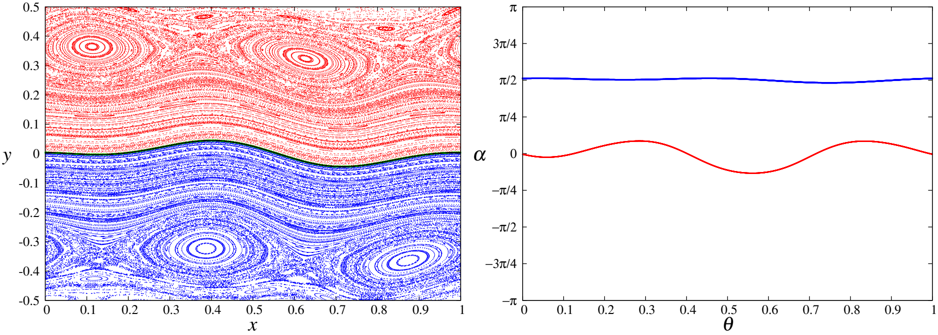




NF= 1024

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

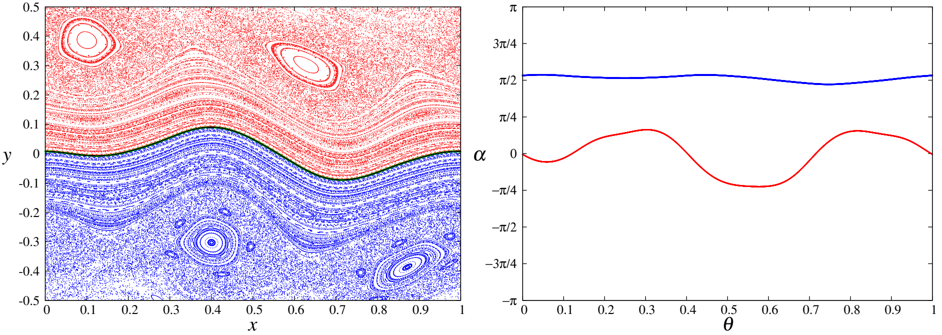
$\varepsilon = 0.500$, $\mu = 0.0006691132$



NF= 1024

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

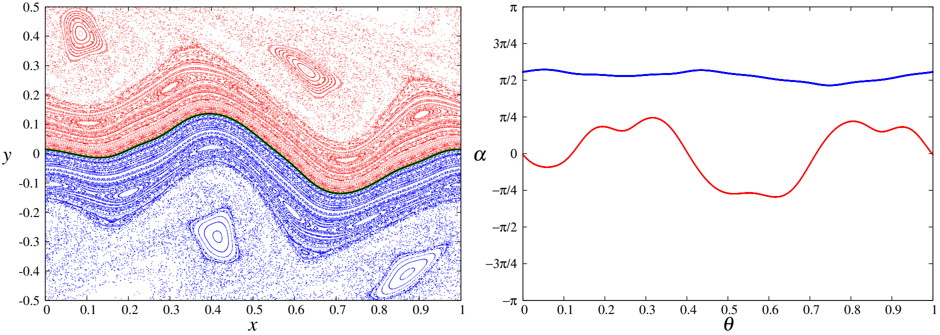
$\varepsilon = 1.000$, $\mu = 0.0026359979$



NF= 1024

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

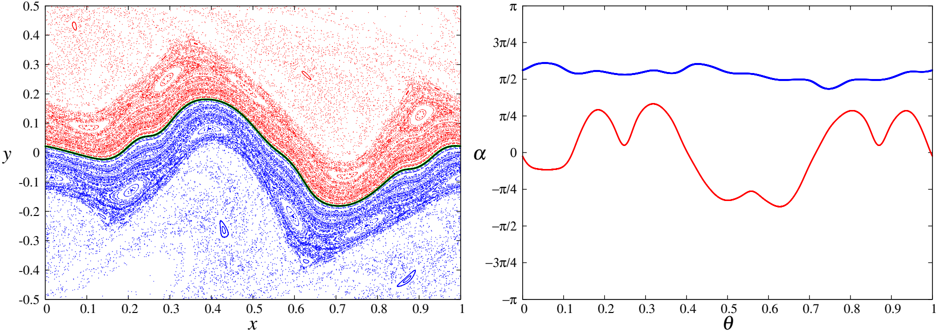
$\varepsilon = 1.500$, $\mu = 0.0057799523$



NF= 1024

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

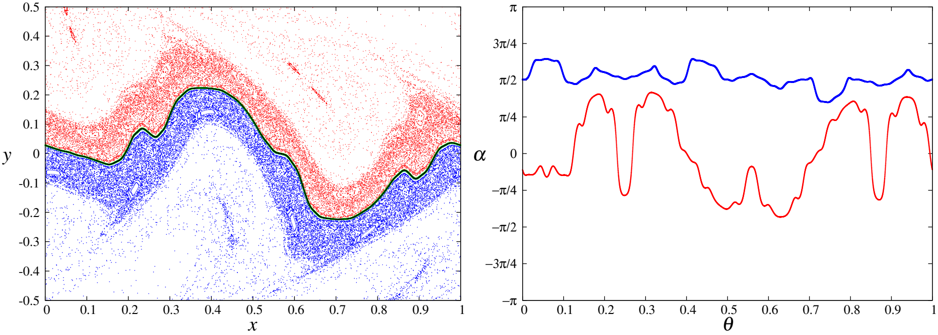
$\varepsilon = 2.000$, $\mu = 0.0099248657$



NF= 1024

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

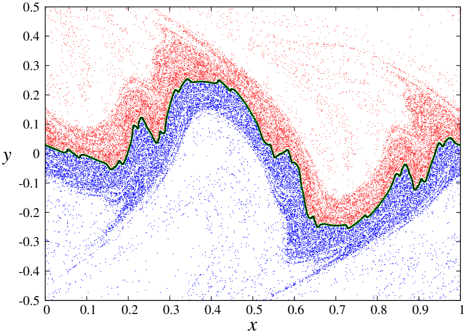
$\varepsilon = 2.500$, $\mu = 0.0150071316$

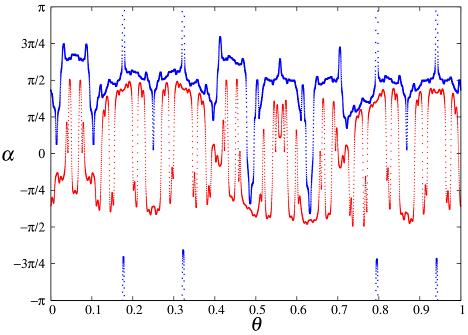


NF= 1024

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

$\varepsilon = 2.800$, $\mu = 0.0187418638$

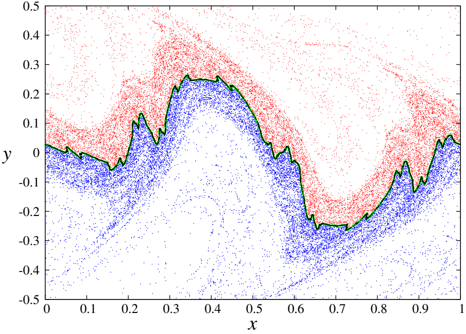


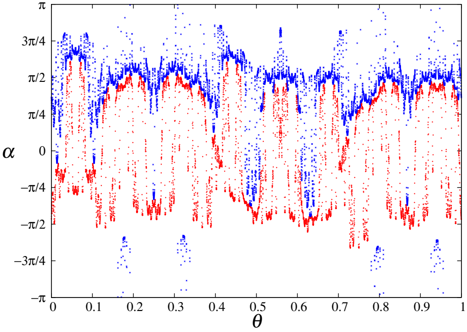


NF= 32768

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

$\varepsilon = 2.860$, $\mu = 0.0195948602$

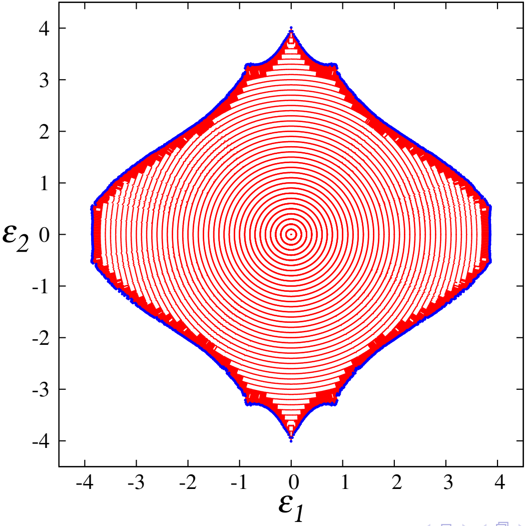




NF= 524288

Example 3: Continuation of a bunch of non-twist tori

Taking $\varphi = 0.000, 0.001, 0.002, \ldots, 1.000$.



Contributions and perspectives

Contributions

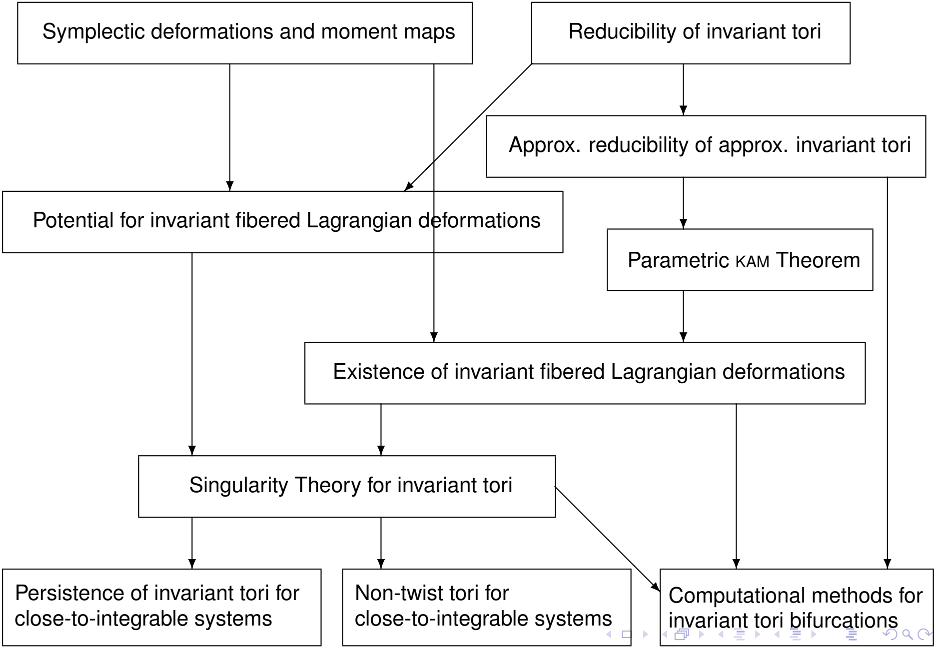
- The infinite dimensional problem of finding invariant tori for Hamiltonian systems is reduced to the finite dimensional problem of finding critical points of the potential.
- Singularity Theory for invariant tori
  - Non-twist tori correspond to degenerate critical points.
  - Applicable to any finite-determined singularity of the frequency map.
- Persistence results for small twist systems.
  - Applicable to perturbation of isochronous systems, introducing a suitable Melnikov potential.

See

A. González-Enríquez, A. Haro, R. de la Llave. *Singularity theory for nontwist KAM tori*. Memoirs of the AMS (2014)

Perspectives

- Study the connections with Symplectic Geometry.
  - The theory is developed using such language.
- Develop efficient numerical algorithms to compute and validate invariant tori and their bifurcations.
  - The method is designed for the cases on which numerical evidence of bifurcations is known but the system is not close to integrable.
  - It is useful for studying of the breakdown of degenerate invariant tori.
- Apply the techniques to “real” problems in plasma physics, oceanography (LCS?), celestial mechanics, etc.


Symplectic deformations and moment maps
Reducibility of invariant tori
Approx. reducibility of approx. invariant tori
Potential for invariant fibered Lagrangian deformations
Parametric KAM Theorem
Existence of invariant fibered Lagrangian deformations
Singularity Theory for invariant tori
Persistence of invariant tori for close-to-integrable systems
Non-twist tori for close-to-integrable systems
Computational methods for invariant tori bifurcations