# THE PRIMITIVE FUNCTION OF AN EXACT SYMPLECTOMORPHISM

## Variational principles, Converse KAM Theory and the problems of determination and interpolation

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Certifico que la present memòria ha estat
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A la Cristina

# Prefaci

*Com que la construcció de tot l'univers és absolutament perfecta i es deguda a un Creador amb coneixement infinit, no res existeix al món que no mostri alguna propietat de màxim o mínim. Així doncs, no pot haver cap dubte sobre la possibilitat que tots els efectes estiguin determinats pels seus designis finals amb l'ajuda del mètode del màxim, de la mateixa manera que ells estan també determinats per les causes inicials.*

## La Geometria de la Natura

Les lleis fonamentals de la Natura, des de la mecànica clàssica, l'òptica geomètrica, la gravetat, l'electromagnetisme fins a la mecànica quàntica, semblen ser Hamiltonianes. Maupertuis ho va explicar tot dient que, suposant que l'Univers tingués un Creador perfecte, llavors ha de ser el millor dels universos possibles, i així doncs hauria d'estar regit per un *principi variacional*. Encara que això ho va dir abans que Hamilton formulés la seva dinàmica, és un fet ben conegut que els principis variacionals i Hamiltonians estan íntimament relacionats. Com diu R.S. MacKay [69], tot plegat és una mica misteriós.

Com que el llenguatge de la *mecànica Hamiltoniana* és el càlcul de formes diferencials i camps vectorials sobre varietats diferenciables [1], la formulació bàsica d'aquest càlcul actua com les *regles gramaticals* [96]. Una conseqüència agradable és la possibilitat d'evitar els càlculs feixucs tan corrents en mecànica analítica. De fet, el primer exemple d'aquest formalisme va aparèixer en un treball de J.L. Lagrange [58] sobre mecànica celest l'any 1808. Lagrange va escriure les equacions del moviment per als elements orbitals $z = (z_1, \ldots, z_6)$ d'un planeta, sota l'efecte de pertorbacions, en la forma

$$\frac{\partial H}{\partial z_i} = \sum_{j=1}^{6} a_{ij}(z) \frac{dz_j}{dt},$$

[1] Un petit resum de geometria diferencial apareix cap al final de la memòria.

on $(a_{i,j})_{i,j=1\div 6}$ és una matriu antisimètrica, i va mostrar que mitjançant una elecció adequada del sistema coordenat es poden escriure les equacions en la forma ara coneguda com *equacions de Hamilton*.

Així doncs, com va comentar Alan Weinstein [98], aquest formalisme Hamiltonià té el paper en Matemàtiques d'un llenguatge que pot facilitar la comunicació entre la Geometria i l'Anàlisi. De fet, la geometrització d'aquest llenguatge és el que avui en dia es coneix com *Geometria Simplèctica*, que ha esdevingut una pròspera branca de les Matemàtiques. La paraula *simplèctic* va ser inventada per H. Weyl [99], el qual va substituir l'arrel grega per la llatina de la paraula *complex*. A ell li devem també la citació següent que il·lustra el tarannà del que estem comentant:

> *A l'interior d'un matemàtic s'estan barallant el dimoni de l'àlgebra abstracta i l'àngel de la geometria.*

El formalisme Hamiltonià/simplèctic ha impregnat altres teories matemàtiques, que en principi estan bastant llunyanes. Com a exemples, citem la teoria de representacions de grups de Lie, la teoria de resolubilitat local d'operadors diferencials lineals i la teoria d'operadors canònics. Des d'un punt de vista més aviat filosòfic, sembla que es pugui *simplectificar* tot. Encara que nosaltres no tractarem d'aquests temes, sí que mostre la importància que té l'estudi de la geometria simplèctica dins les Matemàtiques.

Podem resumir aquestes idees dient que sembla que *Déu és un geòmetra*, i la geometria de la Natura és simplèctica.

## L'estructura de l'espai de fase

Tornant a la mecànica clàssica [5, 1, 61], és una bona idea descriure els estats dels sistemes amb unes coordenades $z = (x, y)$, on $x = (x_1, \ldots, x_d)$ són les coordenades locals sobre una varietat $\mathcal{M}$ (l'*espai de configuració*) i que descriuen les *posicions* dels punts d'aquesta, i $y = (y_1, \ldots, y_d)$ són els corresponents *moments* conjugats, que descriuen covectors (1-formes) sobre tal varietat. És a dir, $z = (x, y)$ són coordenades locals del fibrat cotangent $\mathcal{N} = T^*\mathcal{M}$ de $\mathcal{M}$, l'*espai de fase* del nostre sistema. $d$ és el nombre de graus de llibertat. Veiem que això és una herència directa de les lleis de Newton, que en particular diuen que per determinar el moviment d'un sistema de partícules necessitem les posicions i velocitats en un determinat instant (a part de les seves interaccions, evidentment). L'estructura de l'espai de fase que ara descriurem seria molt diferent si les equacions de Newton fossin de tercer ordre i no de segon, és a dir, si necessitéssim, a més, les acceleracions inicials de les partícules per determinar els seus moviments.

Un sistema dinàmic general ve donat per un camp vectorial sobre l'espai de fase $\mathcal{N}$, que codifica l'evolució infinitesimal de qualsevol quantitat definida sobre ell. És a dir, si $X \in \mathcal{X}(\mathcal{N})$ és un camp vectorial i $F \in C^\infty(\mathcal{N})$ és una funció, llavors la variació infinitesimal $\dot{F}$ de $F$ al llarg de les trajectòries de $X$ (o derivada orbital) ve donada per

$$\dot{F} \;=\; X(F).$$

Aquesta és la versió intrínseca d'un sistema d'equacions diferencials ordinàries

$$\begin{cases} \dot{x}_i = f_i(x, y), \\ \dot{y}_i = g_i(x, y), \end{cases}$$

on $i = 1 \div d$, i

$$X = \sum_{i=1}^{d} \left( f_i(x,y)\frac{\partial}{\partial x_i} + g_i(x,y)\frac{\partial}{\partial y_i} \right).$$

D'aquesta manera, escrivim

$$X(F) = \sum_{i=1}^{d} \left( f_i(x,y)\frac{\partial F}{\partial x_i} + g_i(x,y)\frac{\partial F}{\partial y_i} \right).$$

Un axioma fonamental a la descripció dels sistemes físics, i que es podria anomenar el *paradigma de l'Energia* [47], és el següent:

> *Tot sistema físic té una funció definida sobre el seu espai d'estats, anomenada l'Hamiltonià del sistema, que conté tota la seva informació dinàmica.*

Així, doncs, si $\mathcal{N}$ modelitza l'espai d'estats d'una família de sistemes dinàmics, hi ha d'haver per a cada funció $H$ sobre $\mathcal{N}$ un camp vectorial $X_H$ que descrigui un sistema dinàmic. En el cas de la mecànica Hamiltoniana, aquesta associació ve donada geomètricament per una 2-*forma simplèctica* sobre $\mathcal{N}$, és a dir, una 2-forma $\boldsymbol{\omega}$ que és tancada ($\mathrm{d}\boldsymbol{\omega} = 0$) i no degenerada (com a 2-forma sobre cada punt). Es diu que el parell $(\mathcal{N}, \boldsymbol{\omega})$ és una *varietat simplèctica*. La primera condició ve donada pel fet de lligar les diferents 2-formes no degenerades sobre els diferents punts. La condició de no-degeneració implica la paritat de la dimensió de l'espai fàsic i ens permet caracteritzar $X_H$ mitjançant

$$\boldsymbol{\omega}(X_H, Y) = -\mathrm{d}H(Y),$$

on $Y \in \mathcal{X}(\mathcal{N})$ és qualsevol camp. $X_H$ es diu que és un *camp Hamiltonià* i $H$ és la seva *funció de Hamilton*. El seu flux preserva l'estructura simplèctica.

Per comparar dos fluxos Hamiltonians donats per $H_1$ i $H_2$ es pot utilitzar el parèntesi de Lie dels camps corresponents. Com que nosaltres tenim una estructura addicional això, es pot traduir en un parèntesi aplicat no als camps sinó als Hamiltonians mateixos. És el *parèntesi de Poisson*:

$$\{H_1, H_2\} = \boldsymbol{\omega}(X_{H_1}, X_{H_2}) = -\mathrm{d}H_1(X_{H_2}) = \mathrm{d}H_1(X_{H_2}),$$

que satisfà

$$X_{\{H_1,H_2\}} = [X_{H_1}, X_{H_2}].$$

L'estructura d'àlgebra de Lie del conjunt de camps vectorials amb el parèntesi de Lie és heretada pel conjunt de funcions amb el parèntesi de Poisson. Va ser també Lagrange el primer que va utilitzar aquest parèntesi.

Usant unes coordenades adients, dites simplèctiques, podem escriure la 2-forma $\boldsymbol{\omega}$ com

$$\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^{d} \mathrm{d}y_i \wedge \mathrm{d}x_i.$$

Un fet remarcable és que totes les formes simplèctiques es poden escriure localment d'aquesta manera, tal com afirma el teorema de Darboux. Aquesta és una diferència substancial entre la geometria simplèctica i la geometria Riemanniana. Les equacions de Hamilton no són res més que la traducció a coordenades del camp $X_H$:

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial y_i}, \\ \\ \dot{y}_i = -\dfrac{\partial H}{\partial x_i}. \end{cases}$$

El parèntesi de Poisson és escrit com

$$\{H_1, H_2\} \;=\; \sum_{i=1}^{d} \left( \frac{\partial H_1}{\partial y_i} \cdot \frac{\partial H_2}{\partial x_i} - \frac{\partial H_1}{\partial x_i} \cdot \frac{\partial H_2}{\partial y_i} \right).$$

La unitat bàsica estructural de la mecànica Hamiltoniana és una 1-forma sobre l'espai fàsic $\mathcal{N} = T^*\mathcal{M}$, $\boldsymbol{\alpha} \in \Omega^1(T^*\mathcal{M})$, que és caracteritzada per

$$\forall \rho \in \Omega^1(\mathcal{M}) \;\; \rho^* \boldsymbol{\alpha} = \rho,$$

on la part dreta de la igualtat la veiem com una aplicació $\rho : \mathcal{M} \to T^*\mathcal{M}$ (de fet, és una secció diferenciable del fibrat cotangent). Aquesta 1-forma natural rep el nom de *forma de Liouville*, i la seva diferencial $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$ és la forma simplèctica natural sobre el fibrat cotangent. En coordenades cotangents $(x, y)$ la forma simplèctica és

$$\boldsymbol{\omega} \;=\; \mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^{d} \mathrm{d}y_i \wedge \mathrm{d}x_i$$

i la forma de Liouville és

$$\boldsymbol{\alpha} \;=\; y\; \mathrm{d}x = \sum_{i=1}^{d} y_i\; \mathrm{d}x_i.$$

Així doncs, un cas especialment important correspon al fet que la forma simplèctica sigui exacta, és a dir, que existeixi una 1-forma $\boldsymbol{\alpha}$, anomenada *forma d'acció*, que verifiqui $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$. Nosaltres només considerarem aquest cas. De fet, a partir d'ara considerarem que el nostre espai fàsic és $\mathcal{N} = T^*\mathcal{M}$, encara que la majoria de definicions són vàlides en contextos més generals.

## Simplectomorfismes exactes

Nosaltres treballarem amb una versió discreta de la mecànica. És a dir, en lloc de treballar amb fluxos (donats per camps vectorials) ho farem amb difeomorfismes. De fet, passar dels fluxos als difeomorfismes és fàcil, emprant seccions de Poincaré. Així, per exemple, si el nostre camp $X = X(z, t)$ és $T$-periòdic en el temps, i el seu flux és $\varphi_{t,t_0}$, llavors podem considerar l'aplicació $F = \varphi_{T,0}$, que la podem interpretar com l'aplicació de Poincaré associada a la secció $\Sigma = \{(z, 0) \in \mathcal{N} \times \mathbb{S}\}$, on hem ampliat l'espai de fase a $\mathcal{N} \times \mathbb{T}$ i aquí $\mathbb{T} = \mathbb{R}/(T\mathbb{Z})$.

Com que els nostres camps són Hamiltonians, llavors els seus fluxos preserven l'estructura simplèctica. En general, un difeomorfisme que preserva aquesta estructura s'anomena *simplectomorfisme*. Aquest terme va ser introduït per Souriau, i equival al de *transformació canònica*, utilitzat a la mecànica analítica. Així doncs, un simplectomorfisme $F : \mathcal{N} \to \mathcal{N}$ és un difeomorfisme que verifica

$$F^*\boldsymbol{\omega} \;=\; \boldsymbol{\omega}.$$

Com que la nostra estructura simplèctica és exacta, és a dir, té una primitiva $\boldsymbol{\alpha}$, llavors la 1-forma $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}$ és tancada:

$$\begin{aligned} 0 &= F^*\mathrm{d}\boldsymbol{\alpha} - \mathrm{d}\boldsymbol{\alpha} \\ &= \mathrm{d}(F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}). \end{aligned}$$

Si, en particular, aquesta forma és exacta llavors direm que el nostre simplectomorfisme és exacte. Això vol dir que hi ha una funció $S : \mathcal{N} \to \mathbb{R}$ que satisfà l'*equació d'exactitud*

$$F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} \;=\; \mathrm{d}S,$$

i s'anomena la *funció primitiva* de $F$. No cal dir que està definida llevat de constants.

Un fet que nosaltres trobem curiós és que molts autors es refereixen a $S$ com la *funció generatriu* de $F$, quan en realitat no el genera! De fet, $S$ genera una família de simplectomorfismes, tots amb la mateixa funció primitiva. Podem dir abreujadament que

$S$ determina $F$ llevat de difeomorfismes sobre la base.

La raó és que un difeomorfisme sobre $\mathcal{M}$ pot ser elevat a un simplectormorfisme exacte sobre $T^*\mathcal{M}$, i la funció primitiva corresponent és nul·la. Per això, nosaltres hem cregut més convenient seguir la nomenclatura utilitzada a [7]. Ens plantegem llavors estudiar la natura d'aquesta funció primitiva, i quina informació en podem extreure. Encara que el concepte de funció generatriu (de tipus Lagrangià) pot ser introduït pels simplectomorfismes exactes, la seva existència global restringeix

- els tipus de simplectormorfismes, que han de ser tranversals a la fibració estàndard, que ve donada per les fulles verticals;
- la topologia del nostre espai de fase, perquè l'espai de configuració hauria de ser difeomorf a $\mathbb{R}^d$.

De totes maneres, hem de dir que encara que $\mathcal{M}$ no sigui $\mathbb{R}^d$, es poden considerar tan funcions generatrius locals com multiformes, però la majoria de resultats demanen la seva existència global. Nosaltres no adoptarem aquest punt de vista i treballarem sempre amb la funció primitiva. Recordem també que si $\mathcal{M} = \mathbb{T}^d$, treballarem amb el seu recobridor universal, $\mathbb{R}^d$.

## Algunes qüestions relacionades amb la dinàmica simplèctica

Sigui llavors un simplectomorfisme exacte $F : T^*\mathcal{M} \to T^*\mathcal{M}$, amb funció primitiva $S : T^*\mathcal{M} \to \mathbb{R}$. L'escriurem, utilitzant coordenades cotangents, com

$$\begin{cases} \bar{x} = f(x,y), \\ \bar{y} = g(x,y), \end{cases}$$

(encara que podem definir intrínsecament la component bàsica com $f = q \circ F$, on $q : T^*\mathcal{M} \to \mathcal{M}$ és la projecció).

Com que la funció primitiva no determina el simplectomorfisme, la pregunta que ens podem fer immediatament és:

> quina informació addicional necessitem per obtenir el nostre simplectomorfisme a partir de la seva funció primitiva?

Aquest problema l'hem anomenat el *problema de determinació*, i està relacionat amb el *problema d'interpolació*, que el podem resumir dient que:

> donat un simplectomorfisme $F$, podem trobar un Hamiltonià no autònom $H = H(z,t)$ el flux del qual interpoli $F$ ($\varphi_{1,0} = F$)?

Aquesta segona qüestió va ser tractada per Moser [77] per demostrar la convergència de la forma normal de Birkhoff [19] per una *aplicació del pla que preserva l'àrea* [2] i al voltant d'un punt fix hiperbòlic. Més endavant, altres autors es van preocupar per diferents aspectes del problema, com són Douady [29], Conley i Zehnder [26], Kuksin [55], Kuksin i Pöschel [56]. Darrerament Pronin i Treschev [86], treballant en el cas analític, van demostrar constructivament que si es podia interpolar el nostre simplectomorfisme llavors es podia aconseguir que el Hamiltonià fos periòdic en el temps.

Nosaltres considerarem aquests dos problemes en el cas en què el simplectomorfisme deixa fixa la secció zero (la base) i coneixem la dinàmica sobre aquesta. També seguirem aquesta línea *constructivista* i el que farem serà:

- construir formalment el nostre simplectomorfisme a partir de la dinàmica sobre la secció zero, que és fixa, i la funció primitiva;
- en lloc de demostrar directament l'analiticitat de les sèries [3], trobarem constructivament un Hamiltonià que l'interpoli, i demostrarem que és analític en un entorn de la secció zero i respecte al temps.

Per construir el Hamiltonià utilitzem un *mètode d'homotopia*, i obtenim certa equació en derivades parcials de tipus evolutiu, no lineal. En aquesta equació apareix el que a l'àmbit de la mecànica analítica es coneix com l'*acció elemental* d'un Hamiltonià, i que

[2] Abreujadament: *a.p.a.*
[3] Podríem també haver-ho fet directament, però d'aquesta manera "matem dos pardals d'un tret".

nosaltres veurem que és una *derivació* a l'àlgebra de Lie de funcions (on el producte és el parèntesi de Poisson). És:

$$\boldsymbol{\Lambda}(H) \;=\; \boldsymbol{\alpha}(X_H) - H.$$

Aquest operador és no invertible, és a dir, existeixen "constants d'integració", que són les funcions homogènies de grau 1 a les variables $y$, com es veu fàcilment si l'escrivim amb coordenades cotangents:

$$\boldsymbol{\Lambda}(H)(x,y) \;=\; y \cdot \nabla_y H(x,y) - H(x,y).$$

Això està relacionat amb l'existència de molts simplectomorfismes amb la mateixa funció primitiva. Hem d'aconseguir que l'analiticitat de $H$ respecte al temps arribi una mica més enllà de l'1, almenys en un entorn de la secció zero. El mètode utilitzat per demostrar-ho és bàsicament el clàssic *mètode de les majorants* de Cauchy. El punt clau és aprofitar la distinció canònica que hi ha entre les variables posicions i moments.

El fet de deixar fixa la secció zero i veure què passa al seu voltant de no és tan restrictiu, i el que ens fa falta saber bàsicament és on va a parar una certa varietat Lagrangiana i com es comporten els punts d'aquesta. Una varietat Lagrangiana és una subvarietat de $\mathcal{N}$, de dimensió $d$, on s'anul·la la forma simplèctica aplicada als seus vectors tangents. Un exemple trivial ve donat precisament per la secció zero d'un fibrat cotangent, i uns teoremes de Weinstein [97, 98] diuen que qualsevol varietat Lagrangiana es comporta, localment, com la secció zero del seu fibrat cotangent. Són una generalització del teorema de Darboux. Seguint la línia del que estem explicant, les nostres varietats Lagrangianes seran exactes. Una varietat Lagrangiana és exacta si la forma d'acció $\boldsymbol{\alpha}$ sobre la varietat, que és tancada, és exacta. Les construccions anteriors permetrien construir moltes dinàmiques al voltant d'aquests tipus de varietats.

Ara sorgeix una altra pregunta:

> quines propietats tenen les varietats Lagrangianes exactes invariants per l'acció d'un simplectomorfisme exacte $F$?

Algunes propietats ja les hem pogut observar en estudiar els dos problemes anteriors. Per exemple, si $\mathcal{M} = \mathbb{R}^d$, $F = (f,g) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ és un simplectomorfisme amb $S$ com a funció primitiva i $F$ deixa fixa la secció $\{y = 0\}$ llavors obtenim que:

- $S(x,0)$ és una funció constant;
- $\forall x \in \mathbb{R}^d \;\; \dfrac{\partial S}{\partial x}(x,0) = 0, \; \dfrac{\partial S}{\partial y}(x,0) = 0.$

La primera propietat es pot generalitzar fàcilment a qualsevol varietat Lagrangiana exacta, i el que diu és que si és exacta invariant per $F$ llavors li podem assignar una quantitat conservada. La segona propietat diu particularment que si considerem $S$ com una família $x$-parametritzada de funcions, llavors per a cada $x$ el punt corresponent de la base, $(x,0)$, és un punt crític de $S(x,\cdot)$. El recíproc també és cert si la nostra aplicació és monòtona, és a dir, transversal a la foliació estàndard:

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \; \left|\frac{\partial f}{\partial y}(x,y)\right| \neq 0$$

(la simetrització d'aquesta matriu de derivades parcials rep el nom de *torsió*). Això ho generalitzarem per als grafs Lagrangians invariants, però també podríem considerar qualssevol varietats Lagrangianes utilitzant foliacions transversals adequades.

Un exemple especialment important apareix en el cas que tots els *punts crítics fibrats* siguin mínims, perquè llavors les òrbites minimitzen certa acció. Les òrbites d'un simplectomorfisme exacte satisfan un principi variacional, de la mateixa manera que les trajectòries d'un sistema mecànic clàssic satisfan el *principi de l'acció estacionària*. Els principis variacionals discrets són una eina poderosa a l'hora de demostrar l'existència de punts fixos, òrbites periòdiques, quasiperiòdiques, homoclíniques, etc. Va ser Poincaré [85] el primer que els va considerar en certs problemes de mecànica celest, i després l'han seguit molts més autors. Per exemple, han estat fonamentals per a la demostració de l'existència d'òrbites quasiperiòdiques en certes aplicacions que preserven àrea (les anomenades aplicacions *twist*). Aquestes òrbites, que minimitzen una acció (donada mitjançant una funció generatriu Lagrangiana), o bé corresponen a corbes invariants o bé a conjunts invariants Cantorians (anomenats conjunts d'Aubry-Mather) [13, 71].

Generalment, per definir principis variacionals es necessita l'existència d'una funció generatriu global que, com ja hem comentat, restringeix la topologia de l'espai de configuració i el nostre simplectomorfisme. Nosaltres hem evitat l'ús de la funció generatriu i hem utilitzat la funció primitiva, de manera que els nostres principis variacionals són vàlids per a qualselvol sistema mecànic discret (és a dir, un simplectomorfisme exacte sobre $T^*\mathcal{M}$). També hem adoptat un punt de vista radicalment diferent: en lloc d'utilitzar els principis variacionals per trobar òrbites nosaltres els utilitzem per obtenir informació d'elles mateixes. A més, en un cert sentit, són principis variacionals locals. Finalment, i en relació amb el tema dels grafs Lagrangians invariants, amb les nostres construccions podem generalitzar alguns resultats de Mather [73], Herman [40] i MacKay, Meiss i Stark [68].

Així doncs, pensem que els nostres principis variacionals són interessants per les raons següents:

- podem treballar sobre qualsevol fibrat cotangent, i són una espècie de lleis de la mecànica clàssica discreta;
- no necessitem la funció generatriu, que no sempre està definida o és difícil de computar (pensem, per exemple, en el cas que el nostre simplectomorfisme vingui donat per un flux Hamiltonià);
- es poden estendre al voltant de qualssevol varietats Lagrangianes exactes, gràcies als teoremes de Weinstein;

Per definir els principis variacionals hem procedit de la manera següent. Aquí utilitzarem coordenades cotangents $(x, y)$ o, millor, treballarem a $\mathbb{R}^d \times \mathbb{R}^d$.

1. Considerem primer dues posicions $\mathbf{x}_m, \mathbf{x}_n \in \mathbb{R}^d$ on $n > m+1$, que volem unir mitjançant un tros d'òrbita.
2. Definim llavors el conjunt de cadenes que connecten ambdós punts. Una cadena és una seqüència
$$(x_m, y_m), (x_{m+1}, y_{m+1}), \dots (x_{n-1}, y_{n-1})$$

que satisfà

- $x_m = \mathbf{x}_m$,
- $\forall i = m \div n{-}2,\ f(x_i, y_i) = x_{i+1}$,
- $f(x_{n-1}, y_{n-1}) = \mathbf{x}_n$.

3. L'acció sobre aquest conjunt no és res més que la suma

$$\boldsymbol{S}_{m,n}(x_m, y_m, x_{m+1}, y_{m+1}, \ldots, x_{n-1}, y_{n-1}) \;=\; \sum_{i=m}^{n-1} S(x_i, y_i).$$

4. Finalment obtenim que les òrbites que connecten les posicions $\mathbf{x}_m, \mathbf{x}_n$ són extremals de l'acció (definida sobre el conjunt de cadenes), i que el recíproc és cert si el nostre simplectomorfisme és monòton. Llavors té sentit dir que una òrbita és *minimitzant* si cadascun del seus segments minimitza l'acció corresponent.

Quina és la interpretació física d'aquesta construcció? Bé, considerem un sistema mecànic continu i periòdic en el temps, donat per un Hamiltonià

$$H : T^*\mathcal{M} \times \mathbb{T} \longrightarrow \mathbb{R},$$

on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Sigui $F = \varphi_{1,0}$ el seu flux a temps el període. La seva funció primitiva és, utilitzant coordenades cotangents,

$$\begin{aligned} S(x, y) &= \int_0^1 \mathbf{\Lambda}(H_t)(x(t), y(t))dt \\ &= \int_0^1 \left( y(t) \cdot \frac{\partial H}{\partial y}(x(t), y(t), t) - H(x(t), y(t), t) \right) dt, \end{aligned}$$

on $(x(t), y(t)) = \varphi_{t,0}(x, y)$ és el flux. Llavors, una cadena és una "òrbita" del nostre sistema mecànic la velocitat de la qual és sobtadament canviada cada període. Es tracta de suavitzar les punxes, i això s'aconsegueix extremitzant l'acció.

Si considerem cadenes de longitud 1, però en aquest cas imposem que els punts $(x, y)$ vagin a parar a la mateixa fibra, i l'acció és la mateixa funció primitiva, llavors el que estem buscant són punts fixos. És a dir, els punts fixos són punts crítics de la funció primitiva restringida al *conjunt transformat verticalment* $K = \{(x, y) \mid f(x, y) = x\}$. De fet, hem retrobat una construcció que ja va utilitzar Moser [79] per al cas de simplectomorfismes exactes definits al fibrat cotangent d'un tor, i que després va ser utilitzada per Arnaud [3].

Per seguir un ordre lògic, a la memòria hem descrit primer aquests principis variacionals per a punts fixos i després hem considerat els relatius a òrbites. També hem dedicat una mica de temps a la relació entre el caràcter dinàmic i l'extremal d'un punt fix, encara que ja hi ha molts resultats sobre el tema [53, 66, 3]. Nosaltres hem considerat també alguns casos degenerats, corresponents a punts fixos no monòtons.

Finalment, hem demostrat que el caràcter extremal d'una òrbita i d'un graf Lagrangià invariant és invariant sota canvis de variable a l'espai de configuració i translacions fibrades de l'espai de fase. La interpretació física d'aquest resultat és que les lleis de la mecànica discreta són independents de les coordenades de l'espai de configuració i certs "observadors privilegiats". Aquest tipus d'invariància està geomètricament connectada amb l'elecció d'una 1-forma natural a l'espai de fase, $\boldsymbol{\alpha} = y\,\mathrm{d}x$, i a la distinció concomitant entre variables posició i moment que això implica. Recordem, però, que la dinàmica dels sistemes és independent de les coordenades del nostre espai de fase. Així, per exemple, els multiplicadors de Floquet associats a una òrbita periòdica són invariants per a qualssevol canvis de variable.

## Aplicacions a la teoria KAM inversa

Les varietats Lagrangianes són interessants des d'un punt de vista de la teoria dels sistemes dinàmics perquè apareixen tot sovint quan es consideren, per exemple:

- els tors invariants de la teoria de Kolmogorov [52], Arnold [4] i Moser [78], coneguda abreujadament com *teoria KAM*, que són Lagrangians perquè la dinàmica sobre ells ve donada per rotacions ergòdiques [40, 41];

- les varietats estable i inestable d'un punt hiperbòlic, que són Lagrangianes perquè les dinàmiques corresponents colapsen al punt fix quan iterem cap a endavant i cap a enrere el nostre simplectomorfisme, respectivament.

Nosaltres considerarem més aviat el primer exemple. En relació amb el segon, hem de dir que apareixen en la teoria del trencament de separatrius, originada per Poincar 'e i desenvolupada posteriorment per Melnikov i Arnold. Només cal comentar que en el cas simplèctic no s'utilitza un vector de Melnikov sinó una funció de Melnikov (un potencial) per mesurar el trencament, i que originàriament s'utilitzava per la seva definició la funció generatriu. Va ser Easton [30] el primer que va considerar la funció primitiva en la definició d'aquest potencial, i les seves fórmules van ser generalitzades per Delshams i Ramírez-Ros [28].

Per il·lustrar les idees bàsiques de la teoria KAM considerem la ben coneguda aplicació estàndard, que és una a.p.a. tipus twist i va ser introduïda per Chirikov [24]. És:

$$\begin{cases} x' = x + y - \dfrac{K}{2\pi}\sin(2\pi x) \pmod 1 \\ \\ y' = y - \dfrac{K}{2\pi}\sin(2\pi x) \end{cases},$$

on l'espai de fase és el cilindre $\mathbb{T} \times \mathbb{R}$ que està coordenat per les variables angle-acció $(x, y)$. $K$ és un paràmetre pertorbatiu, i quan és zero la nostra aplicació esdevé integrable:

$$\begin{cases} x' = x + y \pmod 1 \\ \\ y' = y \end{cases}.$$

Això vol dir que l'espai de fase està foliat per tors (bé, corbes) invariants $\{y = y_0\}$, i la dinàmica sobre aquests ve donada per rotacions. Les *corbes invariants rotacionals*

[4], perquè envolten el cilindre, estan etiquetades per les freqüències corresponents $y_0$, i n'hi ha de dos tipus:

- $y_0 \in \mathbb{Q}$, i llavors contenen òrbites periòdiques;
- $y_0 \in \mathbb{R} \setminus \mathbb{Q}$, i contenen òrbites quasiperiòdiques que les omplen densament.

Quan $K$ és incrementada a partir de zero, la pregunta és quina quantitat de l'estructura integrable persisteix. Experimentalment veiem que la majoria d'òrbites encara semblen pertànyer a c.i.r., i això és el que precisament prediu la teoria KAM: la majoria de tors invariants persisteix si la pertorbació $K$ és prou petita. Un fet realment destacable és que la persistència d'aquestes corbes invariants depèn de quant llunyanes estan les corresponents freqüències dels nombres racionals. Aquest grau d'irracionalitat es tradueix en el que s'anomena *condició diofàntica*. Un nombre $\omega$ és diofàntic si existeixen constants $C > 0$, $\tau \geq 1$ tals que per a totes les fraccions $\frac{p}{n} \in \mathbb{Q}$

$$|n\omega - p| \quad > \quad \frac{C}{n^\tau}.$$

Els nombres diofàntics són durs d'aproximar-los per racionals. Això també està connectat amb els seus tipus de desenvolupament en fracció contínua. Així, el nombre *més irracional* és la *raó àuria*

$$\gamma \quad = \quad \frac{\sqrt{5}+1}{2} = [1; 1, 1, 1, \ldots],$$

que satisfà la condició diofàntica amb $C = \gamma^2$ i $\tau = 1$.

Ara ens podem fer la pregunta oposada: com ha de ser de gran ha la pertorbació per què ja no existeixi cap corba invariant rotacional? I el que és més: quina és l'última corba invariant? També ens podem preguntar quan es trenca una certa corba invariant, fixant-nos en la seva freqüència. El conjunt d'eines i criteris dissenyats per tal de resoldre aquests problemes s'anomena teoria KAM inversa (de l'anglès *converse KAM theory*, seguint MacKay, Meiss i Stark [68]). Al contrari que la teoria KAM, la teoria KAM inversa és no pertorbativa, i és capaç de donar condicions perquè per un cert punt de l'espai de fases no passi una corba invariant.

Mentre que hi ha molts treballs referents a la teoria KAM inversa per als simplectomorfismes en dimensió baixa ($d = 1$), no n'hi ha tants que tractin les dimensions altes ($d > 1$), entre els quals destaquem els de MacKay, Meiss i Stark [68], Herman [40, 41, 42, 43] i Tompaidis [94, 95]. L'espai de fase que es considera és el fibrat cotangent d'un tor, $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$, també anomenat *anell* o *cilindre*, l'espai recobridor del qual no és altre que $\mathbb{R}^d \times \mathbb{R}^d$. El problema principal és que, mentre que per les aplicacions que preserven l'àrea de tipus twist les corbes invariants han de ser grafs, per un teorema de Birkhoff [19], no hi ha un equivalent per a dimensions altes i ens hem de restringir a aquells tors Lagrangians que són grafs. D'altra banda, tenim les aplicacions que no són twist, o aquelles on la monotonia canvia de signe, o no és ni positiva ni negativa (en el cas $d > 1$). Un altre problema és que no hi ha un clar anàleg multidimensional del desenvolupament en fracció contínua.

[4] Abreujadament: *c.i.r.*

Podem agrupar les diferents tècniques i criteris de la teoria KAM inversa en els grups següents.

- **Criteris Lipschitzians.** Per un altre teorema de Birkhoff podem fitar el pendent que ha de tenir una c.i.r. per una a.p.a. tipus twist. Llavors es poden obtenir criteris restrictius per a la no-existència d'aquestes corbes. Per exemple, Mather [72] va trobar que per a l'aplicació estàndard no existeix cap c.i.r. si $|K| \geq \frac{4}{3}$ i després MacKay i Percival [67] ho van refinar, utilitzant l'ajuda d'un ordinador, per obtenir rigorosament una fita $|K| \geq \frac{63}{64}$, i Jungreis [48] va millorar aquests resultats rigorosos. Herman [40, 41] va demostrar similars desigualtats Lipschitzianes per als grafs Lagrangians invariants per simplectomorfismes monòtons positius, i nosaltres relacionarem els seus resultats amb els principis variacionals.

- **Criteris variacionals.** A [67], MacKay i Percival van relacionar també el seu criteri de l'encreuament de cons (de l'anglès, *cone-crossing criterion*) amb els principis variacionals d'Aubry-Mather. Després, per a dimensions altes, MacKay, Meiss i Stark [68] van implementar un mètode per detectar si per un punt de l'espai de fase és impossible que passi un graf Lagrangià, i el van interpretar també de manera geomètrica relacionant-lo amb els "pendents" dels plans Lagrangians tangents. En els dos casos es necessita que el simplectormorfisme satisfaci fortes condicions de positivitat, i que estigui llavors definit mitjançant una funció generatriu Lagrangiana. Es tracta de detectar si un determinat segment d'òrbita minimitza o no una certa acció, i això es verifica estudiant una certa matriu Hessiana. És curiós que es necessitin aquestes condicions globals per al simplectomorfisme i de fet, a l'hora d'implementar el criteri, només es necessiti detectar si una certa condició local se satisfà o no.

  Nosaltres hem seguit més aviat la línia de definicions de Herman, i hem considerat simplectormorfismes monòtons positius [40, 41]. Hi ha exemples senzills de simplectomorfismes monòtons positius que no són twist, i no estan definits per funcions generatrius Lagrangianes. Les definicions que hem donat nosaltres són locals i permeten estudiar diferents regions de l'espai de fase. Podem concloure que

  > si un graf Lagrangià invariant viu en una regió monòtona positiva, llavors és minimitzant, i les seves òrbites són minimitzants.

  A partir d'aquí, podem fer càlculs semblants als que apareixen a [68], però, insistim, amb condicions menys restrictives. En especial, podem utilitzar el que nosaltres hem anomenat la *iteració MMS*, que és un mètode implementat a [68] per tal de determinar si una certa matriu tridiagonal per blocs i simètrica és definida positiva o no. [5]

- **Criteris de tipus Greene.** A [36], Greene va proposar un criteri per detectar quan una certa c.i.r. d'una a.p.a. es trenca en augmentar el valor de la pertorbació. El seu mètode estava basat en l'estudi de l'estabilitat d'òrbites periòdiques

[5] Al final de la memòria també hem escrit un petit resum sobre matrius simètriques definides positives.

properes. Greene va descobrir que, per a l'aplicació estàndard, l'última corba invariant té freqüència $\omega = \gamma$ (el nombre auri) i que es trenca per a un valor crític $K_\gamma \simeq 0'971635406$ (de fet aquest valor va ser trobat per MacKay [63]). Ell va raonar que si un conjunt d'òrbites periòdiques, les freqüències $r_i = \frac{p_i}{n_i}$ de les quals tendeixen cap a la freqüència $\omega$,

$$\lim_{i\to\infty} r_i = \omega,$$

tenen residus entre 0 i 1 (són el·líptiques), llavors la corba invariant corresponent a $\omega$ deu existir. Aquesta *conjectura del residu* va ser demostrada per MacKay [65] i Falconini i de la Llave [32] en alguns casos. Llavors ell considerà els valors crítics $K_{r_i}$, on l'òrbita periòdica corresponent té residu 1 (que correspon a una bifurcació de doblament de període), i observà que tendien cap a $K_\omega$. A més, quan es considera el nombre auri (o, més en general, qualsevol nombre noble) i la seqüència d'aproximacions racionals és la donada pels convergents de la fracció contínua, llavors es comprova que la seqüència de valors crítics tendeix geomètricament cap al paràmetre de trencament de la corba invariant. Això està relacionat amb la varietat estable d'un punt fix d'un cert *operador de renormalització* a l'espai d'aplicacions twist que preserven àrea, tal com va estudiar numèricament MacKay [63].

Per a dimensions altes la situació no és tan clara. Primer de tot, falta un clar candidat de mètode d'aproximació de vectors irracionals que generalitzi les fraccions contínues. Nosaltres hem considerat el mètode de Jacobi-Perron, seguint Tompaidis [95]. El mateix Tompaidis va considerar un anàleg del mètode de Greene [94] i el va aplicar a un exemple 3-dimensional d'aplicació que preserva el volum [95]: l'aplicació estàndard rotacional. Un altre problema és que el comportament de renormalització és més complicat i difícil de detectar. Això ajudaria a millorar les estimacions dels valors crítics de trencament dels tors.

Nosaltres hem desenvolupat un mètode amb la mateixa filosofia, però en lloc de considerar l'estabilitat de les òrbites periòdiques hem considerat el seu caràcter extremal. Hem treballat amb simplectomorfismes monòtons positius. Des d'un punt de vista heurístic, ens hem basat en el fet que si una òrbita és minimitzant (i les que estan sobre els tors ho són, almenys en certs casos), llavors qualsevol segment d'òrbita suficientment proper és també minimitzant. Encara que les òrbites el·líptiques no són minimitzants, sí que ho són segments suficientment petits d'aquestes. Recordem que totes les òrbites d'un simplectomorfisme monòton positiu i integrable són minimitzants.

- **Criteris obstruccionals.** Considerem una a.p.a. i suposem que la varietat inestable d'alguna òrbita periòdica es talla amb la varietat estable d'una altra. Llavors no hi pot haver cap c.i.r. continguda entre ambdues òrbites. Com que aquestes interseccions *heteroclíniques* es poden calcular numèricament, ho podem utilitzar com un criteri pràctic de no-existència de c.i.r., tal com van fer Olvera i Simó [82]. Aquestes varietats també s'utilitzen per fitar les anomenades *ressonàncies*, que són conjunts de punts de l'espai fàsic que es comporten de manera semblant. La *teoria del transport* (vegeu, per exemple, [76]) estudia el moviment d'aquests conjunts, i es pregunta quant tarda un conjunt de punts a desplaçar-se d'una regió de l'espai de fase a una altra.

Quan treballem en dimensions altes no podem utilitzar les varietats estable i inestable d'òrbites periòdiques hiperbòliques, perquè no separen l'espai. El que podem fer és considerar òrbites de tipus el·líptic-hiperbòlic, amb només dues direccions hiperbòliques, i les seves *varietats central-estable i central-inestable*. Aquestes varietats són, doncs, de codimensió 1, i nosaltres pensem que poden ser útils tant per explicar el mecanisme del trencament dels tors invariants com per estudiar el transport. Són l'esquelet del nostre sistema dinàmic. Com a exemple, hem considerat una aplicació 4-dimensional, l'aplicació de Froeschlé, i hem estudiat la zona de ressonància associada a l'origen, que és un punt fix el·líptic. Veurem que sembla que aquesta regió estigui fitada per les varietats central-estable i central-inestable dels seus companys el·líptic-hiperbòlics. Per això, hem hagut de desenvolupar aquestes varietats en sèries de potències (de 3 variables) fins a un ordre elevat, perquè són difícils de globalitzar. La visualització d'aquestes varietats es pot fer intersecant-les amb objectes de dimensió més petita, per exemple plans, pels quals les interseccions són, genèricament, corbes. A causa de les interseccions heteroclíniques entre les diferents varietats, hi haurà una estructura molt complicada de plegaments d'aquestes, cosa que donarà lloc a ressonàncies més petites.

En certa manera hem unificat els criteris Lipschitzians-variacionals amb el criteris de Greene. Els primers són equivalents [67, 68], i, a més, permeten fer demostracions rigoroses amb l'ajut de l'ordinador, emprant l'anàlisi intervalar (encara que nosaltres no ho hem fet). Els segons no permeten fer demostracions rigoroses, però donen estimacions molt bones dels valors crítics dels trencaments dels tors. Com ja hem comentat, hem implementat un criteri de tipus Greene però amb caràcter variacional. La unificació d'aquests amb els criteris obstruccionals vindria donada per un estudi complet de la relació dinàmica-extremalitat.

Per il·lustrar totes aquestes idees hem considerat exemples 2D, 3D i 4D. En tots ells hem aplicat primer els criteris variacionals per descartar zones de l'espai de fase que no continguin tors invariants (tipus graf), seguint la línia de [68]. Hem considerat exemples de diferents tipus: twists, monòtons positius no twist, que canvien el signe de la monotonia, etc. Els exemples 2D que hem considerat són de la família de l'aplicació estàndard, els 4D de la família de l'aplicació de Froeschlé i el 3D és l'aplicació estàndard rotacional. Així, per exemple, si considerem una a.p.a. no monòtona, llavors hem observat que les c.i.r. que travessen les corbes no monòtones (on falla la monotonia) no semblen grafs i tenen plegaments. A més, sembla que aquestes siguin més difícils de trencar. Pensem que la majoria d'aquestes corbes són, de fet, definides (positives o negatives), però fa falta considerar coordenades adients.

Per provar el nostre mètode de Greene variacional hem considerat primer l'aplicació estàndard, perquè ha estat molt estudiada. També hem observat els típics comportaments associats a la renormalització associada al trencament de les corbes nobles. També hem considerat una altra de la seva família: l'aplicació estàndard exponencial, que és monòtona positiva però no twist. Per als exemples 4D hem considerat l'aplicació de Froeschlé

$$\begin{cases} {y'}_1 = y_1 - \dfrac{K_1}{2\pi}\sin(2\pi x_1) - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \\ \\ {y'}_2 = y_2 - \dfrac{K_2}{2\pi}\sin(2\pi x_2) - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \\ \\ {x'}_1 = x_1 + {y'}_1 \pmod 1 \\ \\ {x'}_2 = x_2 + {y'}_2 \pmod 1 \end{cases}$$

i dos vectors de rotació diferents: un parell quadràtic $(\sqrt{2}-1, \sqrt{3}-1)$ i un vector auri (per l'algorisme de Jacobi-Perron). Depenent dels valors dels paràmetres $K_1$ i $K_2$, i considerant el paràmetre $\lambda$ com a paràmetre pertorbatiu, hem observat també diferents tipus de trencaments. Per fer-ho, hem calculat òrbites periòdiques de períodes grans i les hem continuat respecte a $\lambda$. El mètode per calcular les òrbites periòdiques és una espècie de tir paral·lel, perquè permet calcular-les amb més cura. La pregunta que ens hem formulat és:

quins són els equivalents en dimensions altes dels conjunts d'Aubry-Mather?

En el cas 2D aquests conjunts tenen dimensió de Hausdorff zero (en el cas hiperbòlic), tal com va demostrar MacKay [64], però aquí semblen tenir o bé dimensió zero o bé dimensió 1. En aquest segon cas també hem advertit ressonàncies associades a òrbites periòdiques de períodes baixos, que donen lloc a la possibilitat de considerar els criteris obstruccionals per explicar trencament del tors. Finalment, hem considerat també l'exemple 3D de Tompaidis i el vector auri. L'aplicació estàndard rotacional és una aplicació que depèn de dos paràmetres $K$ i $\lambda$ i una rotació $\omega$, està definida sobre el cilindre $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ coordenat per $(x, y, \theta)$ i és

$$\begin{cases} y' = y - \dfrac{1}{(2\pi)}\sin(2\pi x)(K + \lambda\cos(2\pi\theta)) \\ \\ x' = x + y' \pmod 1 \\ \\ \theta' = \theta + \omega \pmod 1 \end{cases} .$$

Per això, hem hagut de desenvolupar primer una teoria variacional per als simplectomorfismes exactes *no autònoms*, que són uns simplectomorfirmes que depenen d'una variable *temporal* (de l'anglès, *exact symplectic skew-products*). Això no ho hem fet amb tot detall, perquè és similar a la teoria variacional ja construïda.

Hem de remarcar que existeixen altres criteris per estudiar el trencament dels tors invariants, com aquells basats en l'estimació dels radis de convergència dels desenvolupaments en sèries de Fourier dels tors KAM (com va fer Percival [84]) o l'anàlisi de freqüències de Laskar [60].

## Esquema general de la memòria

Hem dividit la memòria en quatre parts ben diferenciades.

**PART I. Geometria exactosimplèctica** (introducció dels problemes)

Aquesta part conté les eines bàsiques de la geometria simplèctica i planteja els quatre problemes que tractarem al llarg de la memòria:

1. el problema de determinació,
2. el problema d'interpolació,
3. el problema variacional,
4. el problema del trencament de tors invariants.

**PART II. Sobre la varietat simplèctica estàndard** (part analítica)

Aquí hem treballat a $\mathbb{R}^d \times \mathbb{R}^d$, és a dir, hem fet un tractament coordenat dels resultats. Primer relacionem les funcions generatrius amb la funció primitiva i després resolem formalment el problema de determinació. Després tractem diferents principis variacionals: per als punts fixos, per a les òrbites periòdiques i per als segments orbitals. La seva invariància respecte a certs tipus de transformacions de l'espai de fase és demostrada, donant una interpretació física. Finalment donem les propietats bàsiques dels grafs Lagrangians invariants, especialment aquella que diu que les òrbites sobre un graf minimitzant són minimitzants.

**PART III. Sobre el fibrat cotangent** (part geomètrica)

Els tres primers capítols segueixen més o menys la línia dels tres precedents, amb la diferència fonamental que aquí considerem qualsevol fibrat cotangent. Fem, llavors, un tractament intrínsec. El quart capítol d'aquesta part està dedicat a resoldre el problema d'interpolació en el cas analític.

**PART IV. Aplicacions** (part numèrica)

Aquesta última part tracta de les aplicacions a la teoria KAM inversa, o del trencament dels tors invariants. Primer donem una llista d'exemples que més endavant utilitzarem. Després generalitzem la teoria KAM inversa de [68] i la relacionem amb la teoria Lipschitziana de Birkhoff i Herman [40, 41]. Llavors implementem el nostre criteri de Greene variacional i l'apliquem a diferents exemples. També estudiem els equivalents dels conjunts d'Aubry-Mather en dimensió alta (bé, = 4). Després apliquem aquesta metodologia a l'aplicació estàndard rotacional (3D), indicant abans la teoria necessària. Llavors donem algunes idees de com generalitzar els criteris obstruccionals a dimensions altes, hi ho mostrem amb un petit exemple. Finalment retrobem algunes formes normals de Birkhoff utilitzant la nostra metodologia basada en la funció primitiva i expliquem una mica com es podria considerar la nostra teoria tenint en compte foliacions Lagrangianes arbitràries.

## Aportacions més rellevants

Per acabar, parlarem de les aportacions més rellevants d'aquesta tesi, i quina és la feina que encara ens queda per fer.

- Primer de tot, pensem que l'aportació més important és l'ús sistemàtic de la funció primitiva d'un simplectomorfisme exacte. Les eines analítiques, geomètriques i numèriques emprades al llarg de la tesi giren al voltant de la funció primitiva i les seves propietats. Aquestes provenen de l'estructura de l'espai de fase, donada per una forma d'acció.

- L'ús de la funció primitiva ens ha perm
'es introduir principis variacionals en contextos més generals i amb hipòtesis més febles de les que usualment s'exigeixen, com l'existència d'una funció generatriu global.

- Així, hem establert una espècie de principis variacionals de la mecànica discreta (estudi dels simplectomorfismes exactes definits sobre un fibrat cotangent). La geometrització d'aquests principis variacionals ve donada per la forma de Liouville i la foliació estàndard associada a aquesta.

- Hem donat també una interpretació variacional dels grafs Lagrangians invariants. Els seus punts són crítics fibra a fibra d'una certa funció relacionada amb la funció primitiva i amb el propi graf. Això ens permet també classificar variacionalment els diferents grafs Lagrangians invariants. Trobarem resultats per als grafs que són minimitzants (o maximitzants), en particular, que les seves òrbites són minimitzants (o maximitzants). Això generalitza alguns resultats de Mather [73], Herman [40, 41] i MacKay, Meiss i Stark [68].

- Per donar condicions d'existència de tors invariants per simplectomorfismes definits a $\mathbb{T}^d \times \mathbb{R}^d$ hem utilitzat aquests mètodes variacionals. Llavors, hem relacionat les línies d'investigació de [68] i [40, 41], amb la diferència principal que nosaltres hem utilitzat la funció primitiva en lloc de la funció generatriu, que no sempre existeix.

- El tractament *local* dels nostres principis variacionals ens permet estudiar les regions de l'espai de fase on se satisfan certes condicions de positivitat. Ho podem resumir dient que

  > si un graf Lagrangià invariant viu en una regió monòtona positiva, llavors és minimitzant, i les seves òrbites són minimitzants.

  A partir d'aquí, podem fer càlculs semblants als que apareixen a [68], però, insistim, amb condicions menys restrictives. Aquest teorema és important perquè ens permet donar condicions suficients per saber si un graf Lagrangià és minimitzant, sense tenir la seva expressió explícita, és clar.

- Un altre punt és el desenvolupament de criteris de tipus Greene per detectar acuradament el trencament dels tors, però nosaltres hem aprofitat les propietats extremals i no les dinàmiques de les òrbites periòdiques. El test del mètode amb l'aplicació estàndard ha donat bons resultats, i apareixen també comportaments de renormalització. Hem aplicat també aquests mètodes a aplicacions 4D.

- Per "veure" el trencament d'aquest tors hem hagut de calcular òrbites periòdiques de periodes grans (de l'ordre de $10^5$). Per això hem utilitzat un mètode de tir paral·lel. Els resultats concorden amb els obtinguts amb el nostre mètode variacional. A més, hem detectat diferents tipus de trencament dels tors, és a dir, de formació de can-tors (de l'anglès, *cantori*). Aquests tipus de fenòmens haurien d'ésser estudiats en el futur.

- Hem explicat també un possible mecanisme de trencament d'aquests tors, associat a les interseccions heteroclíniques de varietats invariants de codimensió 1 (varietats central-estable i central-inestable d'òrbites de tipus el·líptic-hiperbòlic). Això ja havia estat ben estudiat en dimensió baixa, i pensem que pot ser una bona explicació del fenomen en dimensions altes, així com també del transport. Encara que això no ho hem desenvolupat completament, pensem que l'exemple que hem introduït és prou instructiu.

- Finalment, hem aplicat aquesta metodologia a l'estudi de tors invariants per simplectomorfismes *quasiperiòdics* (és a dir, no autònoms on la variable temporal és un angle que es mou quasiperiòdicament). L'exemple 3D que hem considerat va ser tractat ja per Tompaidis [95]. Els resultats que hem obtingut en aplicar el nostre criteri variacional de Greene per estudiar el trencament d'un tor auri 2D concorden bastant (amb la precisió que podem) amb els que va trobar ell.

- Una part important de la tesi està dedicada als exemples. Com a models de simplectomorfismes que no són twist, hem introduït les aplicacions exponencial estàndard, quadràtica estàndard i trigonomètrica estàndard. Com a test dels nostres mètodes hem utilitzat la ben coneguda aplicació estàndard. Els acoblaments entre aquestes aplicacions ens han donat una gran varietat d'exemples 4D, similars a l'aplicació de Froeschlé. Com a exemple de simplectomorfisme quasiperiòdic hem considerat l'aplicació estàndard rotacional, però també podríem haver considerat exemples similars als anteriors.

Queden, és clar, molts problemes per resoldre. El primer correspon al cas en què tenim una aplicació que preserva l'àrea, la torsió canvia de signe, i la corba invariant que estem considerant passa per regions de monotonia de signe diferent. Aquesta corba pot tenir plecs (no ser transversal a la foliació estàndard) i sembla que sigui més difícil de trencar que les que són *positives* o *negatives*. Possiblement, la majoria d'aquests tipus de corbes tinguin signe definit en coordenades adients. El segon es presenta quan treballem en dimensions altes i, encara que la torsió sigui no degenerada, és indefinida. En aquest aspecte, Herman té alguns resultats [43]. Nosaltres sabem com són les òrbites dels tors, des d'un punt de vista extremal, una vegada hem fet un pas de la forma normal de Birkhoff, però, és clar, això no és suficient. S'hauria d'estudiar quins són els índexs de les seves òrbites. A més, ho hauríem de lligar tot amb la dinàmica al voltant d'aquests tors. D'altra banda, tenim l'aplicació d'operadors de renormalització en dimensions altes, associats a aproximacions racionals multidimensionals. Ja hem dit que també seria molt interessant explicar el fenomen del trencament de tors invariants en termes geometricoobstruccionals, i no només els relacionats amb les òrbites periòdiques, sinó també amb altres objectes com són els tors isotròpics invariants (que, en el cas 4D, corresponen a corbes invariants).

Respecte a l'existència de punts fixos, pensem que seria interessant treballar més aquest aspecte, perquè es poden considerar altres espais de configuració: $\mathbb{S}^2$, SO(3), etc. Per exemple, podrien ser utilitzats per comptar òrbites periòdiques de sistemes mecànics periòdics en el temps, mitjantçant la teoria de Morse i potser poden ser utilitzats per detectar les bifurcacions d'aquests punts fixos a partir dels canvis geomètrics al conjunt transformat verticalment. De totes maneres, aquests resultats d'existència donats per implicacions topològiques no són constructius, i el problema essencial és de caràcter local.

Continuem parlant ara d'altres aportacions.

- Un altre punt important és l'enfrontament funció primitiva/funció generatriu. Ja hem dit que no sempre és possible obtenir la funció generatriu, i això pot ser un problema a l'hora d'estudiar la dinàmica al voltant de tors que no siguin definits. Nosaltres hem avançat una mica en aquesta direcció. Per obtenir la dinàmica al voltant d'un tor invariant l'únic que ens fa falta és la dinàmica sobre aquest (que, en principi, pot ser qualsevol, però si és un tor KAM ha de ser una translació ergòdica) i la funció primitiva.

- Una altra manera d'obtenir la dinàmica és interpolant-la per un flux Hamiltonià. De fet, això ho hem utilitzat per demostrar l'analiticitat de les sèries. És important el fet que les demostracions són constructives i que les recurrències poden ser implementades en un ordinador. A més, podem aconseguir que el Hamiltonià interpolador sigui periòdic en el temps, mitjançant el mètode de mitjanes de Pronin i Treschev [86].

- A les demostracions ha estat fonamental l'aprofitament de l'estructura geomètrica de l'espai de fase. A part de la forma de Liouville i la seva foliació associada (l'estàndard), han estat clau les propietats del que nosaltres hem anomenat *derivada de Liouville*, que a l'àmbit de la mecànica analítica es coneix com l'acció elemental (d'un Hamiltonià). Aquest estatus especial que li hem volgut donar prové precisament del fet que nosaltres hem considerat l'acció elemental com un operador a l'espai de funcions i hem vist que és una derivació. A més, aquest operador pot ser associat a qualsevol varietat simplèctica exacta (no fa falta que sigui un fibrat cotangent), o millor, a qualsevol forma d'acció.

Aquestes construccions poden esdevenir interessants perquè permeten *inventar* moltes dinàmiques al voltant de varietats Lagrangianes invariants. Per exemple, si la varietat bàsica és un tor necessitem programar un manipulador algebraic de sèries de Fourier-Taylor. Podem posar qualsevol dinàmica sobre el tor, com una translació ergòdica, un difeomorfisme d'Anosov, etc. També estem treballant en això (es poden aconseguir exemplets fàcils si el tor té dimensió 1). Seria interessant aplicar-ho a l'estudi de tors indefinits, i veure els *canals d'escapament* que va trobar Herman [43]. Ja hem dit que hem demostrat l'analiticitat de les solucions del problema de determinació i d'interpolació, però el cas diferenciable resta obert (cf. [16]).

Des d'un punt de vista geomètric, l'objete important en la nostra teoria és la forma de Liouville, que precisament s'anul·la sobre la secció zero i sobre els vectors tangents a la foliació estàndard del fibrat cotangent. Aquesta foliació és transversal a la secció zero. Suposem que tot això es pot generalitzar mitjançant l'ús de foliacions Lagrangianes arbitràries, transversals a les nostres varietats invariants Lagrangianes. Un altre possible camí és considerar varietats Lagrangianes sobre el nostre fibrat cotangent que estiguin definides per les anomenades *famílies de Morse* o *funcions de fase*, que són una espècie de funcions generatrius que tenen uns paràmetres addicionals que permeten que les varietats es pleguin (no siguin transversals a la foliació estàndard). Hem pogut generalitzar a aquest context alguns resultats relacionats amb la caracterització dels grafs Lagrangians invariants, però encara no hem trobat la manera de desenvolupar-ho. També hem estés alguns resultats al cas $\sigma$-simplèctic (és a dir, quan $F^*\boldsymbol{\omega} = \sigma\boldsymbol{\omega}$, on $\sigma \in \mathbb{R}$ o, en el cas complex, $\sigma \in \mathbb{C}$) i, en particular, al cas antisimplèctic, que correspon a $\sigma = -1$ (cf. [23]).

Encara que la nostra teoria l'apliquem principalment al voltant de qualsevol secció zero d'un fibrat cotangent, moralment ho fem al voltant de qualsevol varietat Lagrangiana exacta. Per exemple, al voltant d'un tor Lagrangià, o d'un tros de la varietat estable d'un punt fix hiperbòlic, o al voltant d'un tros de varietat estable d'un tor hiperbòlic de dimensió baixa (aquests tipus de varietats s'utilitzen per explicar el fenomen conegut com a *difusió d'Arnold*). Utilitzant les nostres construccions, hem retrobat formes normals per a aquests exemples.

Finalment, pensem que aquest treball pot ser interessant pel conjunt de tècniques geomètriques, analítiques i numèriques que hem estudiat i relacionat, a les quals hem intentat donar una certa estructura.

# Agraïments

Abans de continuar amb el desenvolupament de la memòria m'agradaria recordar totes les persones que d'alguna manera o una altra m'han ajudat.

Primer de tot, agrair la inestimable ajuda del meu professor Carles Simó, que em va introduir en aquesta àrea de recerca i que amb el seu encoratjament, i, sobretot, amb la seva paciència, ha fet possible aquest treball. He abocat molts dels seus ensenyaments en aquest treball, els quals no apareixen a cap llibre o article, sinó a les seves classes, seminaris, xerrades, etc. També agraeixo al Departament de Matemàtica Aplicada i Anàlisi de la Universitat de Barcelona els mitjans, no només materials sinó també humans, que ha posat a la meva disposició. En aquest aspecte, dono les gràcies especialment als meus companys de despatx, i amics, Gerard Albà, Miquel Àngel Andreu, Inma Baldomà, Xavi Tolsa i Joan Vidal, que han sabut aguantar les meves manies i per la gran quantitat d'estones agradables que hem passat tots plegats. Agraeixo també la paciència que ha tingut Pau Martín a ajudar-me a corregir la tesi, i els coneixements en informàtica de José María Mondelo. També vull donar les gràcies a la nostra secretària del departament, la Nati Civil, per la seva amabilitat a resoldre sempre els meus futils problemes burocràtics. Un record especial també pel que era membre d'aquest departament, August Palanques, amb qui vaig compartir durant alguns anys l'ensenyança d'assignatures relacionades amb la mecànica analítica i que, malauradament, no està ja entre nosaltres.

Agraeixo l'amabilitat mostrada pel professor Robert MacKay en acollir-me al seu departament. El mes i mig llarg que vaig passar al Departament de Matemàtica Aplicada i Física Teòrica de la Universitat de Cambridge em va ajudar moltíssim al desenvolupament d'aquest treball. Els seus articles han estat també d'una inestimable ajuda. Per aquest motiu també dono les gràcies al professor Michael R. Herman, encara que no el conec personalment. Em falta molt per aprendre dels seus articles. Així mateix agraeixo al professor Rafael de la Llave els seus valuosos comentaris. Això es fa extensiu també al gran nombre de persones que, a través dels seus articles, xerrades, comentaris, etc. m'han ajudat en algun moment. En especial al grup de Sistemes Dinàmics de la Universitat de Barcelona i de la Universitat Politècnica de Catalunya.

Finalment, dono les gràcies als meus amics i a la meva família, que en els moments de decaïment, cansament, histèria, etc. m'han donat suport adequadament. Especialment a la dona que comparteix la meva vida, la Cristina, que m'ha ajudat i animat en tot moment, a la qual demano disculpes per les poques estones que l'he pogut dedicar. Un record molt especial pel meu pare, el qual mai podrà llegir aquestes línies, però que segur que allà on estigui estarà orgullós de mi.

# Preface

*Since the construction of the entire universe is absolutely perfect and is due to a Creator with infinite knowledge, nothing exist in the world which does not exhibit some property of maximum or minimum. Therefore, there cannot be any doubt whatsoever about the possibility that all the effects are determined by their final aims with the help of the maxima method, in the same way in which they are also determined by the initial causes.*

## The Geometry of Nature

The fundamental laws of Nature, from classical mechanics, geometric optics, gravity, electromagnetism to, even, quantum mechanics, seem to be Hamiltonian. Maupertuis explained it by saying that, assuming the universe had a perfect Creator then it must be the best possible universe, so everything should be governed by a *variational principle*. Although he said this before Hamilton formulated his dynamics, it is well known that the variational and Hamiltonian principles are quite related. As R.S. MacKay says [69], all of this is a bit mysterious.

Since the language of *Hamiltonian Mechanics* is the calculus of differential forms and vector fields on smooth manifolds, the basic formulation of this calculus is like 'grammatical rules' [96]. [1] A pleasant consequence is the possibility of avoiding the messy calculations so usual in analytical mechanics. In fact, the first example about this formalism appeared in a J.L. Lagrange's work [58] on celestial mechanics in 1808. He wrote the equations of motion for the orbital elements $z = (z_1, \ldots, z_6)$ of a planet, under the effect of perturbations, in the form

$$\frac{\partial H}{\partial z_i} = \sum_{j=1}^{6} a_{ij}(z) \frac{dz_j}{dt},$$

where $(a_{i,j})_{i,j=1\div 6}$ is a skew-symmetric matrix, and he showed that a suitable change of variables put these equations in the form now known as Hamilton's equations.

[1] A small summary about differential geometry appears at the end of this thesis.

So then, as A. Weinstein said [98], the Hamiltonian formalism plays the role in mathematics of a language which can facilitate communication between geometry and analysis. In fact, the geometrization of this language is called *symplectic geometry*, which has become an important branch of mathematics. The word *symplectic* was invented by H. Weil [99], who substituted Greek for Latin roots in the word *complex* to obtain a term which would describe a group related to line complexes but which would not be confused with complex numbers. Next citation is also owed to H. Weil, and it reflects that we are saying:

> *Inside a mathematician are fighting the devil of abstract algebra and the angel of geometry.*

The Hamiltonian/symplectic formalism has impregnated other theories, which were far enough as, for instance, the theory of representations of Lie groups, the theory of local solvability of linear differential operators, the theory of a canonical operator, and others. From a philosophical point of view, it seems that *all* can be symplectified. Although we shall not deal with these subjects, they show the importance of the study of symplectic geometry inside mathematics.

We can summarize these ideas by saying that *God is a geometer* and the geometry of the world is symplectic.

## The structure of phase space

Turning to classical mechanics [5, 1, 61], it is a good idea to describe the states of the systems with coordinates $z = (x, y)$, where $x = (x_1, \ldots, x_d)$ are the local coordinates on a manifold $\mathcal{M}$ (the *configurations space*) and which describe the *positions* of the points, and $y = (y_1, \ldots, y_d)$ are the corresponding *momentum*, which are covectors (1-forms) on such a manifold. That is to say, $(x, y)$ are the local coordinates of the cotangent bundle $\mathcal{N} = T^*\mathcal{M}$ of $\mathcal{M}$, the *phase space* of our system. $d$ is the number of degrees of freedom. This is a heritage of Newton's laws of motion, which particularly means that if we want to determine the motion of a system of particles then we need their positions and velocities in a certain time (and their interactions, of course). So then, the structure of the phase space that we are going to describe would be very different if we also need the initial accelerations in order to determine the motion.

A dynamical system is given by a vector field on the phase space $\mathcal{N}$, that encodes the infinitesimal evolution of any quantity defined on it. That is, if $X \in \mathcal{X}(\mathcal{N})$ is a vector field and $F \in C^\infty(\mathcal{N})$ is any function, then the infinitesimal change $\dot{F}$ of $F$ along the trajectories of $X$ (or orbital derivative) is given by

$$\dot{F} \;=\; X(F).$$

This is the intrinsic version of the system of ordinary differential equations

$$\begin{cases} \dot{x}_i = f_i(x, y) \\ \dot{y}_i = g_i(x, y) \end{cases},$$

where $i = 1 \div d$, and

$$X = \sum_{i=1}^{d} \left( f_i(x,y)\frac{\partial}{\partial x_i} + g_i(x,y)\frac{\partial}{\partial y_i} \right).$$

Then, we write

$$X(F) = \sum_{i=1}^{d} \left( f_i(x,y)\frac{\partial F}{\partial x_i} + g_i(x,y)\frac{\partial F}{\partial y_i} \right).$$

A fundamental axiom in the description of physical systems, that we could call the *Energy paradigm* [47], is the following:

> *For every physical system there is a function defined on its space of states, called the energy or Hamiltonian of the system, containing all its dynamical information.*

So then, if $\mathcal{N}$ models the state space of a family of dynamical systems, then, there is an assignment to any function $H$ on $\mathcal{N}$ of a vector field $X_H$ describing a dynamical system. In Hamiltonian mechanics, this assignment is geometrically given by a *symplectic* 2*-form* on $\mathcal{N}$, that is, a 2-form $\boldsymbol{\omega}$ which is closed ($\mathrm{d}\boldsymbol{\omega} = 0$) and non degenerate (as a 2-form on each point). Then, the pair $(\mathcal{N}, \boldsymbol{\omega})$ is called a *symplectic manifold.* The first condition is given in order to join the different non degenerate 2-forms of the different points. The non-degeneracy condition implies that our manifold has even dimension and let us to characterize $X_H$ by means of

$$\boldsymbol{\omega}(X_H, Y) = -\mathrm{d}H(Y),$$

where $Y \in \mathcal{X}(\mathcal{N})$ is any vector field. $X_H$ is called the *Hamiltonian vector field* associated to the *Hamiltonian function* $H$. Its flow preserves the symplectic structure.

If we want to compare two Hamiltonian flows given by the corresponding Hamiltonians $H_1$ and $H_2$, we can use the Lie bracket of the corresponding vector fields. Since we have an additional structure, we can translate it to a bracket applied to the Hamiltonians. It is the Poisson bracket:

$$\{H_1, H_2\} = \boldsymbol{\omega}(X_{H_1}, X_{H_2}) = -\mathrm{d}H_1(X_{H_2}) = \mathrm{d}H_1(X_{H_2}),$$

that satisfies

$$X_{\{H_1,H_2\}} = [X_{H_1}, X_{H_2}] .$$

The Lie algebra structure of the set of vector fields is then inherited by the set of functions. We point out that it was also Lagrange the first who used the *Poisson bracket.*

Using suitable coordinates, called *symplectic coordinates*, we write the symplectic 2-form as

$$\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^{d} \mathrm{d}y_i \wedge \mathrm{d}x_i.$$

An outstanding fact is that all the symplectic forms can be written locally in this way, thanks to Darboux's theorem. This is an essential difference between symplectic geometry and Riemannian geometry. Hamilton's equations are only the translation to these coordinates of the vector field $X_H$:

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial y_i}, \\ \\ \dot{y}_i = -\dfrac{\partial H}{\partial x_i}. \end{cases}$$

The Poisson bracket is written as

$$\{H_1, H_2\} \;=\; \sum_{i=1}^{d} \left( \frac{\partial H_1}{\partial y_i} \cdot \frac{\partial H_2}{\partial x_i} - \frac{\partial H_1}{\partial x_i} \cdot \frac{\partial H_2}{\partial y_i} \right).$$

The basic structural unit of Hamiltonian mechanics is a 1-form $\boldsymbol{\alpha} \in \Omega^1(T^*\mathcal{M})$ on the phase space $\mathcal{N} = T^*\mathcal{M}$, uniquely characterized by

$$\forall \rho \in \Omega^1(\mathcal{M}) \quad \rho^*\boldsymbol{\alpha} = \rho.$$

This natural 1-form is known as the *Liouville form* and its differential $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$ is the canonical symplectic form on the cotangent bundle. In cotangent coordinates they are given by

$$\boldsymbol{\omega} \;=\; \mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^{d} \mathrm{d}y_i \wedge \mathrm{d}x_i$$

and

$$\boldsymbol{\alpha} \;=\; y\,\mathrm{d}x = \sum_{i=1}^{d} y_i \mathrm{d}x_i.$$

So then, an important case in symplectic geometry corresponds to the fact that the symplectic form be exact, that is, there is a 1-form $\boldsymbol{\alpha}$ called *action form* satisfying $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$. We only shall consider this case. In fact, in this introduction our phase space is $\mathcal{N} = T^*\mathcal{M}$, although many definitions are useful in more cases.

## Exact symplectomorphisms

We study a discrete version of mechanics. That is, instead of working with flows (given by vector fields), we shall consider diffeomorphisms. In fact, they are quite related, via the Poincaré section. For instance, if our vector field $X = X(z,t)$ is $T$-periodic in time and its flow is $\varphi_{t,t_0}$, then we can consider the map $F = \varphi_{T,0}$, that is the Poincarè map associated to the section $\Sigma = \{(z,0) \in \mathcal{N} \times \mathbb{T}\}$, where we have extend the phase space to $\mathcal{N} \times \mathbb{T}$ and $\mathbb{T} = \mathbb{R}/(T\mathbb{Z})$.

Since our vector fields are Hamiltonian then their flows preserve the symplectic structure. In general, a diffeomorphism which preserves such structure is called *symplectomorphism*. This term was introduced by Souriau, and corresponds to *canonical transformation*, used in analytical mechanics. Therefore, a symplectomorphism $F : \mathcal{N} \to \mathcal{N}$ is a diffeomorphism that satisfies

$$F^*\boldsymbol{\omega} = \boldsymbol{\omega}.$$

Since our symplectic structure is exact, and the primitive 1-form is $\boldsymbol{\alpha}$, then the 1-form $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}$ is closed:

$$\begin{aligned} 0 &= F^*\mathrm{d}\boldsymbol{\alpha} - \mathrm{d}\boldsymbol{\alpha} \\ &= \mathrm{d}(F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}). \end{aligned}$$

In particular, if this 1-form is exact we shall say that our symplectomorphism is exact, and then there is a function $S : \mathcal{N} \to \mathbb{R}$ satisfying the *exactness equation*

$$F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S.$$

It is called the *primitive function* of $F$ and, of course, it is defined up to constants.

A curious fact is that many authors refer to that function as the *generating function* of $F$, but really this function does *not* generate $F$! In fact, it generates a family of symplectomorphisms. We can say briefly that

$S$ determines $F$ up to diffeomorphisms on the basis.

The reason is that any diffeomorphism on $\mathcal{M}$ can be *lifted* to an exact symplectomorphism on $T^*\mathcal{M}$, and the corresponding primitive function is zero. By this reason, we have followed the terminology used in [7]. We wonder about the nature and the properties of the primitive function, and what kind of information we can get from it.

Although Lagrangian generating functions can be introduced for exact symplectomorphism, its existence restricts

- the kind of symplectomorphisms, which must be transversal to the standard foliation of the cotangent bundle;
- the topology of our phase space, because the configuration space must be diffeomorphic to $\mathbb{R}^d$.

Anyway, although $\mathcal{M}$ is not $\mathbb{R}^d$, we can consider local or many-valued generating functions, but many results ask for its global existence. We shall not take this point of view and we shall work with the primitive function. Recall that if, for instance, $\mathcal{M} = \mathbb{T}^d$, we can consider its universal covering, $\tilde{\mathcal{M}} = \mathbb{R}^d$.

# Some questions related with symplectic dynamics

Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, and $S : T^*\mathcal{M} \to \mathbb{R}$ be its primitive function. We shall use cotangent coordinates $(x, y)$ and $F$ is given, then, by

$$\left\{ \begin{array}{l} \bar{x} = f(x, y) \\ \bar{y} = g(x, y) \end{array} \right. ,$$

(although we can define the basic component by $f = q \circ F$, where $q : T^*\mathcal{M} \to \mathcal{M}$ is the projection).

Since the primitive function does not determine our symplectomorphism, the question we can ask ourselves is:

> what additional information do we need in order to obtain $F$ from $S$?

We have called this question the *determination problem*, and it is related to the *interpolation problem*, that we can summarize by:

> given a symplectomorphism $F$, can we get a time-dependent Hamiltonian $H = H(z,t)$ whose flow interpolate $F$, that is $\varphi_{1,0} = F$?

This question was studied by Moser [77] to prove the convergence of the expansions in the Birkhoff normal form [19] for an *area preserving map* [2] around a hyperbolic fixed point. Later, other authors worry about different aspects of the problem, as Douady [29], Conley and Zehnder [26], Kuksin [55], Kuksin i Pöschel [56]. Lastly Pronin and Treschev [86], working on analytic set up and with compact manifolds, proved constructively that if our symplectomorphism can be interpolated then we can get the Hamiltonian be periodic in time.

We shall consider the two problems in the case that our symplectomorphism fixes the zero-section and we know the dynamics on it. We shall also take a constructive point of view. The process will be:

- to construct formally our symplectomorphism from the dynamics on the zero-section and the primitive function;
- instead of proving directly the analyticity of the expansion, we shall find constructively a Hamiltonian that interpolates it, and it is analytic in a neighborhood of the zero-section and respect to a big enough time.

To construct the Hamiltonian we use a *homotopy method*, and we obtain a certain evolutionary partial differential equation, which is not lineal. In this equation it appears what in analytic mechanics is known as the *elementary action* of a Hamiltonian, and we see is a derivation in the Lie algebra of functions (endowed with *the Poisson bracket*). This derivation is

$$\mathbf{\Lambda}(H) \;=\; \boldsymbol{\alpha}(X_H) - H.$$

This operator is not invertible, and the 'integration constants' are the homogeneous functions of degree 1 in the $y$ variables, which is easily proved by mean of cotangent coordinates:

$$\mathbf{\Lambda}(H)(x,y) \;=\; y \cdot \nabla_y H(x,y) - H(x,y).$$

This is related to the existence of many exact symplectomorphisms with the same primitive function. Recall that we must get the analyticity of $H$ with respect to time

[2] in short *a.p.m.*

be just a little bit more than 1, at least in a small neighborhood of the zero-section. The method used is the classical method of majorants due to Cauchy. The key point is to take account of the canonical distinction between position and momentum variables.

Leaving the zero-section fixed and see what happens around it is not so restrictive, and the basic fact is where a certain Lagrangian manifold goes. A Lagrangian manifold is a $d$-submanifold of $\mathcal{N}$ such that the symplectic form vanish on its tangent vectors. A straightforward example is given by the zero-section of a cotangent bundle, and a Weinstein's theorem [97, 98] says that this is in fact the universal model of a Lagrangian manifold. Our Lagrangian manifolds will be exact, that is, the action form on such manifolds, which that is, in fact, exact. The previous constructions let us generate many dynamics around this kind of manifolds.

Now, we have another question:

> What are the properties of the exact Lagrangian manifolds, invariant under the action of our exact symplectomorphism $F$?

Some properties can be seen after studying the two previous problems. For instance, if $\mathcal{M} = \mathbb{R}^d$ and $F = (f, g) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is our symplectomorphism, being $S$ its primitive function, and the zero-section $\{y = 0\}$ is fixed by $F$, then we obtain that:

- $S(x, 0)$ is a constant function;
- $\forall x \in \mathbb{R}^d \ \dfrac{\partial S}{\partial x}(x, 0) = 0, \ \dfrac{\partial S}{\partial y}(x, 0) = 0.$

The first property can be easily generalized to any exact Lagrangian manifold, and it shows that we can assign a conserved quantity to it. The second one means that if we consider $S$ as a $x$-parametrized family of functions, then for each $x$ the corresponding point of the Lagrangian manifold, $(x, 0)$, is a critical point of $S(x, \cdot)$. The converse is also true if our map is monotone, that is, it is transversal respect to the standard foliation:

$$\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \ \left| \frac{\partial f}{\partial y}(x, y) \right| \neq 0$$

(The symmetrization of this matrix of partial derivatives is known as *torsion*). This property can be also applied to any invariant exact Lagrangian graphs, and even it could also work for any invariant exact Lagrangian manifolds by means of suitable transversal foliations.

A specially important example corresponds to the case in which all the 'fibered' critical points are minimum, because the orbits minimize a certain action. The orbits of an exact symplectomorphism on the cotangent bundle satisfy a variational principle (in a similar way the trajectories of a classical mechanical system satisfy the *stationary action principle*). Discrete variational principles are a powerful tool when we want to prove the existence of fixed points, periodic orbits, quasi-periodic orbits, homoclinic orbits, etc. Poincarè [85] was the first person who used these methods in certain problems of celestial mechanics, and they have been used by many authors. For instance, they have been fundamental to prove the existence of quasi-periodic orbits in certain a.p.m. (the *twist* ones). These orbits minimize a certain action, and they correspond to invariant curves or invariant Cantor sets (Aubry-Mather sets) [13, 71].

In general, we need the existence of a global generating function in order to define variational principles, but this fact restricts the topology of our configuration space and our symplectomorphism. We avoid to use the generating functions and we use the primitive function, and our variational principles work for any discrete mechanical system (that is, an exact symplectomorphism on $T^*\mathcal{M}$). We shall take a different point of view: instead of using variational principles to find orbits, we use them to extract information about them. They are, in a sense, local variational principles. Finally, turning to the invariant Lagrangian graphs, our constructions let us generalize some results by Mather [73], Herman [40] and MacKay, Meiss and Stark [68].

Henceforth, we think that our variational principles are interesting because:

- they work in any cotangent bundle, resembling the laws of discrete classical mechanics;
- we do not need the generating function, which does not always exist or is difficult to compute (for instance, if our map is given by a Hamiltonian flow);
- we can extend them around any exact Lagrangian manifold, thanks to Weinstein's theorems.

In order to define those variational principles we have followed the next steps. Here we use cotangent coordinates $(x, y)$ and, in fact, we work on $\mathbb{R}^d \times \mathbb{R}^d$.

1. First, consider two positions $\mathbf{x}_m, \mathbf{x}_n \in \mathbb{R}^d$, where $n > m+1$, that we want to join by means of a piece of orbit (of length $n-m$).

2. Then, we define the set of chains connecting both points, being these chains the sequences
$$(x_m, y_m), (x_{m+1}, y_{m+1}), \ldots (x_{n-1}, y_{n-1})$$
satisfying
   - $x_m = \mathbf{x}_m$,
   - $\forall i = m \div n-2,\ f(x_i, y_i) = x_{i+1}$,
   - $f(x_{n-1}, y_{n-1}) = \mathbf{x}_n$.

3. The action over this set is the sum
$$\boldsymbol{S}_{m,n}(x_m, y_m, x_{m+1}, y_{m+1}, \ldots, x_{n-1}, y_{n-1}) \quad = \quad \sum_{i=m}^{n-1} S(x_i, y_i).$$

4. Finally, we obtain that the orbits connecting the two positions $\mathbf{x}_m, \mathbf{x}_n$ are extremal for the action (defined on the set of chains), and the converse is true if our symplectomorphism is monotone. Then, an orbit is *minimizing* if every of its segments minimizes the corresponding action.

What is the physical interpretation of this construction? Consider a mechanical system given by a time-periodic Hamiltonian

$$H : T^*\mathcal{M} \times \mathbb{T} \longrightarrow \mathbb{R},$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $F = \varphi_{1,0}$ be its time-periodic flow. Its primitive function is, using cotangent coordinates,

$$\begin{aligned} S(x,y) &= \int_0^1 \mathbf{\Lambda}(H_t)(x(t),y(t))dt \\ &= \int_0^1 \left( y(t) \cdot \frac{\partial H}{\partial y}(x(t),y(t),t) - H(x(t),y(t),t) \right) dt, \end{aligned}$$

where $(x(t),y(t)) = \varphi_{t,0}(x,y)$ is the flow. Therefore, a chain is an 'orbit' of our mechanical system whose velocity is suddenly changed each period of time. We want to smooth the peaks, and we obtain it by extremizing the action.

If we consider chains of length 1, that is, points $(x,y)$, but we impose that they go to the same fiber and the action is the primitive function, then we look for fixed points. That is, fixed points are critical points of the primitive function restricted to the *fiberwise transformed set* $K = \{(x,y) \mid f(x,y) = x\}$. In fact, we have found a construction already used by Moser [79] in the case of exact symplectomorphisms defined on the cotangent bundle of a torus, and used later by Arnaud [3].

To follow a logic order, in the thesis we have described firstly the variational principles for fixed points and afterwards we have considered those related with orbits. We have also devoted some time to the relationship between the dynamical character and extremal character of a fixed point, although there are many results about this subject [53, 66, 3]. We have also considered some degenerate cases, that correspond to non-monotone fixed points.

Finally, we have proved that the extremal characters of an orbit and an invariant exact Lagrangian graph are invariant under changes of variables in our configuration space and fiberwise translations on the phase space. The physical interpretation is that the laws of discrete mechanics are independent of the coordinates in the configuration space and certain 'privileged observers'. From a geometrical point of view, this is connected to the election of a natural 1-form in the phase space, $\boldsymbol{\alpha} = y\, \mathrm{d}x$, and the concomitant distinction between position and momentum variables that this implies. On the other hand, recall that the dynamics of the systems are independent of any coordinates on the phase space. Then, for instance, the Floquet multipliers associated to a periodic orbit are invariant under any change of variables.

## Applications to converse KAM theory

Lagrangian manifolds are interesting from a dynamical point of view because they appear often in the theory of dynamical systems. For instance:

- the invariant tori of the theory by Kolmogorov [52], Arnold [4] and Moser [78], known briefly as *KAM theory*, which are Lagrangian because their dynamics are given by ergodic rotations, as Herman proved [40, 41];

- the stable and unstable manifolds of a hyperbolic fixed point, which are Lagrangian because the corresponding dynamics collapses to the fixed point when we iterate our map or its inverse, respectively.

We shall consider the first example. About the second one, they appear in the theory of splitting of separatrices, founded by Poincarè and developed later by Melnikov and Arnold. In the symplectic case one use a Melnikov function rather than a Melnikov vector in order to measure the breakdown, and in principle one uses generating functions. Easton [30] already used the primitive function for the definitions of that potential, and his formulae was generalized by Delshams and Ramírez-Ros [28].

In order to show the main ideas of KAM theory we consider now the well known *standard map*, which is a twist a.p.m. introduced by Chirikov [24]. It is:

$$\begin{cases} x' = x + y - \dfrac{K}{2\pi}\sin(2\pi x) \pmod 1 \\ \\ y' = y - \dfrac{K}{2\pi}\sin(2\pi x) \end{cases},$$

where the phase space is the cylinder $\mathbb{T} \times \mathbb{R}$, whose coordinates are the angle-action variables $(x, y)$. $K$ is a perturbative parameter, and for $K = 0$ our map is integrable:

$$\begin{cases} x' = x + y \pmod 1 \\ \\ y' = y \end{cases}.$$

That is to say, our phase space is foliated by invariant tori $y = y_0$, and their dynamics are given by rotations. These rotational invariant curves [3], because they encircle the cylinder, are labeled by the corresponding frequencies $y_0$, and there are two types:

- $y_0 \in \mathbb{Q}$, and then the orbits are periodic;
- $y_0 \in \mathbb{R} \setminus \mathbb{Q}$, and contain quasi-periodic orbits that densely fill the curve.

When $K$ is increased from zero, the question is about how much of the integrable structure survive. Experimentally, we see that the major part of orbits still belongs to r.i.c., and this is exactly that KAM theory says: the major part of tori persists if the perturbation $K$ is small enough. A remarkable fact is that the persistence of these invariant curves depend on the 'distance' of their frequencies to the rational numbers. The irrationality degree is measured with the called *diophantine condition*. A number $\omega$ is diophantine if there are two constants $C > 0$ and $\tau \geq 1$ such that for all the fractions $\frac{p}{n} \in \mathbb{Q}$

$$|n\omega - p| \;>\; \frac{C}{n^\tau}.$$

[3] in short: *r.i.c.*.

The diophantine numbers are difficult to approximate by rationals. This is also related to the continued fraction expansion of a number. So then, the 'more irrational' number is the *golden mean*

$$\gamma \;=\; \frac{\sqrt{5}+1}{2} = [1;1,1,1,\ldots],$$

that satisfies a diophantine condition with $C = \gamma^2$ and $\tau = 1$.

Now, we ask ourselves about the converse question: how large should be the perturbation to break all the r.i.c.? Another question is: what is the 'last' r.i.c.? We can also consider when a certain r.i.c. with a certain frequency breaks down. The set of criteria and tools performed in order to solve these kind of problems is called *converse KAM theory*, following MacKay, Meiss and Stark [68]). On the contrary to KAM theory, converse KAM theory is non perturbative, and it is able of giving conditions to say that for a certain point of phase space does not belong to a r.i.c. (or, in higher dimensions, to a torus).

While there are many results about converse KAM theory in low dimension ($d = 1$), this is not the case in higher dimensions ($d > 1$). We emphasize MacKay, Meiss and Stark [68], Herman [40, 41, 94, 43] and Tompaidis [94, 95]. The phase space that one consider is the cotangent bundle of a torus, $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$, named $d$-annulus or $d$-cylinder, whose covering space is $\mathbb{R}^d \times \mathbb{R}^d$. The main problem is, while for twist a.p.m. the r.i.c. are graphs (thanks to a Birkhoff's theorem [19]), there is not an equivalent statement in higher dimensions. We must only pay attention to those Lagrangian tori which are graphs. On the other side, we have the maps which are not twist, or those in which the torsion changes its sign, or it is non degenerate but not definite (it can happen for $d > 1$). Moreover, the multidimensional generalization of the continued fraction expansion is not so clear.

We can group the different techniques and criteria of converse KAM theory in the following groups.

- **Lipschitz criteria.** Thanks to another Birkhoff's theorem one can obtain Lipschitz bounds on slopes of a r.i.c. for a twist a.p.m., and then obtain restrictive criteria for the non-existence of those curves (as Birkhoff [19], Herman [39] and Mather [72] studied this, we shall refer to this theory as *BHM Theory*). For instance, Mather [72] found that for the standard map there are no r.i.c. if $|K| \geq \frac{4}{3}$ and later MacKay and Percival [67] refined it, with the aid of a computer, in order to obtain a rigorous bound $|K| \geq \frac{63}{64}$, and Jungreis [48] improved these rigorous results. Herman [40, 41] also proved similar Lipschitz inequalities for the invariant Lagrangian graphs of monotone positive symplectomorphisms. We shall relate some of his results to variational principles.

- **Variational criteria.** In [67], MacKay and Percival related also their *cone-crossing criterion*) with Aubry-Mather's variational principles. Later, in higher dimensions, MacKay, Meiss and Stark [68] performed a method to detect if a point of phase space does not belong to a Lagrangian graph, and they also gave a geometrical explanation by means of the 'slopes' of Lagrangian planes. In both cases the symplectomorphism should satisfy strong positivity conditions, being defined by a Lagrangian generating function. The question was to detect if a certain segment of orbit minimizes or not a certain action, and this is easily verified by considering a certain Hessian matrix. It was curious that one need global conditions on our symplectomorphism while one only check local conditions.

We have followed the definition of positivity given by Herman, and we have considered monotone positive symplectomorphisms [40, 41]. There are examples of monotone positive symplectomorphisms that are not twist and they are not defined by Lagrangian generating functions. Our definitions are local, and they let us to study suitable pieces of phase space. As a conclusion we have that

> if an invariant Lagrangian graph lives in a monotone positive region, then it is minimizing, and all their orbits are minimizing.

From here, we can do similar calculations to those given in [68], but with less restrictive conditions. Specially, we can use what we have called the *MMS iteration*, that is, an algorithm performed by MacKay, Meiss and Stark [68] in order to determine if a certain block-tridiagonal symmetric matrix is positive definite. [4]

- **Greene-like criteria.** In [36], Greene proposed a criterion to detect when a certain r.i.c. of an a.p.m. breaks when the perturbation is increased. His method was founded in the study of the stability of nearby periodic orbits. Greene discovered that, for the standard map, the last invariant curve has frequency $\gamma$ (the golden mean) and it breaks for a critical value $K_\gamma \simeq 0.971635406$ (this value was obtained by MacKay [63]). He reasoned that if there is a set of periodic orbits whose frequencies $r_i = \frac{p_i}{n_i}$ limit on the frequency $\omega$,

$$\lim_{i\to\infty} r_i = \omega,$$

  and they have residues between 0 and 1 (they are elliptic periodic orbits), then the invariant circle exists. This *residue conjecture* has been proven by MacKay [65] and Falconini and de la Llave [32] in some cases. Then, he considered the critical values $K_{r_i}$ where the corresponding periodic orbits have residue 1 (which corresponds to a period doubling bifurcation), and observed that they limit on $K_\omega$. Moreover, when one considers the golden mean (or any noble number) and the sequence of rational approximations is given by the convergent of the corresponding continued fraction, then the sequence of critical values geometrically limits on $K_\omega$. This is related to the stable manifold of a fixed point of a certain *renormalization operator* in the space of twist a.p.m., as MacKay [63] numerically studied.

  In higher dimensions the situation is not so clear. First of all, we have not a definite candidate of rational approximation of irrational vectors which generalizes the continued fractions. For instance, Tompaidis [94] extended the Greene method

[4] At the end of this thesis we have written a small summary about positive definite symmetric matrices.

and he applied it to a 3-dimensional volume preserving map [95] the rotational standard map. He used the Jacobi-Perron algorithm in order to approximate the irrational vectors. We have followed him. Another problem is the renormalization behaviour which is more complicated and difficult to detect. This should help us to improve the estimation of the critical values of breakdown of the tori.

We have developed a method with the same flavour, but instead of considering the stability of periodic orbits we have considered their extremal character. We have also worked with monotone positive symplectomorphisms. From a 'naïve' point of view, we have used that if an orbit is minimizing (and the orbits on the tori are minimizing), then any segment of orbit close enough to it is also minimizing. Although elliptic periodic orbits are not minimizing, small enough segments of them are. Recall that all the orbits of a monotone positive integrable symplectomorphism are minimizing.

- **Obstructional criteria.** Consider an a.p.m. and suppose that the unstable manifold of a periodic orbit cuts the stable manifold of another one. Then there can be no r.i.c. contained between them. Since these heteroclinic connections can be numerically computed, then one can use them as a practical criterion, as Olvera and Simó did [82]. On the other side, these manifolds are also used to bound the named resonances, which are regions on the phase space whose points have similar behaviour. *Transport theory* (see, for instance, [76]) deal with the motion of these sets.

  When we work in higher dimensions we can not use the stable and unstable manifolds of hyperbolic periodic orbits, because they do not separate the phase space in connected components. Codimension-1 manifolds are needed, as the center-stable and center-unstable manifolds of elliptic hyperbolic periodic orbits (with only two hyperbolic directions). We think that they can be useful to explain the mechanism of the breakdown and to study the transport. They are the skeleton of our dynamical system. As an example, we have considered a 4D map, the Froeschlé map, and we have studied the resonance region associated to the origin, that is an elliptic point. We shall see that this region is bounded by the center-stable and the center-unstable manifolds of its two elliptic-hyperbolic companions. To do this, we have expanded these manifolds in power series (with 3 variables) until a high degree, because they are difficult to globalize. Their visualization can be obtained by intersection of them with planes. Such intersections are, generically, curves. Since different invariant manifolds cut between them, there is a complex structure of folds and bags, which give smaller resonance regions.

In some sense, we have unified the Lipschitz/variational criteria with Greene criteria. The first ones are equivalent [67, 68] and let us do rigorous proofs with the aid of a computer, by using interval arithmetic (although we have not done this). The second ones do not let us do rigorous proofs, but they give accurate estimates of the critical values of breakdown. As we have already seen, we have performed a variational Greene criterion. The unification of all of these criteria with obstructional criteria would be given by a complete study of the relationship between dynamics and extremality.

In order to show these ideas we consider two, three and four dimensional examples. In all cases we have applied first the variational criteria in a similar way than [68], in order to eliminate the regions in phase space that do not contain invariant tori (like graphs). We have considered different kind of examples: twists, monotone positive but not twist, non monotone, etc. The 2D examples belong to the family of the standard map, the 4D ones belong to the family of the Froeschlé map, and in the 3D case we have taken the rotational standard map. For instance, if we consider a non monotone a.p.m., then we see that the r.i.c. which crosses the non monotone curves (where monotonicity fails) are not graphs, have folds and they are more difficult to destroy. We suppose that the major part of these curves have definite sign, but we should consider suitable coordinates.

To check our variational Greene method, we have firstly considered the standard map, because it is well known. We have also observed the typical self-similarity behaviour associated to the breakdown of noble curves. We have also applied it to another 2D map, the exponential standard map, which is monotone positive but not twist. For the 4D dimensional examples we have taken the Froeschlé map, that is defined on $\mathbb{T}^2 \times \mathbb{R}^2$ and is

$$\begin{cases} {y'}_1 = y_1 - \dfrac{K_1}{2\pi}\sin(2\pi x_1) - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \\ {y'}_2 = y_2 - \dfrac{K_2}{2\pi}\sin(2\pi x_2) - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \\ {x'}_1 = x_1 + {y'}_1 \pmod 1 \\ {x'}_2 = x_2 + {y'}_2 \pmod 1 \end{cases},$$

and two different rotation vectors: a quadratic pair $(\sqrt{2}-1, \sqrt{3}-1)$ and the golden vector (for the Jacobi-Perron algorithm), which is a cubic pair. Depending on the values of the parameters $K_1$ and $K_2$, and considering $\lambda$ as a perturbation parameter, we have seen different kinds of breakdown. We have computed periodic orbits with 'big' periods and we have continued them respect to $\lambda$. We have used a parallel-shooting like method. The question that we ask ourselves is:

How are the Aubry-Mather sets in higher dimensions?

In the 2D case these sets have zero Hausdorff dimension (in the hyperbolic case), as MacKay proved [64], but here they seem to have zero or 1 dimension. We have also seen certain resonance regions associated to periodic orbits of low period, and this carry out to consider obstructional criteria in order to explain the mechanism of breakdown. Finally, we have also considered the Tompaidis 3D example and the golden vector. The rotational standard map depends on two parameters $K$ and $\lambda$ and a rotation $\omega$, it is defined on the cylinder $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ endowed with the coordinates $(x, y, \theta)$, and it is

$$\begin{cases} y' = y - \dfrac{1}{(2\pi)} \sin(2\pi x)(K + \lambda \cos(2\pi\theta)) \\ x' = x + y' \pmod 1 \\ \theta' = \theta + \omega \pmod 1 \end{cases} .$$

In order to do this, he have had to develop a variational theory for the named exact symplectic skew-products. This part have not been given with detail, because is similar to the variational theory for exact symplectomorphisms.

We remark that there are other non-existence criteria, as those based in the computation of the radius of convergence of Fourier series of KAM tori (done, for instance, by Percival [84]) and the frequency analysis by Laskar [60].

# General summary of the thesis

We have divided this thesis in four parts.

**PART I. Exact symplectic geometry** (introduction of the problems)

This part contains the basic tools of symplectic geometry and outline the four subjects that we have study along the thesis:

1. the determination problem,
2. the interpolation problem,
3. the variational problem,
4. the breakdown problem.

**PART II. On the standard symplectic manifold** (analytical part)

We recall the necessary tools to work on $\mathbb{R}^d \times \mathbb{R}^d$. That is, we perform a coordinate treatment of the results. First of all, we relate different kinds of generating functions to the primitive function and later we solve formally the determination problem. Then we introduce different variational principles: for fixed points, periodic orbits and orbital segments. Their invariance under certain kind of transformations of phase space is proved, and we interpret physically such results. Finally, we give the basic properties of invariant exact Lagrangian graphs, obtaining, at last, that if our graph is minimizing then its orbits are minimizing.

**PART III. On the cotangent bundle** (geometrical part)

The first three chapters are similar to the three previous ones, with the difference that we do an intrinsic treatment of the results, by considering any cotangent bundle. The fourth chapter in this part deal with the solution of the interpolation problem, given in analytic set up.

**PART IV. Converse KAM theory** (numerical part)

The last part deal with the applications to converse KAM theory. First of all, we give a small list of different examples that we shall study later. Then, we generalize converse KAM theory by [68] and we related it to the Lipschitz theory by Birkhoff and Herman [40, 41]. Then, we perform our variational Greene method and apply it to different examples. Also we study numerically the Aubry-Mather sets in higher dimensions. After this, we apply our methods to the rotational standard map, that is a symplectic skew product. Then, we give some ideas about the geometrical obstructions for existence of invariant tori, showing them with a simple example. We also find some known Birkhoff normal forms using our methods. Finally, we explain briefly how our theory can be used for arbitrary Lagrangian foliations.

# Main achievements

Now, we summarize what are the main conclusions of this thesis, and further questions that we leave till the future.

- First of all, we think that our main contribution is the systematic use of the primitive function of an exact symplectomorphism. The analytical, geometrical and numerical tools used along this thesis take into account the properties of this primitive function. In fact, they come from the structure of the phase space, given by an action form.

- This use let us to introduce variational principles in more general contexts and with weaker hypotheses that one usually demands, as the existence of a global generating function.

- We have stated variational principles of discrete mechanics (that is, the study of exact symplectomorphisms on a cotangent bundle). The geometrization of these principles come from the Liouville form and the standard foliation associated.

- We also have given a variational interpretation of the invariant exact Lagrangian graphs. Their points are fibered critical points of a certain function. This let us to classify from a variational point of view the different graphs. We consider mainly the minimizing graphs, and obtain that their orbits are minimizing. This is a generalization of some results by Mather [73], Herman [40, 41] and MacKay, Meiss and Stark [68].

- In order to give existence conditions of invariant tori for symplectomorphisms defined on $\mathbb{T}^d \times \mathbb{R}^d$, we have used these variational methods. Then, we have related [68] and [40, 41], with the main difference that we have not used the Lagrangian generating functions, which do not always exist.

- The 'local' treatment of our variational principles let us to study the regions of phase space which satisfy certain positiveness conditions. We can summarize that if an invariant Lagrangian graph lives in a monotone positive region, then it is minimizing, and all their orbits are minimizing.

From this point, we can perform similar computations to [68], but with less restrictions.

- Another point is the development of Greene-like criteria to detect when a certain torus breaks down. We have used the extremal properties instead of the dynamical properties of the close periodic orbits. We have checked our method with the standard map. We have also applied it to a 4D example.
- In order to 'see' the breakdown of invariant tori we have had to compute periodic orbits with long periods ($\approx 10^5$). We have used a parallel shooting method. The results agree with our variational method. Moreover, we have detected different kinds of breakdown. These phenomena should be studied in the future.
- We have explained a possible mechanism of breakdown, associated to the intersections between codimension-1 invariant manifolds. Although we have not developed this completely, this work is in progress. We think that the example that we have considered is sufficiently instructive.
- Finally, we have applied our methodology to a broader class of maps, the symplectic skew-products. The 3D example that we have considered was already used by Tompaidis [95]. Our results agree with his.
- An important part of this thesis is devoted to the examples. As models of non twist area preserving maps we have introduce the exponential standard map, the quadratic standard map and the trigonometric standard map. We have also considered the standard map. The couplings of these maps give us many 4D examples, like the Froeschlé map. As a 3D example (of symplectic skew-product) we have taken the rotational standard map, but we could also consider other examples.

About these methods we have some problems to solve. The first one appear, for instance, when we have an a.p.m. whose monotonicity changes its sign and the r.i.c. that we are studying cross regions with monotonicity of different sign. These curves have folds and are more difficult to destroy. Possibly the major part of these curves have definite sign in suitable coordinates. The second one appears when we work in higher dimensions and the monotonicity, although is non degenerate, is undefinite. Herman has some results about this [43]. We know how the orbits are on these tori, from a variational point of view, once we have made one step of the Birkhoff normal form. Of course, this is not enough. Moreover, another deep problem is about the application of renormalization operators in higher dimensions, associated to multidimensional rational approximations. Finally, we would like to explain the breakdown from a geometrical/obstructional point of view, not only considering invariant manifolds associated to periodic orbits, but also to isotropic tori.

On the other side, we have extended the variational principles to any cotangent bundle and they can work as the laws of discrete mechanics. It should be interesting to consider different configuration spaces, as $\mathbb{S}^2$, SO(3), etc. For instance, one could look for fixed points of symplectomorphism defined on their cotangent bundles (and they could correspond to periodic orbits of mechanical systems). It should be also interesting to detect the bifurcations of such fixed points (if our symplectomorphism depends on parameters, of course) in terms of topological transformations of the fiberwise transformed set. Anyway, these results are not constructive and the nature of the problem is local.

We continue with other contributions.

- Other important point is the confrontation primitive function/generating function. We have also pointed out that the generating function is not always computable, and it could be a problem in order to study undefinite invariant tori. In order to obtain the dynamics around an invariant torus we only need its dynamics (and if it is a KAM torus its dynamics is given by a rotation) and the primitive function.

- We can also obtain the dynamics by interpolation with a Hamiltonian flow. We have used it in order to prove the analyticity of the expansions (the differentiable case remains open –cf. [16]–). It is important the fact that the proofs are constructive, and the recurrences can be carried out with the aid of a computer. Periodicity in time can be got applying some constructive results by Pronin and Treschev [86].

- The geometrical structure of the phase space has been the key point of our proofs. In addition to the Liouville form and its associated foliation, the Liouville derivative that we have defined has been very useful. We have obtained that we can associate a derivation to any exact symplectic manifold, and in the case of the cotangent bundle (endowed with the Liouville form) is the Liouville derivative (also known as the elementary action).

These constructions let us to 'invent' many kinds of dynamics around invariant Lagrangian manifolds. For instance, if the basic manifold is a torus, we need to programme an algebraic manipulator of Fourier-Taylor series. We can put any dynamics on the torus, as an ergodic translation, an Anosov's diffeomorphism, etc. This work is also in progress (one can get easy examples if the torus has dimension 1). It could be also interesting to apply it to the study of undefinite tori, and 'see' the 'escape lines' that Herman found [43].

From a geometrical point of view, the main object of our theory is the Liouville form, which vanishes on the zero-section and on the tangent vectors to the standard foliation of the cotangent bundle. This foliation is transversal to the zero-section. We suppose that it can be generalized by means of suitable foliations transversal to our invariant Lagrangian manifolds. Other kind of generalization is to consider Lagrangian manifolds defined by the called *Morse families* or *phase functions*, which are similar to the generating functions but contain additional parameters that let the foldings in our manifolds. Although we have to be able to extend some results to this context, we still do not know how to develop them. Moreover, we can also consider other maps, as the $\sigma$-symplectic ones (which satisfy $F^*\boldsymbol{\omega} = \sigma\boldsymbol{\omega}$, where $\sigma \in \mathbb{R}$ or, in the complex case, $\sigma \in \mathbb{C}$), or, in particular, the antisymplectic maps, which have $\sigma = -1$ (cf. [23]).

We remark that we have worked around any zero-section of any cotangent bundle and, morally speaking, around any exact Lagrangian manifold. For instance, around a Lagrangian torus, or a piece of stable manifold of a hyperbolic fixed point, or a piece of stable manifold of a hyperbolic lower dimensional (isotropic) torus (useful in the explanation of *Arnold diffusion*). We have used our methods in order to find the known Birkhoff normal forms for these examples.

Finally, we think that this thesis is interesting because we relate different analytical, geometrical and numerical techniques. We have tried to give them a certain structure.

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# Contents

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# Part I

# EXACT SYMPLECTIC GEOMETRY

# Chapter 1

# Exact symplectomorphisms

We recall the elementary definitions and results about exact symplectic manifolds and exact symplectomorphisms. For the sake of simplicity, all the objects (manifolds, functions, diffeomorphisms, ...) will be $C^\infty$. Moreover, all the manifolds will be connected manifolds.

Using the nomenclature of [7], we associate to an exact symplectomorphism a *primitive function*, also called *generating function* by other authors. As we shall see, this function does not generate our symplectomorphism, and because of this we shall adopt the first name.

Finally, we state the *determination problem*, which deal with the additive information that we need in order to determine our exact symplectomorphism from its primitive function.

## 1.1 Exact symplectic manifolds

In accordance with the standard definitions, a *symplectic structure* on a manifold $\mathcal{N}$ is given by a differential 2-form $\boldsymbol{\omega} \in \Omega^2(\mathcal{N})$ satisfying the following two properties:

- $\forall z \in \mathcal{N}$, $\boldsymbol{\omega}_z$ is non degenerate ($\forall X_z \in T_z\mathcal{N} \setminus \{0\}\ \exists Y_z \in T_z\mathcal{N} \mid \boldsymbol{\omega}_z(X_z, Y_z) \neq 0$),
- $\boldsymbol{\omega}$ is closed ($\mathrm{d}\boldsymbol{\omega} = 0$).

We say that $(\mathcal{N}, \boldsymbol{\omega})$ is a *symplectic manifold*, and that $\boldsymbol{\omega}$ is a *symplectic form*. The nondegeneracy condition implies that the dimension of $\mathcal{N}$ is even ($\dim \mathcal{N} = 2d$) and the map

$$\begin{array}{rccc} \boldsymbol{\omega}^\flat: & T\mathcal{N} & \longrightarrow & T^*\mathcal{N} \\ & X & \longrightarrow & {}^\flat X = -\mathrm{i}_X\boldsymbol{\omega} \end{array}$$

is an isomorphism of vector bundles (its inverse is denoted by $\boldsymbol{\omega}^\sharp$: ${}^\flat X = \rho \Leftrightarrow {}^\sharp\rho = X$).

If, moreover, $\boldsymbol{\omega}$ is exact ($\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$, for some Pfaffian form $\boldsymbol{\alpha}$ on $\mathcal{N}$), we say that $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$ is an *exact symplectic manifold* and $\boldsymbol{\alpha}$ is its *action form.*

The most important examples of symplectic manifolds correspond to exact symplectic manifolds. Examples of non exact symplectic manifolds are given, for instance, by orientable compact surfaces, taking their area elements as their symplectic 2-forms.

### Examples

1) **The standard symplectic manifold**

The *standard symplectic structure* on $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, endowed with the *position-momentum* coordinates $(x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_d)$, is given by $\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^d \mathrm{d}y_i \wedge \mathrm{d}x_i$, and it is exact, with $\boldsymbol{\alpha} = y\ \mathrm{d}x = \sum_{i=1}^d y_i\ \mathrm{d}x_i$ as action form.

This is a local model of all the symplectic manifolds of dimension $2d$, according to *Darboux's Theorem* (see, for instance, [2], p. 463). That is to say, if $(\mathcal{N}, \boldsymbol{\omega})$ is a symplectic manifold and $z \in \mathcal{N}$, there exists a local coordinate chart on $z$, given by $(x, y)$ (the *canonical coordinates* or *symplectic coordinates*), in which $\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x$.

'Exotic' symplectic structures on $\mathbb{R}^4$ have been constructed. That is, there exist symplectic structures on $\mathbb{R}^4$ which are not (globally) equivalent to the standard one (see, for instance [8], p. 81).

2) **The cotangent bundle**

In classical mechanics, the *cotangent bundle* (the *phase space*) of a manifold $\mathcal{M}$ (the *configuration space*) is the more celebrated example [34, 5].

Let $\mathcal{M}$ be a $d$-dimensional differentiable manifold and $T^*\mathcal{M}$ its cotangent bundle, whose projection is $q : T^*\mathcal{M} \to \mathcal{M}$. We know that we can define a differentiable structure on $T^*\mathcal{M}$ by means of the *cotangent charts* $\mathcal{U} \times \mathbb{R}^{\lceil}$, where each $\mathcal{U}$ is a chart of $\mathcal{M}$. We write the corresponding coordinates as $(x, y)$.

In order to define an exact symplectic structure on $T^*\mathcal{M}$ we begin by defining an 1-form $\boldsymbol{\alpha} \in \Omega^1(T^*\mathcal{M})$ (called *Liouville form*). It is defined on each 'point' $\rho_x \in T^*\mathcal{M}$ (where $x \in \mathcal{M}$) by

$$\boldsymbol{\alpha}_{\rho_x} \hat{X}_{\rho_x} \quad = \quad \rho_x q_*(\rho_x) \hat{X}_{\rho_x},$$

for any $\hat{X}_{\rho_x} \in T_{\rho_x} T^*M$ (then, $q_*(\rho_x)\hat{X}_{\rho_x} \in T_x M$ and we can apply $\rho_x$). Moreover, $\boldsymbol{\alpha}$ is the unique Pfaffian form on $T^*\mathcal{M}$ which satisfies

$$\rho^* \boldsymbol{\alpha} = \rho, \ \forall \rho \in \Omega^1(\mathcal{M}),$$

where in the right term we see $\rho$ as a map $\rho : \mathcal{M} \to T^*\mathcal{M}$ (in fact, $\rho$ is a section of the cotangent bundle).

Finally, $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$ is the *canonical* symplectic structure on $T^*\mathcal{M}$, and it is exact. In cotangent coordinates on $T^*\mathcal{M}$, $(x, y) \in U \times \mathbb{R}^d$, these forms are:

$$\boldsymbol{\alpha} = y\ \mathrm{d}x, \ \boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x.$$

3) **The tangent bundle of a Riemannian manifold**

If we have a Riemannian metric g on a manifold $\mathcal{M}$, we can transport the canonical symplectic structure from $T^*\mathcal{M}$ to $T\mathcal{M}$, using the isomorphism of vector bundles:

$$\begin{array}{rccc} \mathrm{g}^\flat : & T\mathcal{M} & \longrightarrow & T^*\mathcal{M} \\ & X & \longrightarrow & \mathrm{i}_X \mathrm{g} \end{array}.$$

So, if $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$ is the canonical symplectic structure of $T^*\mathcal{M}$, then $\hat{\boldsymbol{\omega}} = (\mathrm{g}^\flat)^*\boldsymbol{\omega}$ defines an exact symplectic structure on $T\mathcal{M}$, with $\hat{\boldsymbol{\omega}} = \mathrm{d}\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\alpha}} = (\mathrm{g}^\flat)^*\boldsymbol{\alpha}$ (this is a particular case of the *Legendre transformation*).

$\triangleleft$

[1]

## 1.2 Exact symplectomorphisms

### 1.2.1 Definitions

Let $F : \mathcal{N} \to \mathcal{N}$ be a diffeomorphism, where $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$ is an exact symplectic manifold. There are three important properties that $F$ can satisfy:

- $F$ is a *symplectomorphism*: $F^*\boldsymbol{\omega} = \boldsymbol{\omega}$.
  Then $0 = F^*\boldsymbol{\omega} - \boldsymbol{\omega} = F^*\mathrm{d}\boldsymbol{\alpha} - \mathrm{d}\boldsymbol{\alpha} = \mathrm{d}(F^*\boldsymbol{\alpha} - \boldsymbol{\alpha})$, and hence $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}$ is a closed 1-form.

- $F$ is an *exact symplectomorphism*: $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}$ is exact.
  Then $\exists S : \mathcal{N} \to \mathbb{R} \mid F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S$.

- $F$ is an *actionmorphism*: $F^*\boldsymbol{\alpha} = \boldsymbol{\alpha}$.

#### Exactness equation

Given a symplectomorphism $F$, we shall refer to the first-order partial differential equation on $\mathcal{N}$

$$F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} \;=\; \mathrm{d}S$$

as the exactness equation of $F$. The necessary conditions of existence of solutions of this equation are satisfied.

#### The primitive function

If the exactness equation is solvable, with $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S$ for a certain function $S$, we shall say that $S$ is a *primitive function* of $F$, and we shall write $\mathit{pf}(F) = S$. Obviously, $S$ is defined up to constants.

#### Remarks

i) We can do the definitions with different symplectic manifolds, but the maps need to be immersions. The definitions are similar.

ii) (Exact) symplectomorphisms are also called *(globally) canonical transformations*, while actionmorphisms are also known as *homogeneous canonical transformations* or *Mathieu transformations* [100, 61].

[1] In the sequel, while $\triangleleft$ means 'end of remarks' or 'end of examples', $\square$ means 'end of proof'.

iii) In the literature, the primitive function is often called generating function. As we shall see, this function does not generate the symplectomorphism, but a family of symplectomorphisms. By this reason, we have followed the nomenclature used in [7].

iv) Examples of symplectomorphisms on the cotangent bundle of the torus, the annulus $\mathbb{A}^d = T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$, are given in Appendix A.

◁

### 1.2.2 Composition formulae

The behavior of the primitive function by composition, inversion and conjugation is given by the next proposition.

**Proposition 1.1** *:*

*Let $(\mathcal{N}, \mathrm{d}\boldsymbol{\alpha})$ be an exact symplectic manifold, and let $F, G : \mathcal{N} \longrightarrow \mathcal{N}$ be two exact symplectomorphisms, with $pf(F) = S$ and $pf(G) = T$.*
*Then:*

1. $pf(G \circ F) = S + T \circ F$
2. $pf(F^{-1}) = -S \circ F^{-1}$
3. $pf(G^{-1} \circ F \circ G) = S \circ G + T - T \circ G^{-1} \circ F \circ G$

*Proof*:

It is enough to prove 1:

$$(G \circ F)^* \boldsymbol{\alpha} = F^* \circ G^* \boldsymbol{\alpha} = F^*(\boldsymbol{\alpha} + \mathrm{d}T) = \boldsymbol{\alpha} + \mathrm{d}S + \mathrm{d}(F^*T) = \boldsymbol{\alpha} + \mathrm{d}S + \mathrm{d}(T \circ F).$$

□

**Remarks**

i) Last formula is useful in order to obtain normal forms, as we can see in Appendix F.

ii) If we apply the composition formulae to the composition of $n$ exact symplectomorphisms $F_1, \dots F_n$ with corresponding primitive functions $S_1, \dots S_n$, we have

$$pf(F_n \circ \dots \circ F_1) = \sum_{i=0}^{n-1} S_{i+1} \circ F_i \dots \circ F_1.$$

In particular, $pf(F^n) = \sum_{i=0}^{n-1} S \circ F^i$.

◁

## 1.3 The determination problem

As an immediate application of the previous formulae, we see that if we compose $F$ with an actionmorphism $L$, the primitive function is not changed:

$$(L\circ F)^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = F^*L^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S.$$

In fact, all the exact symplectomorphisms with primitive function equal to $S$ are obtained in this way.

**Proposition 1.2** *:*

*Let $(\mathcal{N}, \mathrm{d}\boldsymbol{\alpha})$ an exact symplectic manifold.*
*Let $F : \mathcal{N} \longrightarrow \mathcal{N}$ be an exact symplectomorphism, with $pf(F) = S$. Then:*

$$\{G \mid pf(G) = S\} \quad = \quad \{L\circ F \mid pf(L) = 0\}$$

*Proof*:

Let be $G \mid pf(G) = S$ and define $L = G\circ F^{-1}$.
Then:

$$pf(L) = pf(G\circ F^{-1}) = pf(F^{-1}) + pf(G)\circ F^{-1} = -S\circ F^{-1} + S\circ F^{-1} = 0.$$

□

Thus, an exact symplectomorphism is determined by its primitive function up to actionmorphisms 'by the left'.

In order to determine an exact symplectomorphism from its primitive function we need some additional information, as, for instance, the image of a certain Lagrangian submanifold (Section 4.3). As we shall see, this problem is related with the solution of a certain evolution problem and a derivation in the Lie algebra of functions (endowed with the Poisson bracket) (see Section 2.4 and Chapter 10).

## 1.4 On the symplectic product

As it is well known, given symplectic manifolds one can construct a new one by direct product. So that, if $\mathcal{N} = \mathcal{N}_1 \times \ldots \times \mathcal{N}_q$ is the product of $q$ symplectic manifolds $(\mathcal{N}_1, \boldsymbol{\omega}_1)$, ..., $(\mathcal{N}_q, \boldsymbol{\omega}_q)$ and $\pi_1, \ldots, \pi_q$ are the corresponding projections, then the 2-form on $\mathcal{N}$ given by

$$\boldsymbol{\Omega} \quad = \quad \sum_{i=1}^{q} \pi_i^* \boldsymbol{\omega}_i$$

is symplectic.

Moreover, if the symplectic forms $\boldsymbol{\omega}_i$ are exact, with $\boldsymbol{\omega}_i = \mathrm{d}\boldsymbol{\alpha}_i$ $\forall i = 1 \div q$, then $\boldsymbol{\Omega}$ is also exact, that is, $\boldsymbol{\Omega} = \mathrm{d}\boldsymbol{A}$ with

$$\boldsymbol{A} \quad = \quad \sum_{i=1}^{q} \pi_i^* \boldsymbol{\alpha}_i.$$

**Remarks**

i) Many times is more convenient to take the symplectic 2-form on the product of two symplectic manifolds $(\mathcal{N}_1, \boldsymbol{\omega}_1)$ and $(\mathcal{N}_2, \boldsymbol{\omega}_2)$ as the difference of the pull-backs of $\boldsymbol{\omega}_1$ and of $\boldsymbol{\omega}_2$ instead of their sum.

ii) For the sake of simplicity, we shall write $\boldsymbol{\omega}_i = \pi_i^* \boldsymbol{\omega}_i$ and $\boldsymbol{\Omega} = \sum_{i=1}^q \boldsymbol{\omega}_i$.

◁

Hence, given several symplectomorphisms we can define a new one on the direct product of the corresponding symplectic manifolds.

**Proposition 1.3** :

*Let $\mathcal{N} = \prod_{i=1}^q \mathcal{N}_i$ be the product of $q$ symplectic manifolds $(\mathcal{N}_i, \boldsymbol{\omega}_i)$ $(i = 1 \div q)$, and $\boldsymbol{\Omega} = \sum_{i=1}^q \boldsymbol{\omega}_i$ be the symplectic 2-form defined on $\mathcal{N}$.*

*Suppose we have $q$ symplectomorphisms $F_i : \mathcal{N}_i \to \mathcal{N}_i$ $(i = 1 \div q)$.*

*Then:*

- *The diffeomorphism on $\mathcal{N}$*

$$F \;=\; F_1 \times \ldots \times F_q$$

*(i.e.: $\pi_i \circ F = F_i \circ \pi_i$, $\forall i = 1 \div q$) is symplectic.*

- *If the symplectic forms $\boldsymbol{\omega}_i$ are exact, with $\boldsymbol{\omega}_i = \mathrm{d}\boldsymbol{\alpha}_i$ and the symplectomorphism $F_i$ are exact, with primitive functions $S_i$, respectively, then the symplectomorphism $F$ is exact, and its primitive function is:*

$$S = \sum_{i=1}^{q} S_i \circ \pi_i .$$

*(or $S = \sum_{i=1}^q S_i$, for short).*

An example of actionmorphism is given by the $q$-rotation $R_q$ defined on the direct product $q$ times of an exact symplectic manifold $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$ with itself. The proof of the following result is also straightforward.

**Proposition 1.4** :

*Let $(\mathcal{N}, \boldsymbol{\omega})$ be a symplectic manifold, and let $\boldsymbol{\Omega}_q$ be the symplectic form induced on $\mathcal{N}^q = \prod_{i=1}^q \mathcal{N}$:*

$$\boldsymbol{\Omega}_q \;=\; \sum_{i=1}^{q} \pi_i^* \boldsymbol{\omega}$$

*(where $\pi_i$ is every projection).*
*We consider the $q$-rotation $R_q : \mathcal{N}^q \to \mathcal{N}^q$, which is defined by*

$$R_q(z_1, z_2, \ldots, z_q) = (z_q, z_1, \ldots, z_{q-1}).$$

*Then:*

- *$R_q$ is a symplectomorphism.*
- *If $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$, then $R_q$ is an actionmorphism.*

Given a diffeomorphism $F : \mathcal{N} \to \mathcal{N}$, we note that a $q$-periodic orbit corresponds to a fixed point of $F^q$, of course, but also to a fixed point of $R_q \circ F^{\times q}$ (where $F^{\times q} = F \times \ldots_q \times F$) . In both cases, the (exact) symplectic character is preserved.

**Proposition 1.5** *:*

*Let $F : \mathcal{N} \to \mathcal{N}$ be a symplectomorphism on the symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$. We consider on $\mathcal{N}^q$ the symplectic form $\boldsymbol{\Omega}_q$. We define $F^{\times q}$ as the diffeomorphism defined on $\mathcal{N}^q$ by*

$$F^{\times q}(z_1, \ldots, z_q) = (F(z_1), \ldots, F(z_q)).$$

*We also define the diffeomorphism on $\mathcal{N}^q$: $F_q = R_q \circ F^{\times q} = F^{\times q} \circ R_q$. Thus*

$$\begin{array}{cccc} F_q : & \mathcal{N}^q & \to & \mathcal{N}^q \\ & (z_0, z_1, \ldots, z_{q-1}) & \to & (F(z_{q-1}), F(z_0), \ldots, F(z_{q-2})) \end{array}.$$

*Then:*

- *The fixed points of $F_q$ are in correspondence with the q-periodic orbits of $F$ (the correspondence is q-to-1). Moreover, $F_q$ commutes with $R_q$: $F_q \circ R_q = R_q \circ F_q = R_q \circ F^{\times q} \circ R_q$.*
- *$F_q$ is a symplectomorphism.*
- *If $\boldsymbol{\omega}$ is exact ($\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$) and $F$ is exact, being $S$ its primitive function, then $F_q$ is exact, and its primitive function is:*

$$\begin{array}{cccc} S_q : & \mathcal{N}^q & \to & \mathbb{R} \\ & z = (z_0, \ldots, z_{q-1}) & \to & S_q(z) = \sum_{i=0}^{q-1} S(z_i) \end{array}.$$

*Moreover, the primitive function is $\mathbb{Z}_q$-invariant, i.e., $S_q \circ R_q = S_q$ and $R_q^q = id$.*

**Remarks**

i) The formula of the primitive function of $F_q$ is very closed to the formula of the primitive function of $F^q$.

ii) These ideas give a parallel shooting method for the search for periodic orbits of a diffeomorphism. This method is some times more adequate numerically (see Section C.1.2), and preserves the symplectic character.

◁

Given a $q$-periodic point $z \in \mathcal{N}$ of an exact symplectomorphism $F : \mathcal{N} \to \mathcal{N}$ of an exact symplectic manifold $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$ (i.e. $F^q(z) = z$), we define

- the *action* along the periodic orbit, as

$$S_q(z) = \sum_{i=0}^{q-1} S(F^i(z));$$

- the *averaged action* along the periodic orbit, as

$$\hat{S}_q(z) = \frac{1}{q} S_q(z).$$

A question is how the action along a periodic orbit changes by an exact symplectic change of variables (cf. [62]).

**Proposition 1.6** *:*

*Let $F : \mathcal{N} \to \mathcal{N}$ be an exact symplectomorphisms on the exact symplectic manifold $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$, with pf$(F) = S$. We conjugate $F$ by another exact symplectomorphism $G : \mathcal{N} \to \mathcal{N}$, with pf$(G) = T$.*
*Define $\bar{F} = G^{-1} \circ F \circ G$, being $\bar{S}$ its primitive function.*
*Then:*

*There exists a constant $C \in \mathbb{R}$ such that for all q-periodic point $z \in \mathcal{N}$ of $F$, so $\bar{z} = G^{-1}(z)$ is a q-periodic point of $\bar{F}$, and the corresponding averaged actions differs by $C$:*

$$\hat{\bar{S}}_q(\bar{z}) \;=\; \hat{S}_q(z) + C.$$

*Proof*:

We know that a primitive function of $\bar{F}$ is

$$S \circ G + T - T \circ \bar{F},$$

and hence

$$\bar{S} = S \circ G + T - T \circ \bar{F} + C,$$

for some constant $C \in \mathbb{R}$.

Then:

$$\begin{aligned}
\bar{S}_q(\bar{z}) &= qC + \sum_{i=0}^{q-1} (S \circ G(\bar{F}^i(\bar{z})) + T(\bar{F}^i(\bar{z})) - T(\bar{F}^{i+1}(\bar{z}))) \\
&= qC + \sum_{i=0}^{q-1} (S \circ G(G^{-1} \circ F^i \circ G(G^{-1}(z))) \\
&= qC + \sum_{i=0}^{q-1} S(F^i(z)) \\
&= qC + S_q(z).
\end{aligned}$$

□

**Remark**

If, for instance, we are studying a neighborhood of an elliptic fixed point and we try to simplify the dynamics by means of successive symplectic changes of variables (a normal form process), fixing the value of the successive primitive functions on that fixed point (being equal to zero), then the periodic orbits around the elliptic fixed point preserve their averaged action. ◁

# Chapter 2

# Hamiltonian flow

*Hamiltonian flow* on an exact symplectic manifold provides a nice example of exact symplectomorphism. We recall the basic definitions and results about this subject, and introduce another important object of this thesis: the *derivation* $\mathbf{\Lambda}$ in the Lie algebra of functions. We state the *interpolation problem* and show a possible method for solve it. Finally, we state the *variational problem* of discrete analytical mechanics.

## 2.1 Hamiltonian vector fields

Through this section, we shall work on a symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$.

### Hamiltonian vector fields

As we know, to a function $H : \mathcal{N} \to \mathbb{R}$ we associate the vector field

$$X_H = {}^{\sharp}\mathrm{d}H.$$

called *Hamiltonian vector field* of *Hamiltonian function* $H$. So then, $X_H$ is uniquely determined by $\mathrm{i}_{X_H}\boldsymbol{\omega} = -\mathrm{d}H$.

In symplectic coordinates $(x, y)$, $\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x$ and

$$X_H = \left( \frac{\partial H}{\partial y} \, , \, -\frac{\partial H}{\partial x} \right)^{\top}.$$

If the function is time-dependent, $H = H_t$ [1], we obtain a *time-dependent Hamiltonian vector field*.

### Poisson bracket

The *Poisson bracket* between two functions $K$,$H$ is defined by

$$\{K, H\} = \boldsymbol{\omega}(X_K, X_H) = -\mathrm{d}K(X_H) = \mathrm{d}H(X_K).$$

[1] The subscript $t$ means the dependence on the time $t$.

In symplectic coordinates:

$$\{K,H\} = \frac{\partial K}{\partial y}\cdot\frac{\partial H}{\partial x} - \frac{\partial K}{\partial x}\cdot\frac{\partial H}{\partial y},$$

where $\cdot$ means the scalar product.

We know that the space of functions $\mathcal{F}(\mathcal{N}) = C^\infty(\mathcal{N},\mathbb{R})$ endowed with the Poisson bracket is a Lie algebra. The relation between the Lie bracket and the Poisson bracket is given by the formula

$$X_{\{K,H\}} = [X_K, X_H]\ .$$

**Lie series.-** Let $f:\mathcal{N}\to\mathbb{R}$ be a function and let $\varphi_t$ be the flow of a Hamiltonian vector field $X_H$. We know the formula:

$$\frac{d}{dt}(f\circ\varphi_t) = \{H,f\}\circ\varphi_t$$

(this formula is also valid for time-dependent Hamiltonians). Therefore, if we suppose analyticity and take the Taylor series in $t$, we obtain the *Lie series*

$$f\circ\varphi_t = \sum_{k\geq 0}\frac{t^k}{k!}L_H^k f,$$

where $L_H^0 f = f$ and $L_H^k f = \{H, L_H^{k-1} f\}$, $\forall k\geq 1$.

## 2.2 Exactness of the Hamiltonian flow

Let $(\mathcal{N},\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$ be an exact symplectic manifold.

Let $H_t$ be a time-dependent Hamiltonian function, $X_t = X_{H_t}$ be the corresponding time-dependent vector field and $\varphi_{t,t_0}$ be its flow (in order to simplify, we can suppose completeness). It is known that the *time-t flow from* $t_0$, $\varphi_{t,t_0}$, is an exact symplectomorphism. We recall the proof.

Applying an elementary result about time-dependent vector fields and forms (see, for instance, [2] p. 307), we have

$$\begin{aligned}\frac{d}{dt}(\varphi_{t,t_0}^*\boldsymbol{\alpha}) &= \varphi_{t,t_0}^*\mathrm{L}_{X_t}\boldsymbol{\alpha}\\ &= \varphi_{t,t_0}^*(\mathrm{i}_{X_t}\mathrm{d}\boldsymbol{\alpha} + \mathrm{d}\mathrm{i}_{X_t}\boldsymbol{\alpha})\\ &= \varphi_{t,t_0}^*\ \mathrm{d}(\mathrm{i}_{X_t}\boldsymbol{\alpha} - H_t))\\ &= \mathrm{d}\ \varphi_{t,t_0}^*(\mathrm{i}_{X_t}\boldsymbol{\alpha} - H_t).\end{aligned}$$

Then,

$$\varphi_{t,t_0}^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}\int_{t_0}^{t}(\mathrm{i}_{X_s}\boldsymbol{\alpha} - H_s)\circ\varphi_{s,t_0}\ ds$$

and we must take

$$S_{t,t_0} = \int_{t_0}^{t} (\mathrm{i}_{X_s}\boldsymbol{\alpha} - H_s)\circ\varphi_{s,t_0}\, ds.$$

as primitive function of $\varphi_{t,t_0}$.

We introduce now a linear operator on the space of functions $\mathcal{F}(\mathcal{N}) = C^\infty(\mathcal{N},\mathbb{R})$:

$$\begin{aligned}\boldsymbol{\Lambda} :\ \mathcal{F}(\mathcal{N}) &\longrightarrow \mathcal{F}(\mathcal{N})\\ H &\longrightarrow \mathrm{i}_{X_H}\boldsymbol{\alpha} - H = \boldsymbol{\alpha}(X_H) - H.\end{aligned}$$

Hence, our primitive functions are

$$S_{t,t_0} = \int_{t_0}^{t} \boldsymbol{\Lambda}(H_s)\circ\varphi_{s,t_0}\, ds.$$

In the autonomous case, if we suppose analyticity, we have (taking $\varphi_t = \varphi_{t,0}$)

$$\boldsymbol{\Lambda}(H)\circ\varphi_t = \sum_{k\geq 0}\frac{t^k}{k!}L_H^k(\boldsymbol{\Lambda}(H)).$$

Finally, if we define $\boldsymbol{\Lambda}_k(H) = L_H^{k-1}(\boldsymbol{\Lambda}(H))$, $\forall k\geq 1$, then:

$$S_t = \sum_{k\geq 0}\frac{t^{k+1}}{(k+1)!}L_H^k(\boldsymbol{\Lambda}(H)) = \sum_{k\geq 1}\frac{t^k}{k!}\boldsymbol{\Lambda}_k(H).$$

We summarize the previous argumentation in the following proposition.

**Proposition 2.1** *:*

*Let $H_t$ be a time-dependent Hamiltonian function and $\varphi_{t,t_0}$ be the corresponding flow (which we suppose defined for all time). Then,*

*the time-$t$ flow from $t_0$, $\varphi_{t,t_0}$, is an exact symplectomorphism with primitive function:*

$$S_{t,t_0} = \int_{t_0}^{t} \boldsymbol{\Lambda}(H_s)\circ\varphi_{s,t_0}\, ds,$$

*where $\boldsymbol{\Lambda}(H) = \boldsymbol{\alpha}(X_H) - H$.*

*Moreover, if $H_t = H$ is autonomous and we suppose analyticity, then,*

$$S_t = \sum_{k\geq 1}\frac{t^k}{k!}\boldsymbol{\Lambda}_k(H),$$

*where the $\boldsymbol{\Lambda}$-functions are defined as*

$$\begin{cases}\boldsymbol{\Lambda}_1(H) = \boldsymbol{\Lambda}(H),\\ \boldsymbol{\Lambda}_k(H) = \{H, \boldsymbol{\Lambda}_{k-1}(H)\} \quad (k>1).\end{cases}$$

**Remark**

If we want to compute numerically the primitive function of a Hamiltonian flow, we just only need to add to our first-order differential equations the equation

$$\frac{\partial S_{t,t_0}}{\partial t} = \boldsymbol{\Lambda}(H_t),$$

and to our set of initial conditions the value $S_{t_0,t_0} = 0$. Then we can integrate the whole equations with our favorite numerical method. ◁

## 2.3 The derivation $\boldsymbol{\Lambda}$

The operator $\boldsymbol{\Lambda}$ will be important in the sequel, but for the moment we shall see that it satisfies nice properties. In particular, it is a derivation on the Lie algebra of functions.

**Proposition 2.2** *:*

*The operator* $\boldsymbol{\Lambda}$ *is a derivation in the Lie algebra* $\mathcal{F}(\mathcal{N})$*:*

- $\boldsymbol{\Lambda}$ *is linear,*
- $\boldsymbol{\Lambda}\{H_1, H_2\} = \{\boldsymbol{\Lambda}(H_1), H_2\} + \{H_1, \boldsymbol{\Lambda}(H_2)\}$.

*Moreover, it verifies:*

- $\mathrm{d}(\boldsymbol{\Lambda}(H)) = \mathrm{L}_{X_H}\boldsymbol{\alpha}$.

*Proof*:

Before proving the product rule we shall prove the last formula:

$$\begin{aligned}
\mathrm{L}_{X_H}\boldsymbol{\alpha} &= \mathrm{d}\circ\mathrm{i}_{X_H}\boldsymbol{\alpha} + \mathrm{i}_{X_H}\circ\mathrm{d}\boldsymbol{\alpha} \\
&= \mathrm{d}(\boldsymbol{\alpha}(X_H)) + \mathrm{i}_{X_H}\boldsymbol{\omega} \\
&= \mathrm{d}(\boldsymbol{\alpha}(X_H)) - \mathrm{d}H \\
&= \mathrm{d}(\boldsymbol{\Lambda}(H)).
\end{aligned}$$

Therefore:

$$\begin{aligned}
\boldsymbol{\Lambda}(\{H_1, H_2\}) &= \boldsymbol{\alpha}([X_{H_1}, X_{H_2}]\,) - \{H_1, H_2\} \\
&= \mathrm{d}(\boldsymbol{\alpha}(X_{H_2}))\; X_{H_1} - \mathrm{L}_{X_{H_1}}\boldsymbol{\alpha}\; X_{H_2} - \{H_1, H_2\} \\
&= \{H_1, \boldsymbol{\alpha}(X_{H_2})\} - \mathrm{d}(\boldsymbol{\Lambda}(H_1))\; X_{H_2} - \{H_1, H_2\} \\
&= \{H_1, \boldsymbol{\Lambda}(H_2)\} + \{\boldsymbol{\Lambda}(H_1), H_2\}.
\end{aligned}$$

□

As an immediate consequence of the previous proposition, we note that the time-$t$ flow of a Hamiltonian vector field given by a function $H$ with constant $\boldsymbol{\Lambda}$-derivative is an actionmorphism (it preserves the action form).

**Corollary 2.1** *:*

> *The Hamiltonian vector field $X_H$ of a function with constant $\mathbf{\Lambda}$-derivative is an infinitesimal automorphism of the action form $\boldsymbol{\alpha}$, that is, $\boldsymbol{\alpha}$ is invariant under $X_H$. In fact, the converse is also true. That is:*
>
> $$\mathrm{d}(\mathbf{\Lambda}(H)) = 0 \Leftrightarrow \forall (t,z) \in \mathcal{D}(X_H), \varphi_t^* \boldsymbol{\alpha}(z) = \boldsymbol{\alpha}(z),$$
>
> *where $\mathcal{D}(X_H)$ is the domain of the flow $\varphi$ of $X_H$.*

In symplectic coordinates $(x,y)$, the $\mathbf{\Lambda}$-derivative of a function $H = H(x,y)$ is

$$\mathbf{\Lambda}(H)(x,y) \quad = \quad y \cdot \nabla_y H(x,y) - H(x,y).$$

## 2.4 The interpolation problem

As the time-1 flow of a Hamiltonian vector field is exact symplectic, a natural question arises:

> Given an exact symplectomorphism, is it the time-1 flow of a time-dependent Hamiltonian vector field?

Once we have interpolated our exact symplectomorphism by a time-independent Hamiltonian flow, next question is:

> can we get our Hamiltonian be 1-periodic in time?

This subject has been studied for many authors, and it has many variants. It is a particular case of the more general problem of *inclusion of a map into a flow.* Moser [77] already dealt with this problem when he proven the analyticity of the Birkhoff normal form around a hyperbolic fixed point of an area preserving map. Douady [29] solved the problem in the smooth symplectic case provided our map is given by a generating function and Conley and Zehnder [26] solved it for smooth diffeomorphism of a torus which leave the center of mass fixed. On the other side, Douady [29], Kuksin [55] and Kuksin and Pöschel [56] solved the problem in analytic set up for maps which are close to integrable ones, but in a non-constructive way.

We shall solve the problem in analytic set up around an invariant exact Lagrangian manifold of our symplectomorphism. In fact, we shall solve the first part of the problem. In some cases we can apply a theorem by Pronin and Treschev [86] in order to get the time be periodic. The proof will be constructive.

Although we shall devote Chapter 10 to this subject, we shall explain here the main ideas. The key point is to apply the homotopy method.

### 2.4.1 Set up

Let $F : \mathcal{N} \to \mathcal{N}$ be an exact symplectomorphism, with $pf(F) = S$. We shall try to look for the exact symplectomorphism as the time-1 flow of a time-dependent Hamiltonian vector field. Hence, this problem is related to the determination problem in Section 1.3.

Let $H : \mathcal{N} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian function, $X_t = X_{H_t}$ be the corresponding vector field and $\varphi_t$ be the corresponding flow from $t_0 = 0$ (i.e. $\varphi_t = \varphi_{t,0}$). We would like

$$\varphi_1^* \boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S.$$

In fact, we impose 'a little more', that $\forall t$

$$\varphi_t^* \boldsymbol{\alpha} - \boldsymbol{\alpha} = t\,\mathrm{d}S$$

(this is the idea of a *homotopy method*). That is to say, we want $S_{t,0} = t \cdot S$ (with the notation of Section 2.2).

Then, deriving the homotopy formula,

$$S = \boldsymbol{\Lambda}(H_t) \circ \varphi_t.$$

Therefore, if $H_0$ satisfies

$$S = \boldsymbol{\Lambda}(H_0)$$

and, moreover:

$$0 = \frac{d}{dt}(\boldsymbol{\Lambda}(H_t) \circ \varphi_t),$$

then $H_t$ is a time-dependent Hamiltonian whose time-1 flow is an exact symplectomorphism with primitive function being equal to $S$. Finally,

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\Lambda}(H_t) \circ \varphi_t) &= \mathrm{d}(\boldsymbol{\Lambda}(H_t))(\varphi_t)\,\frac{\partial \varphi_t}{\partial t} + \frac{\partial}{\partial t}(\boldsymbol{\Lambda}(H_t)) \circ \varphi_t \\ &= \{H_t, \boldsymbol{\Lambda}(H_t)\}(\varphi_t) + \frac{\partial}{\partial t}(\boldsymbol{\Lambda}(H_t)) \circ \varphi_t. \end{aligned}$$

Then, we shall impose that our $t$-dependent function $H_t$ satisfies

$$\left\{ \begin{array}{l} S = \boldsymbol{\Lambda}(H_0) \\ \\ B(H_t, H_t) + \frac{\partial}{\partial t}(\boldsymbol{\Lambda}(H_t)) = 0 \end{array} \right. ,$$

where $B$ is the bilinear operator

$$\begin{array}{rccc} B: & \mathcal{F}(\mathcal{N}) \times \mathcal{F}(\mathcal{N}) & \longrightarrow & \mathcal{F}(\mathcal{N}) \\ & & & \\ & (H_1, H_2) & \longrightarrow & \{H_1, \boldsymbol{\Lambda}(H_2)\}, \end{array}$$

and, in particular,

$$B(H, H) = \boldsymbol{\Lambda}_2(H).$$

The expression of $B$ in canonical coordinates is

$$B(H_1, H_2)(x, y) = y^\top \frac{\partial^2 H_2}{\partial x \partial y} \cdot \frac{\partial H_1}{\partial y}^\top - \frac{\partial H_2}{\partial x} \cdot \frac{\partial H_1}{\partial y} - y^\top \frac{\partial^2 H_2}{\partial y^2} \cdot \frac{\partial H_1}{\partial x}^\top.$$

### 2.4.2 An evolution problem

The previous equation only assures that the time-1 flow has primitive function $S$, and this does not determine the symplectomorphism. This is the effect of the fact that our derivation $\mathbf{\Lambda}$ is not invertible (there exists integration 'constants', i.e., functions with vanishing $\mathbf{\Lambda}$-derivative). In fact, we need to solve equations as

$$\mathbf{\Lambda}(H) = S.$$

Suppose that the space of (smooth) functions $\mathcal{F} = \mathcal{F}(\mathcal{N})$ splits as

$$\mathcal{F} \;=\; \ker \mathbf{\Lambda} \oplus \mathbf{\Lambda}(\mathcal{F})$$

(that is $\mathbf{\Lambda}_| := \mathbf{\Lambda}_{|\mathbf{\Lambda}(\mathcal{F})} : \mathbf{\Lambda}(\mathcal{F}) \to \mathbf{\Lambda}(\mathcal{F})$ is an isomorphism). If we define $S_t = \mathbf{\Lambda}(H_t)$, then we must solve the evolution problem

$$\begin{cases} \dfrac{dS_t}{dt} = -\{\mathbf{\Lambda}_|^{-1}(S_t), S_t\} \\ \\ \text{Cauchy's data: } S_0 = S \end{cases} .$$

Of course, we need that $\mathbf{\Lambda}(\mathcal{F})$ be invariant under these operations.

**An iterative method.-** Suppose we know $S_0$ and we want to search for $S_t$ as a development in powers of the time $t$: $S_t = \sum_{k\geq 0} S_k t^k$. Then, $\forall k \geq 0$,

$$S_{k+1} \;=\; \frac{-1}{k+1} \sum_{u+v=k} \{\mathbf{\Lambda}_|^{-1}(S_u), S_v\}.$$

Finally, we must recover $H_t$ from $S_t = \mathbf{\Lambda}(H_t)$, and choose the correct way in order to get our symplectomorphism $F$.

## 2.5 Mechanical systems and variational principles

A classical mechanical system is given by a time-dependent Hamiltonian on the *phase space* $\mathcal{N} = T^*\mathcal{M}$, i.e., on the cotangent bundle of a manifold $\mathcal{M}$, called the *configuration space*. Hence, we need a function

$$H : T^*\mathcal{M} \times \mathbb{R} \longrightarrow \mathbb{R}.$$

Thus, the *energy-momentum* 1-form $\boldsymbol{\alpha} - Hdt$, also called the *Poincaré-Cartan* 1-form, is correctly defined on the extended phase space $T^*\mathcal{M} \times \mathbb{R}$.

### 2.5.1 Continuous variational principles

Given two basic points $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{M}$ and two times $t_0, t_1 \in \mathbb{R}$, let $\Gamma = \Gamma_{(x_0,t_0),(x_1,t_1)}$ be the set of paths $\gamma : [t_0, t_1] \longrightarrow T^*\mathcal{M}$ such that $q\circ\gamma(t_0) = \mathbf{x}_0$ and $q\circ\gamma(t_1) = \mathbf{x}_1$. On $\Gamma$ we define the *action*

$$\mathcal{A}(\gamma) \;=\; \int_\gamma \boldsymbol{\alpha} - Hdt.$$

The next *principle on stationary action* in phase space was formulated by Poincaré [85, 7].

**Proposition 2.3** *:*

> *The path $\gamma$ is a critical point of the functional $\mathcal{A} : \Gamma \to \mathbb{R}$ iff its trajectory is a solution of Hamilton's equations with Hamiltonian $H$.*

The action on a connecting orbit is

$$\mathcal{A}(\gamma) = \int_{t_0}^{t_1} \mathbf{\Lambda}(H_t)(\gamma(t))dt,$$

and we observe that the action on an integral curve $\varphi_{t,t_0}$ is, in fact, the primitive function associated to the corresponding flow:

$$S_{t_1,t_0}(\rho_0) = \int_{t_0}^{t_1} \mathbf{\Lambda}(H_t)(\varphi_{t,t_0}(\rho_0))dt$$

In this context, $\Lambda(H_t)$ is also known as the *elementary action* of the Hamiltonian $H_t$, and it is useful in order to define the Legendre transformation.

### 2.5.2 The variational problem

Other of the subjects of this thesis will be if we can state variational principles for the orbits of an exact symplectomorphism, i.e., to give a discrete version of continuous variational principles. That is to say, we want to state the laws of the *discrete analytical mechanics*. We *avoid* the use of generating functions, because they are not always defined, and its existence imposes serious restrictions to the topology of the configuration space.

For the sake of simplicity, we shall consider a time-periodic mechanical system, that is to say, a Hamiltonian function

$$H : T^*\mathcal{M} \times \mathbb{S}^1 \longrightarrow \mathbb{R},$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\mathcal{M}$ is the configuration space. Let $F = \varphi_{1,0}$ be the time-1 flow (we suppose that it is defined on whole the phase space). It is an exact symplectomorphism, and its primitive function is

$$S(\rho) = \int_0^1 \mathbf{\Lambda}(H_t)(\varphi_{t,0}(\rho))\ dt.$$

Physically speaking, a $F$-chain (see the Sections 5.4 and 8.2) will correspond to an 'orbit' of our Hamiltonian, in which the velocity is rudely changed every period (as in a maneuver). To extreme the action on the space of $F$-chains corresponds to smoothe the sharps. On an orbit, the continuous action and the discrete action coincide. In order to classify the orbits by their discrete extremal character we must compute the index of a certain symmetric matrix. In order to compute this matrix, we need the differential of the Poincaré map, which is easily computed by means of the variational equations of our Hamiltonian vector field.

Of course, if our Hamiltonian is not time-periodic, or simply we consider different time flows, we can do similar considerations. For instance, we can look for orbits connecting two basic points (in the configuration space), periodic orbits, etc, by means of a kind of parallel shooting method. It is like to 'minimize' the maneuvers.

# Chapter 3

# Exact isotropic immersions

Isotropic manifolds and, in particular, Lagrangian manifolds, are objects dynamically interesting. For instance, in *KAM theory* [1], where the invariant tori are Lagrangian (and the low dimensional tori are isotropic), or in *PMA theory* [2], because the stable and unstable (immersed) submanifolds of a hyperbolic fixed point are Lagrangian.

Another of the subjects of this thesis will be the so called *Converse KAM theory* [68], which is a non-perturbative theory about the non-existence of invariant tori. Although an invariant torus can have any dynamics [42], we shall consider only KAM tori, that is, tori whose dynamics are given by rotations. By another result due to Herman [41, 40], any invariant torus for a certain symplectomorphism in which the dynamics is conjugated to an ergodic translation must be isotropic (Lagrangian if its dimension halves the dimension of the phase space).

In this chapter we begin to generalize a result due to Mather [73]. Given an exact symplectomorphism $F$, we can associate to any $F$-invariant exact isotropic immersion a conserved quantity, with the aid of their primitive functions. In Chapters 6 and 9 we shall obtain more information in some special cases, and it will be useful for Converse KAM theory.

## 3.1 Exact isotropic immersions

### 3.1.1 Definitions

An immersion $\nu : \mathcal{P} \to \mathcal{N}$ of a manifold $\mathcal{P}$ into the symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$ is called *isotropic* iff $\nu^*\boldsymbol{\omega} = 0$. If the dimension of $\mathcal{P}$ halves the dimension of $\mathcal{N}$ we shall say that our immersion is *Lagrangian.*

If the symplectic structure is exact, with $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$, we shall say that our isotropic (or Lagrangian) immersion is exact iff there exists a function $l : \mathcal{P} \to \mathbb{R}$ such that $\nu^*\boldsymbol{\alpha} = \mathrm{d}l$. We shall say that $l$ is a primitive function of the immersion, and it is defined up to constants.

[1] by Kolmogorov, Arnold and Moser.
[2] by Poincaré, Melnikov and Arnold.

Of course, we can fit these definitions to immersed submanifolds and (embedded) submanifolds.

**Examples**

1) Given a function $l : \mathbb{R}^d \to \mathbb{R}$, we know that the immersion

$$\begin{array}{rccc} \nu : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \times \mathbb{R}^d \\ & x & \longrightarrow & (x, \nabla l(x)) \end{array}$$

defines a Lagrangian embedding of $\mathbb{R}^d$ into $\mathbb{R}^d \times \mathbb{R}^d$, and its primitive function is $l$:

$$(\nu^*(y\ \mathrm{d}x))_x X_x = (y\ \mathrm{d}x)_{\nu(x)} \nu_*(x) X_x = \nabla l(x) \cdot X_x = \mathrm{d}l(x) X_x,$$

where $x \in \mathbb{R}^d$ and $X_x \in T_x\mathbb{R}^d \simeq \mathbb{R}^d$.

2) The vertical leaves $x = x_0$ ($x_0 \in \mathbb{R}^d$) on $\mathbb{R}^d \times \mathbb{R}^d$ are also exact Lagrangian. If we parametrize them by $\nu(y) = (x_0, y)$ then their primitive functions are $l(y) = 0$.

3) An example of exact Lagrangian submanifold on the cotangent bundle of a manifold is given by its zero-section. In fact, as Weinstein proved [97], this is the universal model of Lagrangian submanifold, on an open neighborhood of it. It is an extension of the Darboux's theorem.

   The leaves of the standard fibration of the cotangent bundle are also exact Lagrangian, and we note that the Liouville form vanish on them.

$\triangleleft$

### 3.1.2 Invariance of isotropic immersions

Given a diffeomorphism $F : \mathcal{N} \to \mathcal{N}$, we shall say that an immersion $\nu : \mathcal{P} \to \mathcal{N}$ is $F$-*invariant* iff there exists an immersion $\bar{f} : \mathcal{P} \to P$ such that $\nu \circ \bar{f} = F \circ \nu$, called the *dynamics* on the immersion. If the immersion is injective, i.e., $\mathcal{P}$ is an immersed submanifold, then the dynamics is also injective. In such a case, the injective immersion is also $F^{-1}$-invariant iff $\bar{f}$ is a diffeomorphism, and we shall say that our injective immersion $\nu$ (or that our immersed submanifold $P$) is $F, F^{-1}$-*invariant*.

**Remarks**

i) For instance, a fundamental domain of an stable manifold of a fixed point is $F$-invariant, but not $F^{-1}$-invariant.

ii) As $F \circ \nu = \nu \circ \bar{f}$ then $\nu^* \circ F^* = \bar{f}^* \circ \nu^*$. In particular, if $F$ is a symplectomorphism on the symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$, then $\bar{f}^* \circ \nu^* \boldsymbol{\omega} = \nu^* \boldsymbol{\omega}$, and the 2-form on $P$ $\nu^* \boldsymbol{\omega}$ is $\bar{f}$-invariant. If, moreover, $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$, we obtain that $\nu^*(F^*\boldsymbol{\alpha} - \boldsymbol{\alpha}) = \bar{f}^* \circ \nu^* \boldsymbol{\alpha} - \nu^* \boldsymbol{\alpha}$ is a closed Pfaffian form on $\mathcal{P}$. If our immersion $\nu$ is exact isotropic, with $\nu^* \boldsymbol{\alpha} = \mathrm{d}l$, this 1-form is also exact: $\bar{f}^* \circ \nu^* \boldsymbol{\alpha} - \nu^* \boldsymbol{\alpha} = \mathrm{d}(l \circ \bar{f}) - \mathrm{d}l$.

$\triangleleft$

If $F$ is an exact symplectomorphism, we can associate a conserved quantity to any $F$-invariant exact isotropic immersion.

**Proposition 3.1** *:*

*Let* $F : \mathcal{N} \to \mathcal{N}$ *be an exact symplectomorphism of a certain exact symplectic manifold* $(\mathcal{N}, \mathrm{d}\boldsymbol{\alpha})$*, with pf*$(F) = S$.
*Let* $\nu : \mathcal{P} \to \mathcal{N}$ *be a* $F$*-invariant exact isotropic immersion of a connected manifold* $P$*, with* $\nu^* \boldsymbol{\alpha} = \mathrm{d}l$*, and* $\bar{f} : \mathcal{P} \longrightarrow \mathcal{P}$ *be its dynamics.*
*We define the function* $\Phi : \mathcal{P} \to \mathbb{R}$ *by*

$$\Phi = S \circ \nu - (l \circ \bar{f} - l).$$

*Then:*

*The function* $\Phi$ *is constant.*

*Proof*:

As

$$\begin{aligned} \mathrm{d}(S \circ \nu) &= \mathrm{d}(\nu^* S) = \nu^* \mathrm{d}S = \nu^*(F^* \boldsymbol{\alpha} - \boldsymbol{\alpha}) = \bar{f}^* \circ \nu^* \boldsymbol{\alpha} - \nu^* \boldsymbol{\alpha} \\ &= \mathrm{d}(l \circ \bar{f}) - \mathrm{d}l, \end{aligned}$$

we reach $\mathrm{d}\Phi = 0$. □

**Remark**

If our immersion is $F, F^{-1}$-invariant and $\Phi'$ is defined similarly by means of $F^{-1}$ and $\bar{f}^{-1}$, we obtain that

$$\Phi' = -\Phi \circ \bar{f}^{-1}.$$

◁

## 3.2 Families of isotropic immersions

Suppose we have a (smooth) family of exact symplectomorphisms

$$\begin{array}{rccc} F: & \mathcal{N} \times \mathbb{R} & \longrightarrow & \mathcal{N} \\ & (z, \epsilon) & \longrightarrow & F_\epsilon(z), \end{array}$$

with *pf*$(F_\epsilon) = S_\epsilon$, and an (smooth) family of invariant exact isotropic immersions of a certain connected manifold $P$

$$\begin{array}{rccc} \nu: & \mathcal{P} \times \mathbb{R} & \longrightarrow & \mathcal{N} \\ & (z, \epsilon) & \longrightarrow & \nu_\epsilon(z). \end{array}$$

Thanks to the result of the previous section, we know that there exists a family of conserved quantities

$$C_\epsilon = S_\epsilon \circ \nu_\epsilon - (l_\epsilon \circ \bar{f} - l_\epsilon),$$

given by the corresponding functions $\Phi_\epsilon$.

We can associate another conserved quantity to the immersion $\epsilon = 0$: the derivative of $\Phi_\epsilon$ respect to $\epsilon$, in $\epsilon = 0$.

**Proposition 3.2** *:*

> *Let $F_\epsilon : \mathcal{N} \to \mathcal{N}$ be a family of exact symplectomorphisms on an exact symplectic manifold $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$, being $S_\epsilon : \mathcal{N} \to \mathbb{R}$ the corresponding family of primitive functions* [3]*.*
> *Let $\nu_\epsilon : \mathcal{P} \to \mathcal{N}$ be a family of $F_\epsilon$-invariant exact isotropic immersions of a connected manifold $\mathcal{P}$, being $l_\epsilon : \mathcal{P} \to \mathbb{R}$ the corresponding family of primitive functions* [4]*, and $\bar{f}_\epsilon : \mathcal{P} \to \mathcal{P}$ be their dynamics.*
> *We shall denote with superscript 1 the derivatives of any of these maps respect to $\epsilon$, in $\epsilon = 0$. Then:*
>
> *the constant function $\Phi^1 : \mathcal{P} \to \mathbb{R}$ (equal to $C^1$) can be written as*
>
> $$\Phi^1(p) = \hat{S}^1(\nu_0(p)) - (\hat{l}^1(\bar{f}_0(p)) - \hat{l}^1(p)),$$
>
> *where*
>
> $$\hat{S}^1(z) \;=\; S^1(z) - \boldsymbol{\alpha}(F_0(z))\; F^1(z)$$
>
> *and*
>
> $$\hat{l}^1(p) \;=\; l^1(p) - \boldsymbol{\alpha}(\nu_0(p))\; \nu^1(p).$$

*Proof*:

We must derive respect to $\epsilon$, in $\epsilon = 0$, the equality

$$C_\epsilon = \Phi_\epsilon(p) \;=\; S_\epsilon(\nu_\epsilon(p)) + l_\epsilon(p) - l_\epsilon(\bar{f}_\epsilon(p)).$$

On one hand,

$$\begin{aligned} C^1 = \Phi^1(p) \;&=\; S^1(\nu_0(p)) + \mathrm{d}S_0(\nu_0(p))\; \nu^1(p) + \\ &\quad\; l^1(p) - l^1(\bar{f}_0(p)) - \mathrm{d}l_0(\bar{f}_0(p))\; \bar{f}^1(p) \\ &=\; S^1(\nu_0(p)) + {F_0}^*\boldsymbol{\alpha}(\nu_0(p))\; \nu^1(p) - \boldsymbol{\alpha}(\nu_0(p))\; \nu^1(p) + \\ &\quad\; l^1(p) - l^1(\bar{f}_0(p)) - {\nu_0}^*\boldsymbol{\alpha}(\bar{f}_0(p))\; \bar{f}^1(p), \end{aligned}$$

and on the other hand,

$$\begin{aligned} {F_0}^*\boldsymbol{\alpha}(\nu_0(p))\; \nu^1(p) \;&=\; \boldsymbol{\alpha}(F_0(\nu_0(p)))\; F_{0*}(\nu_0(p))\; \nu^1(p), \\ {\nu_0}^*\boldsymbol{\alpha}(\bar{f}_0(p))\; \bar{f}^1(p) \;&=\; \boldsymbol{\alpha}(\nu_0(\bar{f}_0(p)))\; \nu_{0*}(\bar{f}_0(p))\; \bar{f}^1(p). \end{aligned}$$

Finally, as $F_\epsilon \circ \nu_\epsilon = \nu_\epsilon \circ \bar{f}_\epsilon$, then:

$$F^1(\nu_0(p)) + F_{0*}(\nu_0(p))\; \nu^1(p) \;=\; \nu^1(\bar{f}_0(p)) + \nu_{0*}(\bar{f}_0(p))\; \bar{f}^1(p),$$

[3] That we suppose smooth, fixing the value of them for a certain point $z \in \mathcal{N}$.
[4] Idem.

and we arrive to the desired equality:

$$\begin{aligned}
C^1 = \Phi^1(p) &= S^1(\nu_0(p)) + \\
&\quad \boldsymbol{\alpha}(F_0(\nu_0(p)))\ F_{0*}(\nu_0(p))\ \nu^1(p) - \boldsymbol{\alpha}(\nu_0(p))\ \nu^1(p) + \\
&\quad l^1(p) - l^1(\bar{f}_0(p)) - \boldsymbol{\alpha}(\nu_0(\bar{f}_0(p)))\ \nu_{0*}(\bar{f}_0(p))\ \bar{f}^1(p) \\
&= S^1(\nu_0(p)) + \boldsymbol{\alpha}(\nu_0(\bar{f}_0(p)))\ (\nu^1(\bar{f}_0(p)) - F^1(\nu_0(p))) + \\
&\quad l^1(p) - l^1(\bar{f}_0(p)) - \boldsymbol{\alpha}(\nu_0(p))\ \nu^1(p) \\
&= S^1(\nu_0(p)) - \boldsymbol{\alpha}(F_0(\nu_0(p)))\ F^1(\nu_0(p)) + \\
&\quad l^1(p) - l^1(\bar{f}_0(p)) + \\
&\quad \boldsymbol{\alpha}(\nu_0(\bar{f}_0(p)))\ \nu^1(\bar{f}_0(p)) - \boldsymbol{\alpha}(\nu_0(p))\ \nu^1(p).
\end{aligned}$$

□

If the same immersion $\nu$ is invariant for the family $F_\epsilon$ of exact symplectomorphisms, then we obtain:

$$C^1 = S^1(\nu(p)) - \boldsymbol{\alpha}(\nu(\bar{f}_0(p)))\ F^1(\nu(p)).$$

In particular, if our family of exact symplectomorphisms is given by the flow of a time independent Hamiltonian vector field, then we obtain that the immersion is contained on an energy level.

**Corollary 3.1** :

*Let $(\mathcal{N}, \mathrm{d}\boldsymbol{\alpha})$ be an exact symplectic manifold.*
*Let $H : \mathcal{N} \to \mathbb{R}$ be a Hamiltonian function, and $\varphi_t$ be the corresponding flow (that we suppose defined $\forall t$).*
*Let $\nu : \mathcal{P} \to \mathcal{N}$ be an exact isotropic immersion, which is invariant for the Hamiltonian flow.*
*Then:*

*$H \circ \nu$ is constant.*

*Proof*:

Let $F_t = \varphi_{t,0}$ be the time-$t$ flow of our hamiltonian. We know that it is exact symplectic and its primitive function is

$$S_t = \int_0^t \boldsymbol{\Lambda}(H) \circ \varphi_t dt.$$

Hence

$$S^1 = \boldsymbol{\Lambda}(H).$$

Therefore:

$$\begin{aligned} C^1 &= S^1(\nu(p)) - \boldsymbol{\alpha}(\nu(\bar{f}_0(p)))\ F^1(\nu(p)) \\ &= \boldsymbol{\Lambda}(H)(\nu(p)) - \boldsymbol{\alpha}(\nu(p))\ X_H(\nu(p)) \\ &= -H(\nu(p)). \end{aligned}$$

□

## 3.3 Two examples in Dynamics

### 3.3.1 Invariant tori

Let $F : \mathcal{N} \to \mathcal{N}$ be an exact symplectomorphism with $pf(F) = S$ of an exact symplectic $d$-manifold $(\mathcal{N}, \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha})$ such that:

- $F$ has an invariant torus of dimension $k \leq d$, given by the $\mathbb{Z}^k$-periodic immersion $\nu : \mathbb{R}^k \to \mathcal{N}$ (i.e., $\nu$ is 1-periodic in all its variables $\theta = (\theta_1, \ldots, \theta_k)$);
- the dynamics on the torus is an ergodic translation (or shift) by $\boldsymbol{\omega}$, $R_\omega$ ($\omega$ is called the rotation vector of the torus): $R_\omega(\theta) = \theta + \omega$.

As the translation is ergodic ($\forall k \in \mathbb{Z}^d, k\omega \notin \mathbb{Z}$) then it is minimal (all the orbits are dense in the $k$-torus), and the immersion must be isotropic, as Herman proved [40, 41]. As $\nu$ is an isotropic immersion of $\mathbb{R}^k$ it is exact: $\nu^*\boldsymbol{\alpha} = \mathrm{d}l$ for some function $l : \mathbb{R}^k \to \mathbb{R}$. By periodicity, this function is

$$l(\theta) = a \cdot \theta + \bar{l}(\theta),$$

where $a \in \mathbb{R}^k$ and $\bar{l}$ is a $\mathbb{Z}^k$-periodic function.

We know that the function

$$\Phi(\theta) \;=\; S(\nu(\theta)) + l(\theta) - l(\theta + \omega)$$

is equal to a constant $C \in \mathbb{R}$. Hence, $\forall \theta \in \mathbb{R}^k$:

$$\begin{aligned} \sum_{i=1}^{q} S(F^{i-1}(\nu(\theta)) &= \sum_{i=1}^{q} S(\nu(\theta + (i-1)\omega)) \\ &= \sum_{i=1}^{q} (\Phi(\nu(\theta + (i-1)\omega)) \ - l(\theta) + l(\theta + q\omega)) \\ &= qC + a \cdot (\theta + q\omega) + \bar{l}(\theta + q\omega) - a \cdot \theta - \bar{l}(\theta) \\ &= q(C + a \cdot \omega) + \bar{l}(\theta + q\omega) - \bar{l}(\theta), \end{aligned}$$

and then

$$\lim_{q \to \infty} \frac{1}{q} \sum_{i=1}^{q} S(F^{i-1}(\nu(\theta)) \;=\; C + a \cdot \omega.$$

**Remarks**

i) The best approximations to the value $C + a \cdot \omega$ by means of the averages

$$\frac{1}{q}\sum_{i=1}^{q} S(F^{i-1}(\nu(\theta))$$

are given by the best approximations of the rotation vector $\omega$ by rational vectors $\frac{p}{q}$, where $p \in \mathbb{Z}^k, q \in \mathbb{N}^*$. In fact, the error is given by

$$\begin{aligned}\frac{1}{q}(\bar{l}(\theta + q\omega) - \bar{l}(\theta)) &= \frac{1}{q}(\bar{l}(\theta + (q\omega - p)) - \bar{l}(\theta)) \\ &= \frac{1}{q}\frac{\partial \bar{l}}{\partial \theta}(\theta + t(q\omega - p)) \cdot (q\omega - p) \\ &= \frac{\partial \bar{l}}{\partial \theta}(\theta + t(q\omega - p)) \cdot (\omega - \frac{p}{q}),\end{aligned}$$

where $t \in [0, 1]$ is given by the Mean Value Theorem applied to the function $\bar{L}(t) = \bar{l}(\theta + t(q\omega - p))$.

In the 1-dimensional case ($k = 1$), i.e., if we have an invariant circle whose dynamics is a rotation by $\omega \in \mathbb{R}$, then the best approximations are given by the convergents of the corresponding continuous fraction.

ii) As the translation by $\omega$ is ergodic, then the average on the orbit is given by an integral:

$$\lim_{q\to\infty}\frac{1}{q}\sum_{i=1}^{q} S(F^{i-1}(\nu(\theta)) = \int_{\mathbb{T}^k} S\circ\nu.$$

iii) If we have a family of exact symplectomorphisms $F_\epsilon : \mathcal{N} \to \mathcal{N}$ and a family of isotropic immersions $\nu_\epsilon : \mathbb{R}^k \to \mathcal{N}$ giving invariant tori for each $\epsilon$ with the same rotation vector, then we can obtain a similar result:

$$\lim_{q\to\infty}\frac{1}{q}\sum_{i=1}^{q} \hat{S}^1(F_0{}^{i-1}(\nu_0(\theta))) = C^1 + a^1 \cdot \omega.$$

◁

### 3.3.2 Stable and unstable manifolds of a hyperbolic fixed point

Let $F : \mathcal{N} \to \mathcal{N}$ be an exact symplectomorphism, with $pf(F) = S$, and $\nu : P \to \mathcal{N}$ be a $F$-invariant exact isotropic immersions ,with $\nu^*\boldsymbol{\alpha} = \mathrm{d}l$ and dynamics $\bar{f} : P \to P$. If $\nu$ contains a fixed point of $F$, $z_0 = \nu(p_0)$, then the conserved quantity is $C = S(z_0)$.

This is the case of the stable and unstable submanifolds of an elliptic-hyperbolic fixed point $z_0$. They are immersed submanifolds $\mathcal{W}^{s,u}$ given by injective immersions

$\nu^{s,u} : \mathbb{R}^k \to \mathcal{N}$ ($k = d$ if the point is hyperbolic) such that $\nu^{s,u}(0) = z_0$ and $\mathrm{d}\nu^{s,u}(0)(\mathbb{R}^k)$ is the tangent space to $\mathcal{W}^{s,u}$ at $z_0$. Moreover, they are isotropic (Lagrangian, if $k = d$), as can easily proved. For instance, in the stable case, as the dynamics is 'contractive':

$$\nu^{s*}\boldsymbol{\omega} = (F^n \circ \nu^s)^* \boldsymbol{\omega} \longrightarrow 0, \text{ where } n \to \infty.$$

As an easy example, for any point $s \in \mathbb{R}^k$ on the stable manifold, we have that

$$\lim_{q \to +\infty} \frac{1}{q} \sum_{i=1}^{q} S(F^{i-1}(\nu^s(s)) \quad = \quad C.$$

This facts were already know by Poincaré [85] for Hamiltonian flows, and they have been used by Tabacman [93] for the computation of homoclinic orbits and by Delshams and Ramírez-Ros [28] for the definition of a Melnikov potential for the study of the splitting of separatrices. Last authors use similar results to the propositions in the Sections 3.1 and 3.2. Easton [30] had already used the primitive function in order to define a Melnikov potential, but he imposed more restrictions on the Lagrangian manifolds.

## 3.4 Converse KAM theory

While KAM theory obtain many invariant tori for symplectomorphism which are near enough to an integrable one (foliated by invariant tori), Converse KAM theory provides criteria for non existence of such tori. These tori are horizontal, in the sense that we have chosen a direction on our phase space and those tori are transversal to those directions. For instance, if our phase space is the annulus $\mathbb{T}^d \times \mathbb{R}^d$, the direction is in fact given by the distinction between $x$ (angles) and $y$ (actions) coordinates. Our tori are Lagrangian and they are given by

$$y = a + \nabla l(x),$$

where $l$ is a 1-periodic function in all its variables and $a \in \mathbb{R}^d$ is the average of the graph.

Converse KAM theory will be another subject of this thesis. The name have been taken from a paper by MacKay, Meiss and Stark, *Converse KAM theory for symplectic twist maps* [68]. In that paper they found a non-existence criterion of invariant tori through a point of the phase space. They obtained that if a segment of orbit through that point does not satisfy a certain local condition (a certain symmetric matrix is not positive definite) then the point does not belong to a invariant torus. Curiously, that local condition comes from the existence of a global function, the generating function. Although the existence of this function is very useful in many cases, and have been proved when our symplectomorphism satisfy some strong positiveness conditions, there are many cases in which it does not exist or it is not clear.

We shall always attack the problems by means of the primitive function of our symplectomorphism which always exists (well, at least if we work in $\mathbb{R}^d \times \mathbb{R}^d$). For instance, the first proposition in the previous section is a generalization of a result due to Mather [73], obtained by him for sympletic twist maps having an invariant Lagrangian graph (the existence of a global generating function was needed).

# Part II

# ON THE STANDARD SYMPLECTIC MANIFOLD

# Chapter 4

# Symplectomorphisms and generating functions

Along this part we shall work on the standard symplectic manifold $\mathbb{R}^{2d}$. In this chapter we follow the main topics of Chapter 1, but we do it in an independent mode. We recall how to construct symplectomorphisms from generating functions,and we relate them with the primitive functions, which always exist.

In the second part of this chapter, we solve formally the determination problem in an special case, when the $x$-axis is fixed. As we shall see later this can be enough for our purposes, if we already know that such a symplectomorphism exists (for instance, if it is given by a Hamiltonian flow).

## 4.1 Symplectomorphisms

We consider $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ endowed with the *position-momentum* coordinates [1]

$$z = (x, y) = (x^1, \dots, x^d, y^1, \dots, y^d).$$

Any diffeomorphism $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ will be represented as

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} .$$

Moreover, we shall write

$$\mathrm{D}F(z) \quad = \quad \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

(i.e.: $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial y}$, etc).

[1] We recall that the standard symplectic structure on $\mathbb{R}^{2d}$ is is given by $\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x$, and it is exact: $\boldsymbol{\alpha} = y\ \mathrm{d}x$ is the action form. However, we shall not use this language along this part.

### 4.1.1 The symplectic group

We note any matrix $M$ of $M_{2d}(\mathbb{R})$ by $d\times d$ blocks:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $J$ be the symplectic matrix

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},$$

where $I_d$ is the $d\times d$-identity matrix. $Sp(2d)$ is the symplectic group of $\mathbb{R}^{2d}$ (subgroup of $GL(2d,\mathbb{R})$), that is to say, the set of matrices $M \in M_{2d}(\mathbb{R})$ such that [2] $M^\top JM = J$. Then:

$$\begin{array}{rcl} M \in Sp(2d) & \Leftrightarrow & M^\top JM = J \Leftrightarrow A^\top C = C^\top A, B^\top D = D^\top B, A^\top D - C^\top B = I_d, \\ \Updownarrow & & \\ M^\top \in Sp(2d) & \Leftrightarrow & MJM^\top = J \Leftrightarrow AB^\top = BA^\top, CD^\top = DC^\top, AD^\top - BC^\top = I_d. \end{array}$$

Moreover, if $M \in Sp(2d)$, then $|M| = 1$ and

$$M^{-1} = \begin{pmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{pmatrix}.$$

### 4.1.2 Exactness equations

A diffeomorphism $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a *symplectomorphism* iff

$$\forall z \in \mathbb{R}^{2d},\ \mathrm{D}F(z) \in Sp(2d).$$

The *exactness equations* associated to $F$ are the Pfaffian system

$$\left\{ \begin{array}{l} \dfrac{\partial S}{\partial x}(x,y) = g(x,y)^\top \dfrac{\partial f}{\partial x}(x,y) - y^\top \\ \dfrac{\partial S}{\partial y}(x,y) = g(x,y)^\top \dfrac{\partial f}{\partial y}(x,y) \end{array} \right. .$$

Then, since $F$ is symplectic, the *integrability conditions* of our system are satisfied and these equations define a function

$$S : \mathbb{R}^{2d} \to \mathbb{R}$$

related with $F$, called its *primitive function*. Of course, it is defined up to constants but, anyway, we shall write $\mathit{pf}(F) = S$. We remark that we can not recover $f$ and $g$ from $S$. We need more information, because we must solve a system of p.d.e..

[2] While $^\top$ means the transpose, $^{-\top}$ will mean the transpose of the inverse.

### 4.1.3 Lifts and vertical translations

For instance, all the diffeomorphisms of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \phi(x) \\ \mathrm{D}\phi(x)^{-\top} y \end{pmatrix},$$

where $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism, are symplectomorphisms and have primitive function equal to zero. These symplectomorphisms are the *lifts* of diffeomorphisms $\phi$, and they are represented by $\hat{\phi}$. It is easy to see that if we compose on the left our initial symplectomorphism $F$ with a lift $L = \hat{\phi}$ then we obtain another symplectomorphism $\bar{F} = L \circ F$, with the same primitive function. Since our symplectomorphism $\bar{F}$ is given by

$$\begin{cases} \bar{f}(x,y) = \phi(f(x,y)) \\ \bar{g}(x,y) = \mathrm{D}\phi(f(x,y))^{-\top} g(x,y) \end{cases}$$

then

$$\begin{aligned} \bar{g}(x,y)^\top \frac{\partial \bar{f}}{\partial x}(x,y) - y^\top &= g(x,y)^\top \mathrm{D}\phi(f(x,y))^{-1} \mathrm{D}\phi(f(x,y)) \frac{\partial f}{\partial x}(x,y) - y^\top \\ &= \frac{\partial S}{\partial x}(x,y) \end{aligned}$$

and

$$\bar{g}(x,y)^\top \frac{\partial \bar{f}}{\partial y}(x,y) = \frac{\partial S}{\partial y}(x,y).$$

A (symplectic) vertical translation is defined by means of a function $l : \mathbb{R}^d \to \mathbb{R}$, it is denoted by $\tau = \tau_{\nabla l}$ and it is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y + \nabla l(x) \end{pmatrix}.$$

It is a symplectomorphism and its primitive function is just $l$ (it is a function which only depends on the $x$-variables).

### 4.1.4 Monotonicity

We shall say that our diffeomorphism $F$ is *monotone* iff

$$\forall z \in \mathbb{R}^{2d} \ |B(z)| \neq 0.$$

If $F$ is a monotone symplectomorphism, then the matrices $B^{-1}(z)A(z)$ and $D(z)B^{-1}(z)$ are symmetric. Following [40, 41], we shall say that $F$ is *monotone positive* iff some of the next two conditions is verified:

($+_a$) $\forall z \in \mathbb{R}^{2d}$ $B^{-1}(z)A(z)$ is positive definite;

($+_d$) $\forall z \in \mathbb{R}^{2d}$ $D(z)B^{-1}(z)$ is positive definite.

We shall distinguish both types of monotone positiveness writing $(+_a)$or $(+_d)$. We can define monotone negativeness in the same way.

The symmetric matrix

$$T(z) = \frac{1}{2}(B(z) + B(z)^\top)$$

will be called the *torsion* of $F$ at the point $z$. If it is positive definite for all the points, we shall say that our diffeomorphism *has positive torsion*. If the torsion is uniformly positive definite, we shall say that $F$ is a *twist map* and, as Avez proved in [14] (see also [68]), the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$$

is a diffeomorphism on $\mathbb{R}^{2d}$.

**Geometrical meaning for $d = 1$.-** We shall consider two geometric features:

- the transformation of vertical and horizontal vectors by the tangent map associated to our symplectomorphism:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix};$$

- the transformation of the vertical and horizontal foliations, which are composed by the leaves $\{x = x_0\}$ and $\{y = y_0\}$, respectively.

We shall consider three cases:

1. **Positive torsion**: $b > 0$.
   - The vertical vector $(0, 1)$ tilts to the right.
   - The leaves of the vertical foliation are transformed in graphs over $x$, which are transversal to such a foliation.
2. **Monotone $(+_d)$**: $\frac{d}{b} > 0$.
   - The vertical vector $(0, 1)$ tilts to the right-up if $b > 0$ and to the left-down if $b < 0$.
   - The leaves of the vertical foliation are transformed in graphs of increasing functions over $x$, being transversal to the vertical and horizontal foliations.
3. **Monotone $(+_a)$**: $\frac{a}{b} > 0$.
   - If $b > 0$ $(b < 0)$, the vertical and horizontal vectors, $(0, 1)$ and $(1, 0)$, tilt to the right (left).
   - If $b > 0$ $(b < 0)$ the vertical and horizontal leaves are transformed in graphs over $x$.

# 4.2 Generating functions

Sometimes, a symplectomorphism is given by a generating function. Here, we shall recall two examples. While for the definition of the Lagrangian generating function we need $F$ be monotone ($|B| \neq 0$), for the Hamiltonian generating function we need $|D| \neq 0$. Although these conditions are enough in order to define locally our symplectomorphism, we shall do global definitions.

## 4.2.1 Lagrangian generating functions

Suppose $F$ is a monotone symplectomorphism, and $pf(F) = S$. We shall say that it is *strongly monotone* iff

$$\forall x \in \mathbb{R}^d, f(x, \cdot) : \mathbb{R}^d \to \mathbb{R}^d \text{ is a diffeomorphism.}$$

This is the case when $F$ is a symplectic twist map.

Let $\varphi = \varphi(x, x')$ be its inverse, i.e., $\forall x, x' \in \mathbb{R}^d$

$$x' = f(x, \varphi(x, x')).$$

We define the function $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$L(x, x') = S(x, \varphi(x, x')).$$

Therefore, applying the exactness equations, we reach to the relations

$$\begin{cases} y = -\nabla_x L(x, x') \\ \\ y' = \nabla_{x'} L(x, x') \end{cases}.$$

The function $L$ is called a *(global) Lagrangian generating function* of $F$.

Hence, the relationship between the Lagrangian generating function and the primitive function is given by

$$S\left(x, -\nabla_x L(x, x')\right) \;=\; L(x, x').$$

Moreover, the second derivatives of $L$ are given by

$$\frac{\partial^2 L}{\partial x^2} = B^{-1}A, \; \frac{\partial^2 L}{\partial x' \partial x} = -B^{-1}, \; \frac{\partial^2 L}{\partial x'^2} = DB^{-1}.$$

**Remark**

Notice that this expressions appear in Section 4.1.4, in the definitions about monotonicity. ◁

### 4.2.2 Hamiltonian generating functions

As before, if

$$\forall x \in \mathbb{R}^d, g(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^d \text{ is a diffeomorphism.}$$

and $\psi = \psi(x, y')$ is its inverse, i.e., $\forall x, x' \in \mathbb{R}^d$

$$y' = g(x, \psi(x, y')),$$

then we define a function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$H(x, y') = y' \cdot f(x, \psi(x, y') - S(x, \psi(x, y')).$$

Therefore, applying the exactness equations, we reach to the relations

$$\begin{cases} y = \nabla_x H(x, y') \\ x' = \nabla_{y'} H(x, y') \end{cases}.$$

The function $H$ is called a *(global) Hamiltonian generating function* of $F$.

Therefore, the relationship between the Hamiltonian generating function and the primitive function is given by

$$S\left(x, \nabla_x H(x, y')\right) \;=\; y' \cdot \nabla_{y'} H(x, y') - H(x, y').$$

## 4.3 Determination of a symplectomorphism

As we have recalled, we can determine a symplectomorphism by means of a generating function, but this is not always possible. On the other side, the primitive function always exists. But, as we have seen in Section 4.1.2, if we start from it we need some additional information.

In this section we shall see that it is possible to recover an exact symplectomorphism when it fixes the zero-section $\{y = 0\}$, using the primitive function and the dynamics on the zero-section. It is useful when one obtains normal forms (Appendix F) and it could be useful in order to obtain different dynamics around an invariant Lagrangian manifold.

In fact, our assumptions on our symplectomorphism are not so restrictive, and our manifold is a 'formal' cotangent bundle. Weinstein's theorems [98] let us to send via a symplectomorphism a certain neighborhood of any Lagrangian manifold onto a neighborhood of the zero-section of its cotangent bundle. Moreover, using a generalized Poincaré's lemma, he also proved that if our Lagrangian manifold is exact then the symplectomorphism is also exact (between two different manifolds, of course) [3]. Finally, we shall suppose that all the points of the zero-section are fixed. If not, we must compose on the left with the lift of the diffeomorphism on the zero-section.

[3] For these results and their applications to the construction of Morse families see [98, 61].

### 4.3.1 Set up

We shall adopt a formal point of view, in order to understand the nature of the problem. Then, we assume that:

- our manifold $\mathcal{M}$ is $\mathbb{R}^d$;
- the primitive function $S$ is a formal series in $y$:

$$S(x,y) = \sum_n s_n(x) y^n$$

  where the $s_n$ are functions (we use multi-index notation: $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$),
- and all the points of the zero-section are fixed:

$$f(x,0) = x, \;\; g(x,0) = 0.$$

Hence, we want to recover our symplectomorphism $F = (f,g)$ looking for expressions of the form:

$$\begin{cases} f(x,y) = \sum_n f_n(x) y^n \\ \\ g(x,y) = \sum_n g_n(x) y^n \end{cases},$$

where the $f_n$ and $g_n$ are vector functions:

$$f_n = (f_n^1, \ldots, f_n^d)^\top, \;\; g_n = (g_n^1, \ldots, g_n^d)^\top,$$

being $f_0 = x$ and $g_0 = 0$.

### 4.3.2 Iterative process

From the exactness equations we can obtain the relationship between the terms of $f$,$g$ and $S$. In the next formulas, $\sum_i$ means $\sum_{i=1}^d$ and $u, v \in \mathbb{N}^d$ are multi-indices. Firstly, since

$$\begin{aligned} \frac{\partial S}{\partial x_j}(x,y) &= \sum_n \frac{\partial s_n}{\partial x_j}(x) y^n \\ &= \sum_i g^i(x,y) \frac{\partial f^i}{\partial x_j}(x,y) - y_j \\ &= \sum_i \left( \left( \sum_n g_n^i(x) y^n \right) \left( \sum_n \frac{\partial f_n^i}{\partial x_j}(x) y^n \right) \right) - y_j \\ &= \sum_n \sum_i \sum_{u+v=n} \left( \frac{\partial f_u^i}{\partial x_j}(x) g_v^i(x) \right) y^n - y_j, \end{aligned}$$

then $\forall n \in \mathbb{N}^d$, $\forall j = 1 \div d$

$$\frac{\partial s_n}{\partial x_j}(x) = \sum_i \sum_{u+v=n} \frac{\partial f_u^i}{\partial x_j}(x) g_v^i(x) - \delta_{ne_j},$$

where $\delta$ is the Kronecker's delta. Secondly, since

$$\begin{aligned}
\frac{\partial S}{\partial y_j}(x,y) &= \sum_n (n_j+1) s_{n+e_j}(x) y^n \\
&= \sum_i g^i(x,y) \frac{\partial f^i}{\partial y_j}(x,y) \\
&= \sum_i \left(\sum_n g_n^i(x) y^n\right)\left(\sum_n (n_j+1) f_{n+e_j}^i(x) y^n\right) \\
&= \sum_n \left(\sum_i \sum_{u+v=n} (u_j+1) f_{u+e_j}^i(x) g_v^i(x)\right) y^n,
\end{aligned}$$

then

$$(n_j+1) s_{n+e_j}(x) = \sum_i \left(\sum_{u+v=n} (u_j+1) f_{u+e_j}^i(x) g_v^i(x)\right).$$

So then:

- the function $s_0$ is constant, and we can suppose that this constant is zero;
- the functions $s_{e_i}$ vanish.

Therefore, the primitive function verifies

$$\mathrm{D}S(x,0) = 0,$$

and, in particular, is constant (null) on $\{y=0\}$.

In order to find the $x$-functions $f_n$ and $g_n$, we have to solve these equations recurrently by increasing orders. The order 1 equations are, $\forall i, j = 1 \div d$

$$\left\{\begin{array}{l} g_{e_i}^j = \delta_{ij} \\ f_{e_i}^j = (1+\delta_{ij})\ s_{e_i+e_j} \end{array}\right. .$$

Now, we suppose that we already know the terms of order $k-1$ and we have to obtain the terms of order $k$. The equations are, $\forall |n| = k$, $\forall j = 1 \div d$:

$$\left\{\begin{array}{l} g_n^j = G_n^j \\ n_j f_n^j + (n_j+1) \sum_{i \neq j} f_{n-e_i+e_j}^i = F_n^j \end{array}\right. ,$$

where the terms $G_n$ are computed from terms of lower order and the $F_n$ depend on, moreover, the $g_n$. We have obtained a linear system with natural coefficients for the $f_n$. We are going to solve it.

### 4.3.3 Solving the linear systems

Let $N = (N_1, \ldots, N_d)$ be a multi-index subscript (with $|N| = k$) and let $J = 1 \div d$ be a superscript. We want to know how many equations the corresponding $f_N^J$ contain. Every equation is identified by a subscript $n$ and a superscript $j$, and it is written as

$$\sum_i (n_j - \delta_{ij} + 1) f^i_{n-e_i+e_j} \quad = \quad F^j_n.$$

Since

$$f^i_{n-e_i+e_j} = f^J_N \;\Rightarrow\; i = J,\; n = N + e_J - e_j,$$

then $f^J_N$ only appears at the $d$ equations

$$\sum_i (N_j + \delta_{jJ} - \delta_{ij}) f^i_{N+e_J-e_i} \quad = \quad F^j_{N+e_J-e_j},$$

where $j = 1 \div d$.

Notice that all the terms $f$ in these equations are of the type $f^i_{N+e_J-e_i}$, with $i = 1 \div d$. Hence, the terms $f_n$ appear in $d \times d$-blocks, and we have to solve the corresponding linear sub-systems. If any subscript has a negative component, we assume that the corresponding $F$ is equal to 0, and we also deduce that the corresponding $f$ is equal to 0. Adding the $d$ equations,

$$\sum_j F^j_{N+e_J-e_j} = \sum_i \sum_j (N_j + \delta_{jJ} - \delta_{ij}) f^i_{N+e_J-e_i} = |N| \sum_i f^i_{N+e_J-e_i},$$

where $|N| = \sum_i N_i$, and we obtain that

$$S^J_N := \sum_i f^i_{N+e_J-e_i} = \frac{1}{|N|} \sum_j F^j_{N+e_J-e_j}.$$

Finally, since

$$F^J_N = \sum_i (N_J + 1 - \delta_{iJ}) f^i_{N+e_J-e_i} = (N_J + 1) S^J_N - f^J_N,$$

we get

$$f^J_N = (N_J + 1) S^J_N - F^J_N.$$

### 4.3.4 Statement of the result

What we have proved in the last paragraphs can be summarized as follows.

**Theorem 4.1** *:*

*Let $F$ be a 'formal' symplectomorphism on $\mathbb{R}^{2d}$ given by*

$$\begin{cases} f(x,y) = x + \displaystyle\sum_{|n|\geq 1} f_n(x) y^n \\ \\ g(x,y) = \displaystyle\sum_{|n|\geq 1} g_n(x) y^n \end{cases},$$

*being*

$$S(x,y) = \sum_n s_n(x) y^n$$

*its primitive function. We take $f_0(x) = x$ and $g_0(x) = 0$. Then:*

- *The function $s_0$ is constant and the functions $s_{e_i}$ vanish, i.e.,*

$$\mathrm{D}S(x,0) = 0.$$

- *We can recover the $x$-functions $f_n$ and $g_n$ from the $x$-functions $s_n$ by means of the next recurrence:*
  - *Step 1: $\forall i,j = 1 \div d$:*

$$\left\{\begin{array}{l} g^j_{e_i} = \delta_{ij} \\ \\ f^j_{e_i} = (1+\delta_{ij})\ s_{e_i+e_j} \end{array}\right. .$$

  - *Step $k>1$: $\forall |n| = k,\ \forall j = 1 \div d$:*

$$\left\{\begin{array}{l} g^j_n = G^j_n \\ \\ f^j_n = (n_j+1) S^j_n - F^j_n \end{array}\right. ,$$

*where*

$$G^j_n = \frac{\partial s_n}{\partial x_j}(x) - \sum_i \sum_{\substack{u+v=n \\ u\neq 0}} \frac{\partial f^i_u}{\partial x_j}(x) g^i_v(x),$$

$$F^j_n = (n_j+1) s_{n+e_j}(x) - \sum_i \sum_{\substack{u+v=n \\ |v|>1}} (u_j+1) f^i_{u+e_j}(x) g^i_v(x)$$

*and*

$$S^j_n = \frac{1}{k} \sum_i F^i_{n+e_j-e_i}.$$

**Examples**

1) If we work on $\mathbb{T}^d \times \mathbb{R}^d$, the functions $f_n$, $g_n$ and $s_n$ are 1-periodic in all its variables for $|n| > 0$. An example corresponds to the case in which the zero-section is invariant and its dynamics is given by a shift $x \to x + \omega$.

2) Another example is the case in which the dynamics on the zero-section is given by $\phi(x) = \Lambda x$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ and, for instance, $|\lambda_i| < 1$, $\forall i = 1 \div d$. (i.e., the origin is a hyperbolic fixed point and the zero-section is the corresponding stable manifold).

◁

**Remarks**

i) The condition $\forall x \in \mathbb{R}^d, \mathrm{D}S(x,0) = 0$ is necessary and sufficient for the zero-section to be fixed. A similar condition also works for an exact Lagrangian invariant graph for an exact symplectomorphism defined on a cotangent bundle. In particular, we can associate a conserved quantity to such a graph.

ii) If the points of the zero-section are not fixed, and we want to solve the problem directly, then the linear systems are more difficult. Let $\phi(x) = f(x,0)$ be the dynamics on the zero-section. The order 1 equations are, $\forall i,j = 1 \div d$:

$$\begin{cases} \sum_l \frac{\partial \phi_l}{\partial x_j} g^l_{e_i} = \delta_{ij} \\ \sum_l f^l_{e_i} g^l_{e_j} = (1+\delta_{ij})\, s_{e_i+e_j} \end{cases} .$$

The order $k$ equations are $\forall |n| = k, \forall j = 1 \div d$:

$$\begin{cases} \sum_i \frac{\partial \phi_i}{\partial x_j} g^i_n = G^j_n \\ \sum_{i,l} (n_j - \delta_{lj} + 1) f^i_{n-e_i+e_j} g^i_{e_l} = F^j_n \end{cases} .$$

In order to solve these equations, we need, of course,

$$|\mathrm{D}\phi(x)| \neq 0.$$

iii) Following the calculations we can obtain the known normal forms around invariant tori and hyperbolic points, as we have made in Appendix F. We can also obtain normal forms around lower dimensional hyperbolic tori, but we need some reducibility hypotheses.

iv) We must prove the analyticity of the expansions. Instead doing this, we shall obtain the symplectomorphism as the time-1 flow of an analytic Hamiltonian.

◁

## 4.4 Primitive function versus generating functions

As we have said, not all the symplectomorphisms can be generated by a generating function, specially by the Lagrangian ones. For instance, the 'naïve' integrable symplectomorphism on the annulus $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ given by

$$\begin{cases} x' = \omega + x + y^2 \pmod 1 \\ y' = y \end{cases}$$

can not be defined by a Lagrangian generating function, even in a neighborhood of the zero-section, which is the invariant curve we are interested in. If we look for the Hamiltonian generating function, we obtain

$$H(x, y') = (\omega + x)y' + \frac{1}{3}y'^3,$$

and this function is not well defined on the annulus. Its primitive function is

$$S(x, y) = \frac{2}{3}y^3.$$

The method we have introduced allow us to construct any dynamics around a zero-section that we keep fixed. The method can be carried out with the aid of a computer, and, if our basic manifold is a torus, we should perform an algebraic manipulator of Fourier-Taylor series. Moreover, the algorithm is a simple iteration, and we do not have to apply the Implicit Function Theorem, as if we use some kind of generating function. On the other side, we shall also see that any of these dynamics can be generated by a Hamiltonian flow.

As we shall see, the primitive function is nearer to the Lagrangian generating function than the Hamiltonian one. This is due to the choice of privileged directions in our phase space: the vertical ones. Notice that the zero-section is horizontal, that is, it is transversal to the vertical directions, as any graph. If we had prefered the horizontal directions, other kind of primitive function could be defined, and it would be nearer the Hamiltonian generating function (see Appendix G).

# Chapter 5

# Variational principles

We shall consider different variational principles for different 'objects' (fixed points, periodic orbits, orbits) of a certain symplectomorphism $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$.

First, we recall the Lagrange and Hamilton variational principles (if the corresponding generating functions can be defined), and some variational principles using the primitive function: the Poincaré variational principles. After this, we construct the variational principles by restriction of the action to a certain submanifold. The action will be defined by means of the primitive function. This idea was already used by Moser [79] for the search of fixed points.

Although variational principles are a very powerful tool in order to look for certain orbits (for instance, in Aubry-Mather theory [76]), most of the results need the existence of a global generating function (mainly the Lagrange one). We have used the primitive function, which is a global function that always exists. We shall not prove existence theorems of fixed points, homoclinic orbits, etc. (This is the usefulness of the existence of a global generating function, for instance), but we shall use these variational principles in order to obtain information about a given orbit.

Last section is devoted to the the invariance of the extremal character under different canonical transformations: the lifts and the vertical translations. This is connected with the election of privileged directions on our phase space.

## 5.1 Lagrange, Hamilton and Poincaré variational principles

We shall look for fixed points and orbits. In the second case, we shall define the actions in a formal way, and they will be applied to bisequences of points:

- $X = (x_k)_{k \in \mathbb{Z}}$ (configurational bisequence),
- $Z = (z_k)_{k \in \mathbb{Z}}$, where $z_k = (x_k, y_k)$ (complete bisequence).

**Remarks**

i) We can define the actions on finite sequences, fixing the initial and final '$x$'.

ii) It is possible to get the actions in order to look for periodic orbits.

iii) In both cases, we can modify the actions if $F$ is a lift of a symplectomorphism on $\mathbb{T}^d \times \mathbb{R}^d$ or $\mathbb{T}^d \times \mathbb{T}^d$, and we look for periodic orbits of a certain rotation vector.

◁

These are the discrete versions of Lagrange and Hamilton variational principles for the orbits of a Lagrangian and Hamiltonian system. While the Lagrangian 'lives' on the configuration space, the Hamiltonian 'lives' on the phase space of positions and momentums.

- **Lagrange variational principle**
  Let $L$ be the Lagrangian generating function. (We need $\frac{\partial f}{\partial y}$ to be non singular).
  - The fixed points correspond with the stationary points of the action

$$\boldsymbol{l}(x) = L(x, x).$$

  - The configurational orbits correspond with the stationary configurational bisequences of the action

$$\boldsymbol{L}(X) = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}).$$

- **Hamilton variational principle**
  Let $H$ be the Hamiltonian generating function. (We need $\frac{\partial g}{\partial y}$ to be non singular).
  - The fixed points correspond with the stationary points of the action

$$\boldsymbol{h}(x, y) = xy - H(x, y).$$

  - The orbits correspond with the stationary bisequences of the action

$$\boldsymbol{H}(Z) = \sum_{k \in \mathbb{Z}} (x_k y_k - H(x_k, y_{k+1})).$$

Poincaré used variational principles in order to look for periodic orbits of systems related with celestial mechanics. He considered the Poincaré maps of a certain Hamiltonian system, and he looked for fixed points of this map. The primitive function arised on them.

- **First Poincaré variational principle**
  (We suppose $\frac{\partial f}{\partial x}$ to be non singular).
  - The fixed points correspond with the stationary points of the action (see [85, 80])

$$\boldsymbol{p}(x, y) = y(x - f(x, y)) + S(x, y).$$

– The orbits correspond with the stationary bisequences of the action

$$\boldsymbol{P}(Z) = \sum_{k\in\mathbb{Z}} (y_{k+1}(x_{k+1} - f(x_k, y_k)) + S(x_k, y_k)).$$

- **Second Poincaré variational principle**

  – The fixed points correspond with the stationary points of the action

  $$\boldsymbol{p}(x,y) = \frac{1}{2}(y + g(x,y))\ (x - f(x,y)) \ + \ S(x,y),$$

  if $-1$ is not an eigenvalue of $\mathrm{D}F(x,y)$, $\forall(x,y)$ (see [85]).

  – The orbits correspond with the stationary bisequences of the action

  $$\boldsymbol{P}(Z) = \sum_{k\in\mathbb{Z}} (\frac{1}{2}(y_{k+1} + g(x_k, y_k))\ (x_{k+1} - f(x_k, y_k)) \ + \ S(x_k, y_k)),$$

  if a certain infinite matrix is non singular. (If we work with finite sequences, we obtain the condition for a certain finite matrix).

- **Third Poincaré variational principle**
  (We suppose $\frac{\partial g}{\partial y}$ to be non-singular).

  – The fixed points correspond with the stationary points of the action

  $$\boldsymbol{p}(x,y) = g(x,y)\ (x - f(x,y)) + S(x,y).$$

  – The orbits correspond with the stationary bisequences of the action

  $$\boldsymbol{P}(Z) = \sum_{k\in\mathbb{Z}} (g(x_k, y_k)\ (x_{k+1} - f(x_k, y_k)) + S(x_k, y_k)).$$

**Remarks**

i) We observe that all the actions give the same result for an orbit of the symplectomorphism:

$$\sum_{k\in\mathbb{Z}} S(x_k, y_k).$$

For fixed points $(x,y)$ it is

$$S(x,x).$$

ii) Although Poincaré variational principles are written by means of the primitive function, they do not seem to have a strong geometrical meaning. Since $y$ is a momentum (a 1-form) and $x$ is a position (with momentum $y$ on it), what does $y(x - f(x,y))$ mean? We need $x - f(x,y)$ be a vector. In $\mathbb{R}^d$ is clear, but in other manifolds? Possibly we need an additional Riemannian structure on the configuration space.

iii) Although we have stated the variational principles using formal sums, they can be finite is some cases. For instance, if we consider homoclinic orbits to an hyperbolic fixed point, and we give the value 0 to the primitive function in such a point. If that fixed is parabolic the convergence of the expansion depends on the velocity in which the homoclinic point tends to the parabolic fixed point. In other cases suitable corrections should have to be performed, for instance for homoclinic orbits to an invariant curve whose dynamics is given by an irrational rotation. Heteroclinic cases can also be considered.

◁

## 5.2 Fixed points

We have an exact symplectomorphism $F$ given by

$$\left\{ \begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \end{array} \right. ,$$

being $S$ its primitive function. So then, we consider the *fixed action* $\boldsymbol{s}$ as the function $S$ restricted to the *vertically transformed set* $K$, that is, the set of points $(x, y)$ verifying the condition $x = f(x, y)$. Of course, it contains the fixed points. We suppose that this set $K$ is a submanifold of $\mathbb{R}^{2d}$, and impose that the rank of the matrix

$$(I - A \ , \ B)$$

is maximal (equal to $d$) in all its points. This *transversality condition* is satisfied when, for instance, $F$ is monotone, and then the vertically transformed set is, locally, a graph. (This last case appear in the works of Moser [79] and Arnaud [3]).

**Proposition 5.1** *:*

*Let $F = (f, g)$ be a symplectomorphism on $\mathbb{R}^{2d}$, being $S$ its primitive function. Suppose that the vertically transformed set*

$$K \quad = \quad \{(x,y) \in \mathbb{R}^{2d} \mid f(x,y) = x\}$$

*satisfies the transversality condition, and consider $\boldsymbol{s} = S_{|K} : K \to \mathbb{R}$. Then:*

- *The fixed points of $F$ are critical points of $\boldsymbol{s}$.*
- *If $F$ is monotone, the critical points of $\boldsymbol{s}$ are fixed points of $F$.*

*Proof*:

By the Lagrange multipliers method, we must look for the critical points of the function

$$L(x, y, \lambda) = S(x, y) + \lambda \cdot (x - f(x, y)),$$

where $\lambda \in \mathbb{R}^d$. The system of equations is

$$\begin{cases} 0 = \dfrac{\partial L}{\partial x} = \dfrac{\partial S}{\partial x} + \lambda^\top \left( I - \dfrac{\partial f}{\partial x} \right) = (g(x,y) - \lambda)^\top \dfrac{\partial f}{\partial x} + (\lambda - y)^\top, \\ 0 = \dfrac{\partial L}{\partial y} = \dfrac{\partial S}{\partial y} - \lambda^\top \dfrac{\partial f}{\partial y} = (g(x,y) - \lambda)^\top \dfrac{\partial f}{\partial y}, \\ 0 = \dfrac{\partial L}{\partial \lambda} = (x - f(x,y))^\top. \end{cases}$$

Therefore:

- If $(x, y)$ is a fixed point of $F = (f, g)$, then it is a critical point of the function $\boldsymbol{s}$ (having $\lambda = y$).
- We suppose now that $\left| \frac{\partial f}{\partial y} \right| \neq 0$ ($F$ is monotone). Let $(x, y)$ be a critical point of $\boldsymbol{s}$ (in particular, $x = f(x, y)$). Then, the second equation gives

$$\lambda = g(x, y),$$

and the first one gives

$$\lambda = y.$$

□

**Remarks**

i) Of course, if the monotonicity condition is satisfied on the points of the vertically transformed set, we can obtain the same result.

ii) We can obtain other variational principles for fixed points having other functions and other constraints. For instance, we can take the action

$$\hat{\boldsymbol{s}}(x, y) \quad = \quad S(x, y) - y(f(x, y) - x)$$

restricted to the set $\{y = g(x, y)\}$.

◁

### 5.2.1 Extremal character

Given a fixed point $(x_0, y_0)$ of a certain symplectomorphism $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, we shall say that it is a *transversal fixed point* iff the rank of the matrix

$$(I - A \ , \ B)$$

is maximal (equal to $d$) on it. In such a case, the vertically transformed set is regular in a neighborhood of it. We distinguish two cases.

**Monotone case:** $|B| \neq 0$**.-** Hence, we can write the vertically transformed set in a neighborhood of that point as a graph

$$y = \eta(x),$$

and the action is written as

$$\boldsymbol{s}(x) = S(x, \eta(x)).$$

Since the function $\eta$ is implicitly defined by

$$x = f(x, \eta(x)),$$

we can compute its derivatives. They are

$$\frac{\partial \eta}{\partial x}(x) = \left(\frac{\partial f}{\partial y}(x, \eta(x))\right)^{-1} \left(I - \frac{\partial f}{\partial x}(x, \eta(x))\right).$$

Hence

$$\begin{aligned}
\frac{\partial \boldsymbol{s}}{\partial x}(x) &= \frac{\partial S}{\partial x}(x, \eta(x)) + \frac{\partial S}{\partial y}(x_i, \eta(x)) \frac{\partial \eta_i}{\partial x}(x) \\
&= g(x, \eta(x))^\top \frac{\partial f}{\partial x}(x, \eta(x)) - \eta(x)^\top + g(x, \eta(x))^\top \frac{\partial f}{\partial y}(x, \eta(x)) \frac{\partial \eta}{\partial x}(x) \\
&= g(x, \eta(x))^\top - \eta(x)^\top.
\end{aligned}$$

As we see, on the fixed point these derivatives vanish. We are going to compute the Hessian matrix on our fixed point. We shall write $A = A(z_0)$, etc.

$$\begin{aligned}
D^2 \boldsymbol{s}(x_0) &= \frac{\partial g}{\partial x}(x_0, y_0) + \frac{\partial g}{\partial y}(x_0, y_0) \frac{\partial \eta}{\partial x}(x_0) - \frac{\partial \eta}{\partial x}(x_0) \\
&= C + (D - I)B^{-1}(I - A) \\
&= DB^{-1} + B^{-1}A + C - DB^{-1}A - B^{-1} \\
&= DB^{-1} + B^{-1}A - (B^{-1} + B^{-\top}).
\end{aligned}$$

Hence, we have proven the next proposition.

**Proposition 5.2** *:*

*The extremal character of a monotone fixed point is given by the symmetric matrix*

$$H = \hat{A} + \hat{B} + \hat{B}^\top,$$

*where* $\hat{A} = DB^{-1} + B^{-1}A$ *and* $\hat{B} = -B^{-1}$.

**Remark**

Notice that

$$g(x,\eta(x)) - \eta(x) \quad = \quad \nabla \boldsymbol{s}(x),$$

and, hence, the image of the $x$-parametrized manifold

$$x \longrightarrow (x, \eta(x))$$

for $F(x,y) - (x,y)$ is parametrized by

$$x \longrightarrow (x, \nabla \boldsymbol{s}(x)).$$

So then, the image for $F(x,y) - (x,y)$ of the vertically transformed set is a Lagrangian submanifold, and the fixed points correspond with the intersections of this Lagrangian submanifold with the zero-section $\{y = 0\}$. ◁

**Non monotone case:** $|B| = 0$.- The fixed point is degenerate as critical point of the action. For instance, if $I - A$ is regular at the fixed point, then we can write locally the vertical transformed set as a vertical graph

$$x \quad = \quad \nu(y).$$

Proceeding as before, the Hessian matrix at the fixed point $(x_0, y_0)$ is

$$(C^\top (I-A)^{-1}B + D - I)(I-A)^{-1}B,$$

and we see that it is degenerate.

**Proposition 5.3** :

*Non monotone fixed points are degenerate critical points of the fixed action.*

We shall consider now the case $d = 1$. Hence, suppose that in a neighborhood of the fixed point $(x_0, y_0)$ we can write the vertically transformed set as a function $x = \nu(y)$. Then, we must seek the critical points of the function

$$\boldsymbol{s}(y) \quad = \quad S(\nu(y), y).$$

We obtain that:

- $\boldsymbol{s}'(y) = (g(\nu(y),y) - y)\nu'(y)$, and then $\boldsymbol{s}'(y_0) = 0$. Moreover, $\nu'(y) = \frac{f_y(\nu(y),y)}{1-f_x(\nu(y),y)}$ and $\nu'(y_0) = 0$.

- $\boldsymbol{s}''(y) = (g_x(\nu(y),y)\nu'(y) + g_y(\nu(y),y) - 1)\nu'(y) + (g(\nu(y),y) - y)\nu''(y)$, and then $\boldsymbol{s}''(y_0) = 0$ and the critical point is degenerate. Moreover, $\nu''(y_0) = \frac{f_{yy}(x_0,y_0)}{1-f_x(x_0,y_0)}$.

- $\boldsymbol{s}'''(y_0) = 2(g_y(x_0,y_0) - 1)\nu''(y_0) = 2\frac{f_{yy}(x_0,y_0)}{f_x(x_0,y_0)}$, and the fixed point is an inflection point of $\boldsymbol{s}$, provided $f_{yy}(x_0,y_0) \neq 0$.

Note that, if $(x_0, y_0)$ is not a fixed point, but $\boldsymbol{s}'(y_0) = 0$, then it is non degenerate provided $f_{yy}(x_0,y_0) \neq 0$.

We also can use the *bordered Hessian* matrices in order to study the extremal character of fixed points.

### 5.2.2 Dynamical character

The dynamical type of a fixed point is given by the eigenvalues of $M = \mathrm{D}F(x_0, y_0)$. It is well know that its eigenvalues (also called the *multipliers* of the fixed point) appear either in pairs or in quadruplets, since the characteristic polynomial is reflexive (see, for instance, [76])[1]:

$$\lambda \in \sigma(M) \Rightarrow \lambda^{-1} \in \sigma(M).$$

In fact, given an eigenvalue $\lambda \in \mathbb{C}$:

- if $\lambda$ is real, but different from $\pm 1$, then it has a real partner $\lambda^{-1}$, and we shall say that $\{\lambda, \lambda^{-1}\}$ is an *hyperbolic pair*, *with reflection* if $\lambda < 0$ and *without reflection* if $\lambda > 0$;
- if $\lambda = \pm 1$, then it has even multiplicity, and we shall say that it is *parabolic*, *with reflection* if $\lambda = -1$ and *without reflection* if $\lambda = 1$;
- if $\lambda$ is on the unit circle but it is not real, then its partner is $\lambda^{-1} = \bar{\lambda}$, and we shall say that $\{\lambda, \bar{\lambda}\}$ is an *elliptic pair*;
- if $\lambda$ is neither real nor of unit modulus, then there must be a *(complex) hyperbolic quadruplet* of eigenvalues $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$ (of course, this case can occur only for more than 2 dimensions).

Hence, we obtain that $\mathbb{R}^{2d}$ splits in elliptic, hyperbolic and parabolic subspaces $\mathbb{R}^{2d} = E \oplus H \oplus P$ (generically, $\dim P = 0$). The dimensions of these subspaces are called the *elliptic, hyperbolic and parabolic dimensions*, respectively. Furthermore, we can compute a kind of symplectic Jordan normal form, called Williamson normal form [101].

Herman proved that the eigenvalues of the matrix $M$ are those values $\lambda$ such that the determinant of the matrix

$$\begin{aligned} M_\lambda &= B^{-1}A + DB^{-1} - \lambda B^{-1} - \lambda^{-1} B^{-\top} \\ &= \hat{A} + \lambda \hat{B} + \lambda^{-1} \hat{B}^\top \\ &= H + (\lambda - 1)\hat{B} + (\lambda^{-1} - 1)\hat{B}^\top \end{aligned}$$

vanish, provided that $B$ is regular. Notice that $M_1 = H$ and $|M_\lambda| = |M_{\frac{1}{\lambda}}|$, since $H$ is symmetric. Hence, the extremal character contains relevant information about the linearized dynamics around the fixed point.

## 5.3 Periodic orbits

In order to look for the $q$-periodic orbits of an exact symplectomorphism $F$ given by

$$\left\{ \begin{array}{l} x' = f(x, y) \\ y' = g(x, y) \end{array} \right. ,$$

[1] '$\sigma$' means the spectrum of a matrix.

being $S$ its primitive function, we shall consider the exact symplectic product and the exact symplectomorphism $F_q$ (see Section 1.4). We write $F_q$ as

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{q-1} \\ y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{pmatrix} \longrightarrow \begin{pmatrix} f(x_{q-1}, y_{q-1}) \\ f(x_0, y_0) \\ \vdots \\ f(x_{q-2}, y_{q-2}) \\ g(x_{q-1}, y_{q-1}) \\ g(x_0, y_0) \\ \vdots \\ g(x_{q-2}, y_{q-2}) \end{pmatrix}.$$

Then fixed points of $F_q$ correspond to $q$-periodic orbits of $F$, and we applied the results of the previous subsection. The fixed action for $F_q$ is the *periodic action*

$$\boldsymbol{S}_q(x_0, \ldots, x_{q-1}, y_0, \ldots, y_{q-1}) = \sum_{i=0}^{q-1} S(x_i, y_i)$$

and it is restricted to the *loops*. The loops are the $q$-sequences of points such that

- $\forall i = 0 \div q-2, f(x_i, y_i) = x_{i+1}$,
- $f(x_{q-1}, y_{q-1}) = x_0$.

We note that if $F$ is monotone, so is $F_q$.

### 5.3.1 Extremal character

In the previous context, in order to compute the extremal character of a $q$-periodic orbit we need first to compute $\mathrm{D}F_q$. It is the $2qd \times 2qd$ matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & A_{q-1} & 0 & 0 & \ldots & B_{q-1} \\ A_0 & 0 & \ldots & 0 & B_0 & 0 & \ldots & 0 \\ & \ddots & & \vdots & & \ddots & & \vdots \\ & & A_{q-2} & 0 & & & B_{q-2} & 0 \\ 0 & 0 & \ldots & C_{q-1} & 0 & 0 & \ldots & D_{q-1} \\ C_0 & 0 & \ldots & 0 & D_0 & 0 & \ldots & 0 \\ & \ddots & & \vdots & & \ddots & & \vdots \\ & & C_{q-2} & 0 & & & D_{q-2} & 0 \end{pmatrix}.$$

Hence, the extremal character of the $q$-periodic orbit is given by (for $q \geq 3$)

$$\begin{aligned} H_q &= \mathbf{D}\mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{A} - (\mathbf{B}^{-1} + \mathbf{B}^{-\top}) \\ &= \begin{pmatrix} \hat{A}_0 & \hat{B}_0 & & & \hat{B}_{q-1}^\top \\ \hat{B}_0^\top & \hat{A}_1 & \hat{B}_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{B}_{q-3}^\top & \hat{A}_{q-2} & \hat{B}_{q-2} \\ \hat{B}_{q-1} & & & \hat{B}_{q-2}^\top & \hat{A}_{q-1} \end{pmatrix}, \end{aligned}$$

provided that $F$ is monotone at all points of the periodic orbit. We have defined $\hat{A}_i = D_{i-1}B_{i-1}^{-1} + B_{i-1}A_{i-1}^{-1}$ and $\hat{B}_i = -B_i^{-1}$ ($i = 0 \div q-1$, identifying $-1$ with $q-1$).

### 5.3.2 Dynamical character

The dynamical character of a $q$-periodic orbit with initial point in $z_0 = (x_0, y_0)$ is given by the eigenvalues of $M = \mathrm{D}F^q(z_0) = \mathrm{D}F(z_{q-1})\ldots\mathrm{D}F(z_0)$. Using Floquet theory in configuration space [66, 53, 22], the eigenvalues of $M$ are in correspondence with those values of $\lambda$ such that the determinant of the matrix

$$
\begin{aligned}
M_\lambda &= \begin{pmatrix}
\hat{A}_0 & \hat{B}_0 & & & \lambda^{-1}\hat{B}_{q-1}^\top \\
\hat{B}_0^\top & \hat{A}_1 & \hat{B}_1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & \hat{B}_{q-3}^\top & \hat{A}_{q-2} & \hat{B}_{q-2} \\
\lambda\hat{B}_{q-1} & & & \hat{B}_{q-2}^\top & \hat{A}_{q-1}
\end{pmatrix} \\
&= H_q + (\lambda-1)E_{q,1}\otimes\hat{B}_{q-1} + (\lambda^{-1}-1)E_{q,1}^\top\otimes\hat{B}_{q-1}^\top,
\end{aligned}
$$

is equal to zero, where $\otimes$ means the *Kronecker product* and $E_{q,1}$ is the $q\times q$ matrix with 1 in the $(q,1)$-entry and zero otherwise. This is an extension of the Herman result in Section 5.2.2. In particular, $M_1 = H_q$. Since the coefficients of $\lambda - 1$ and $\lambda^{-1} - 1$ are rank $d$ matrices and $|M_\lambda| = |M_{\lambda^{-1}}|$, then the determinant will be a polynomial of degree $d$ in the variable $\lambda + \lambda^{-1}$ (see, for instance, [22] for a proof).

If we group the eigenvalues by reciprocal pairs $\lambda_i, \lambda_i^{-1}$ ($i = 1 \div d$) and we consider the $d$ residues of everyone of the multipliers,

$$
R_i = \frac{1}{4}(2 - \lambda_i - \lambda_i^{-1}),
$$

then

$$
\prod_{i=1}^{d} R_i = \left(\frac{-1}{4}\right)^d |H_q| \prod_{j=0}^{q-1} |B_j|,
$$

as Kook and Meiss proved in [53] when our symplectic map is generated by a Lagrangian generating function.

We wonder also about the dynamical character of the $q$-periodic orbit as fixed point of $F_q$. Applying the Herman result, it is given by the values of $\lambda$ such that the determinant of the matrix

$$
\begin{aligned}
\mathbf{M}_\lambda &= \begin{pmatrix}
\hat{A}_0 & \lambda\hat{B}_0 & & & \lambda^{-1}\hat{B}_{q-1}^\top \\
\lambda^{-1}\hat{B}_0^\top & \hat{A}_1 & \lambda\hat{B}_1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & \lambda^{-1}\hat{B}_{q-3}^\top & \hat{A}_{q-2} & \lambda\hat{B}_{q-2} \\
\lambda\hat{B}_{q-1} & & & \lambda^{-1}\hat{B}_{q-2}^\top & \hat{A}_{q-1}
\end{pmatrix} \\
&= H_q + (\lambda-1)\Gamma_q\otimes\hat{B}_{q-1} + (\lambda^{-1}-1)\Gamma_q^\top\otimes\hat{B}_{q-1}^\top,
\end{aligned}
$$

vanishes, where $\Gamma_q$ is the *fundamental circulant matrix*, which is

$$\Gamma_q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We also obtain $\mathbf{M}_1 = H_q$ and $|\mathbf{M}_\lambda| = |\mathbf{M}_{\lambda^{-1}}|$.

## 5.4 Connecting orbits

Let $F$ be the symplectomorphism in $\mathbb{R}^{2d}$ given by

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases},$$

with $S$ as primitive function.

Given two $x$-points $\mathbf{x}_m, \mathbf{x}_n \in \mathbb{R}^d$, where $n > m+1$, we want to look for the orbits connecting them after $n-m$ steps, i.e., the $(m,n)$-sequences of $\mathbb{R}^{2d}$

$$(x_m, y_m), (x_{m+1}, y_{m+1}), \dots (x_{n-1}, y_{n-1})$$

such that

- $x_m = \mathbf{x}_m$,
- $\forall i = m \div n-2,\ F(x_i, y_i) = (x_{i+1}, y_{i+1})$,
- $f(x_{n-1}, y_{n-1}) = \mathbf{x}_n$.

We shall consider the $(m,n)$-*orbital action*

$$\boldsymbol{S}_{m,n}(x_m, y_m, x_{m+1}, y_{m+1}, \dots, x_{n-1}, y_{n-1}) = \sum_{i=m}^{n-1} S(x_i, y_i),$$

which is restricted to the points satisfying that

- $x_m = \mathbf{x}_m$,
- $\forall i = m \div n-2,\ f(x_i, y_i) = x_{i+1}$,
- $f(x_{n-1}, y_{n-1}) = \mathbf{x}_n$.

This set will be call the *set of chains*, $K_{m,n} = K_{\mathbf{x}_m, \mathbf{x}_n}$. It is a $d(n-m-1)$-submanifold of $\mathbb{R}^{2d(n-m)}$, provided the rank of the matrix

$$\begin{pmatrix} B_m & -I & & & & & \\ & A_{m+1} & B_{m+1} & -I & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_{n-2} & B_{n-2} & -I & \\ & & & & & A_{n-1} & B_{n-1} \end{pmatrix}$$

be maximal ($= n-m$) in all the chains. For instance, this *transversality condition* is satisfied when $F$ is monotone. If this condition is satisfied for a certain connecting orbit, we shall say that it is *transversal.* An orbit will be a *transversal orbit* iff all its segments are transversal.

**Proposition 5.4** *:*

*Let $F = (f, g)$ be a symplectomorphism on $\mathbb{R}^{2d}$, being $S$ its primitive function. Given two $x$-points $\mathbf{x}_m, \mathbf{x}_n \in \mathbb{R}^d$, suppose that the corresponding set of chains, $K_{m,n}$, satisfies the transversality condition, and consider the orbital action $\boldsymbol{S}_{m,n}$ on it. Then:*

- *The connecting orbits are critical chains of $\boldsymbol{S}_{m,n}$.*
- *If $F$ is monotone, the critical chains of $\boldsymbol{S}_{m,n}$ are connecting orbits.*

*Proof*:

By the Lagrange multiplier method, we must seek the critical points of the function

$$L(y_m, x_{m+1}, y_{m+1}, \ldots, x_{n-1}, y_{n-1}, \lambda_1, \ldots, \lambda_n)$$

$$= \sum_{i=m}^{n-1} (S(x_i, y_i) + \lambda_{i+1} \cdot (x_{i+1} - f(x_i, y_i))),$$

where all the variables belong to $\mathbb{R}^d$, and we have taken $x_m = \mathbf{x}_0$ and $x_n = \mathbf{x}_n$. The system of equations is:

$$\begin{cases} 0 = \dfrac{\partial L}{\partial x_i} = (g(x_i, y_i) - \lambda_{i+1})^\top \dfrac{\partial f}{\partial x}(x_i, y_i) + (\lambda_i - y_i)^\top \, (i = m+1 \div n-1), \\[2ex] 0 = \dfrac{\partial L}{\partial y_i} = (g(x_i, y_i) - \lambda_{i+1})^\top \dfrac{\partial f}{\partial y}(x_i, y_i) \, (i = m \div n-1), \\[2ex] 0 = \dfrac{\partial L}{\partial \lambda_i} = (x_i - f(x_{i-1}, y_{i-1}))^\top \, (i = m+1 \div n). \end{cases}$$

Therefore:

- If $(x_i, y_i)_{i=m \div n-1}$ is a connecting orbit between $\mathbf{x}_m$ and $\mathbf{x}_n$, then it is a critical point of the function $\boldsymbol{S}_{m,n}$ ( having $\lambda_i = y_i \; \forall i = m+1 \div n-1$ and $\lambda_n = g(x_{n-1}, y_{n-1})$).
- We suppose now that $\left|\dfrac{\partial f}{\partial y}\right| \neq 0$ ($F$ is monotone). For a critical point of $\boldsymbol{S}_{m,n}$ (in particular, $\forall i = m \div n-1,\; x_{i+1} = f(x_i, y_i)$) the second equations give $\forall i = m \div n-1$

$$\lambda_{i+1} \;=\; g(x_i, y_i)$$

and, therefore, the first ones give $\forall i = m{+}1 \div n{-}1$

$$\lambda_i \;=\; y_i, .$$

Finally, we obtain $\forall i = m \div n{-}2$

$$g(x_i, y_i) \;=\; \lambda_{i+1} = y_{i+1}.$$

□

### 5.4.1 Extremal character

We have seen that the connecting orbits between two $x$-points $\mathbf{x}_m$ and $\mathbf{x}_n$ are critical $F$-chains of a certain action. We wonder about their extremal character, i.e., about the second order derivatives $H_{m,n} = \mathrm{D}^2\boldsymbol{S}_{m,n}$. If $F$ is monotone, or at least it is monotone in the region where the segment lives, then the function $\boldsymbol{S}_{m,n}$ can be locally written in variables $x_{m+1}, \ldots, x_{n-1}$, and we can compute this Hessian matrix.

For the sake of simplicity, we shall consider $m = 0$. Hence, we shall consider the connecting orbits between $\mathbf{x}_0$ and $\mathbf{x}_n$. Since $F$ is monotone, the set of equations

$$f(x_i, y_i) \;=\; x_{i+1} \quad (i = 0 \div n-1)$$

defines implicitly a set of functions

$$\eta_i \;=\; \eta_i(x, x') \quad (i = 0 \div n-1)$$

such that

$$f(x_i, \eta_i(x_i, x_{i+1})) = x_{i+1} \quad (i = 0 \div n-1).$$

Of course, these functions are defined on a neighborhood of a connecting orbit. Their derivatives are given by the equations

$$0 \;=\; \frac{\partial f}{\partial x}(x_i, \eta_i(x_i, x_{i+1})) + \frac{\partial f}{\partial y}(x_i, \eta_i(x_i, x_{i+1}))\,\frac{\partial \eta_i}{\partial x}(x_i, x_{i+1})$$

and

$$I \;=\; \frac{\partial f}{\partial y}(x_i, \eta_i(x_i, x_{i+1}))\,\frac{\partial \eta_i}{\partial x'}(x_i, x_{i+1}).$$

Therefore, we have to compute the critical points of the function

$$\boldsymbol{S}_{0,n}(x_1, \ldots, x_{n-1}) \;=\; \sum_{i=0}^{n-1} S(x_i, \eta_i(x_i, x_{i+1})),$$

where we have taken $x_0 = \mathbf{x}_0$ and $x_n = \mathbf{x}_n$. So then, $\forall i = 1 \div n-1$:

$$\begin{aligned}
\frac{\partial \boldsymbol{S}_{0,n}}{\partial x_i} &= \frac{\partial S}{\partial x}(x_i,\eta_i) + \frac{\partial S}{\partial y}(x_i,\eta_i)\ \frac{\partial \eta_i}{\partial x}(x_i,x_{i+1}) + \frac{\partial S}{\partial y}(x_{i-1},\eta_{i-1})\ \frac{\partial \eta_{i-1}}{\partial x'}(x_{i-1},x_i) \\
&= g(x_i,\eta_i)^\top\ \frac{\partial f}{\partial x}(x_i,\eta_i) - \eta_i^\top + g(x_i,\eta_i)^\top\ \frac{\partial f}{\partial y}(x_i,\eta_i)\ \frac{\partial \eta_i}{\partial x} + \\
&\quad g(x_{i-1},\eta_{i-1})^\top\ \frac{\partial f}{\partial y}(x_{i-1},\eta_{i-1})\ \frac{\partial \eta_{i-1}}{\partial x'} \\
&= g(x_{i-1},\eta_{i-1})^\top - \eta_i^\top .
\end{aligned}$$

Therefore, the orbits are extremals of the action. We are going to compute the second derivative on a connecting orbit. We have written $A_i = \frac{\partial f}{\partial x}(x_i, y_i)$, etc.

- $\forall i = 1 \div n-1$,

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{S}_{0,n}}{\partial x_i^2} &= \frac{\partial g}{\partial y}(x_{i-1},y_{i-1})\ \frac{\partial \eta_{i-1}}{\partial x'}(x_{i-1},x_i) - \frac{\partial \eta_i}{\partial x}(x_i,x_{i+1}) \\
&= D_{i-1}\ B_{i-1}^{-1} + B_i^{-1}\ A_i\ ;
\end{aligned}$$

- $\forall i = 2 \div n-1$,

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{S}_{0,n}}{\partial x_{i-1}\partial x_i} &= \frac{\partial g}{\partial x}(x_{i-1},y_{i-1}) - \frac{\partial g}{\partial y}(x_{i-1},y_{i-1})\ \frac{\partial \eta_{i-1}}{\partial x}(x_{i-1},x_i) \\
&= C_{i-1} - D_{i-1}B_{i-1}^{-1}A_{i-1} = -B_{i-1}^{-\top}\ ;
\end{aligned}$$

- $\forall i = 1 \div n-2$,

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{S}_{0,n}}{\partial x_{i+1}\partial x_i} &= -\frac{\partial \eta_i}{\partial x'}(x_i,x_{i+1}) \\
&= -B_i^{-1}\ ;
\end{aligned}$$

- All the other second derivatives vanish, and, of course, we have obtained that the Hessian matrix is symmetric.

Summarizing:

**Proposition 5.5** *:*

*The Hessian matrix associated to a monotone segment of orbit is given by the block-tridiagonal symmetric matrix*

$$H_{0,n} = \begin{pmatrix}
\hat{A}_1 & \hat{B}_1 & & & \\
\hat{B}_1^\top & \hat{A}_2 & \hat{B}_2 & & \\
 & \ddots & \ddots & \ddots & \\
 & & \hat{B}_{n-3}^\top & \hat{A}_{n-2} & \hat{B}_{n-2} \\
 & & & \hat{B}_{n-2}^\top & \hat{A}_{n-1}
\end{pmatrix},$$

*where the matrices $\hat{A}_i$ and $\hat{B}_i$ are given by*

$$\hat{A}_i = D_{i-1}B_{i-1}^{-1} + B_i^{-1}A_i$$

*and*

$$\hat{B}_i = -B_i^{-1}.$$

Connecting orbits with positive definite Hessian matrix will be interesting in the sequel. All of their points must be monotone, because in other case the Hessian is degenerate.

### 5.4.2 Minimizing orbits

We shall say that a connecting orbit is *minimizing* iff its Hessian matrix is positive definite. Hence, an orbit is *minimizing* iff every segment of it is minimizing (for the corresponding action). We shall say that a point $z$ is minimizing iff the action $W_{0,2}$ with $\mathbf{x}_0 = q{\circ}F^{-1}(z)$ and $\mathbf{x}_2 = q{\circ}F(z)$ is minimized on $z$. Then (in the monotone case), the matrix

$$\hat{A}(z) = D(F^{-1}(z))B(F^{-1}(z))^{-1} + B(z)^{-1}A(z)$$

must be positive definite.

**Remarks**

i) So then, minimizing means non degenerate minimum.

ii) Since the eigenvalues of a matrix depend continuously on its components, if we have a minimizing segment of orbit then another segment of orbit close enough to the first will be also minimizing.

iii) All the subsegments of a minimizing segment are also minimizing.

iv) A minimizing orbit of $F$ is also a minimizing orbit for any power of $F$, because in the second case the chains are defined with more constraints and the primitive function of a power of $F$ is the sum of the primitive function on each point of the segment. That is to say, if $S_q$ is the primitive function of $F^q$, where $q \in \mathbb{N}^*$, we know that

$$S_q = \sum_{i=0}^{q-1} S{\circ}F^i.$$

Then, the $(m, n)$-action associated to $F^q$ applied to the $F^q$-chain

$$(x_m^q, y_m^q, \ldots, x_{n-1}^q, y_{n-1}^q)$$

is

$$\begin{aligned} \boldsymbol{S}_{q;m,n}(x^q_m, y^q_m, \ldots, x^q_{n-1}, y^q_{n-1}) &= \sum_{i=m}^{n-1} S_q(x^q_i, y^q_i) \\ &= \sum_{i=m}^{n-1}\sum_{j=0}^{q-1} S{\circ}F^j(x^q_i, y^q_i), \end{aligned}$$

which is also the $(m, nq)$-action associated to $F$ applied to the corresponding $F$-chain.

v) Let $(x_0, y_0)$ be a $q$-periodic point, with $q \geq 2$. If the corresponding segment of length $q$ minimizes the $q$-periodic action then it also minimizes the $q$-orbital action with $\mathbf{x}_0 = x_0$ and $\mathbf{x}_q = x_0$.

◁

**The MMS iteration.-** In [68] the Hessian matrix is written using the Lagrangian generating function, but if we use the results of Section 4.2 we see that the two matrices coincide. Then, although the Lagrangian generating function does not exist, we can define a (local) extremal behaviour of the orbits.

On the other side, they describe the following method for block-diagonalizing that matrix. They write

$$\begin{pmatrix} \hat A_1 & \hat B_1 & & \\ \hat B_1^\top & \hat A_2 & \hat B_2 & \\ & \ddots & \ddots & \ddots \\ & & \hat B_{n-2}^\top & \hat A_{n-1} \end{pmatrix} =$$

$$\begin{pmatrix} I & & & \\ \hat B_1^\top \hat D_1^{-\top} & I & & \\ & \ddots & \ddots & \\ & & \hat B_{n-2}^\top \hat D_{n-2}^{-\top} & I \end{pmatrix} \begin{pmatrix} \hat D_1 & & & \\ & \hat D_2 & & \\ & & \ddots & \\ & & & \hat D_{n-1} \end{pmatrix} \begin{pmatrix} I & \hat D_1^{-1}\hat B_1 & & \\ & I & \hat D_2^{-1}\hat B_2 & \\ & & \ddots & \ddots \\ & & & I \end{pmatrix},$$

where the diagonal blocks are given by the recurrence

$$\left\{ \begin{array}{l} \hat D_1 = \hat A_1, \\ \\ \hat D_i = \hat A_i - \hat B_{i-1}^\top \hat D_{i-1}^{-1} \hat B_{i-1} \ (i = 2 \div n-1) \end{array} \right. ,$$

provided $\hat D_{i-1}$ is invertible. If the matrix is positive definite then all the symmetric matrices $\hat D_i$ are positive definite.

**1 degree of freedom.-** If $d = 1$, then we can obtain a recurrence for the characteristic polynomials of $H_{0,i}$ $(i > 1)$, that we shall call $p_{i-1}$: $p_{i-1}(x) = \det(xI - H_{0,i})$. Then:

$$\left\{ \begin{array}{l} p_0(x) = 1, p_1(x) = x - \hat a_1; \\ p_i(x) = (x - \hat a_i)p_{i-1}(x) - \hat b_{i-1}^2 p_{i-2}(x) \ (i > 2). \end{array} \right.$$

The sequence of polynomials $\{p_i, p_{i-1}, \ldots, p_1, p_0\}$ is an Sturm sequence for the polynomial $p_i$ (see [11]). In particular, all the eigenvalues of $H_i$ are different (and real, of course), and, moreover, we can compute the number of positive eigenvalues:

> If $p_{i-1}(0) \neq 0$, the number of positive eigenvalues of $H_i$ is equal to the number of changes of sign in the sequence $\{p_0(0), p_1(0), \ldots, p_{i-1}(0)\}$.

This is a particular case of the Sturm's theorem.

Hence, we must compute the number of changes of sign of the sequence

$$r_0 = 1,\ r_1 = -\hat{a}_1;$$
$$\hat{r}_i = -\hat{a}_i r_{i-1} - \hat{b}_{i-1}^2 r_{i-2} \quad (i > 1).$$

**Remark**

This fit into the MMS iteration, by defining (for $i > 1$)

$$\hat{d}_i = -\frac{r_i}{r_{i-1}}.$$

◁

## 5.5 Index, torsion and dynamics

Given a fixed point, we wonder about the relationship among its extremal character as fixed point, periodic orbit or orbit, and its dynamical character. He shall follow with the notation in Section 5.2. We shall consider two examples, but further information can be found in [66, 53, 22, 3].

- About its dynamical character, we shall use the next result due to Herman, who stated that the eigenvalues $\lambda$ of $M$ satisfy

  $$rg(M - \lambda I) = j \Leftrightarrow rg(M_\lambda) = j,$$

  where $M_\lambda = B^{-1}A + DB^{-1} - \lambda B^{-1} - \lambda^{-1}B^{-\top}$. Hence, following Arnaud [3]:

  $$M_\lambda = H + (1 - \lambda^{-1})B^{-\top} + (1 - \lambda)B^{-1}.$$

- As fixed point, its extremal character is given by the matrix

  $$\begin{aligned} H &= DB^{-1} + B^{-1}A - (B^{-1} + B^{-\top}) \\ &= \hat{A} + \hat{B} + \hat{B}^\top. \end{aligned}$$

- As $q$-periodic orbit (for any $q$, $q \geq 3$ – the case $q = 2$ is different –), the character is given by the $dq \times dq$-matrix

  $$\begin{aligned} H_q &= \begin{pmatrix} \hat{A} & \hat{B} & & & \hat{B}^\top \\ \hat{B}^\top & \hat{A} & \hat{B} & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{B}^\top & \hat{A} & \hat{B} \\ \hat{B} & & & \hat{B}^\top & \hat{A} \end{pmatrix} \\ &= I_q \otimes \hat{A} + \Gamma_q \otimes \hat{B} + \Gamma_q^\top \otimes \hat{B}^\top. \end{aligned}$$

The matrix $\Gamma_q$ can be diagonalized by the Fourier matrix [27]. This fact was used in [22] in order to block diagonalize $H_q$, and they found that

$$\sigma(I_q \otimes \hat{A} + \Gamma_q \otimes \hat{B} + \Gamma_q^\top \otimes \hat{B}^\top) \;=\; \bigcup_{j=1}^{q} \sigma(\hat{A} + \omega_q^{j-1}\hat{B} + \bar{\omega}_q^{j-1}\hat{B}^\top),$$

where $\sigma$ means the spectrum of a matrix and $\omega_q = \exp(\frac{2\pi}{q}\mathbf{i})$ is the 'first' $q$-th root of the unity. Note that all the eigenvalues are real, because the matrix is Hermitian.

If $B$ is symmetric we obtain that

$$\sigma(H_q) \;=\; \bigcup_{j=1}^{q} \sigma\left(\hat{A} + 2\gamma_j \hat{B}\right),$$

with $\gamma_j = \cos\left(\frac{2\pi(j-1)}{q}\right)$ $(j = 1 \div q)$.

- Finally, if we want to compute the extremal character of the corresponding segment of length $n+1$ (with $n \geq 1$) then we must consider the $nd \times nd$-matrix

$$H_{0,n+1} \;=\; \begin{pmatrix} \hat{A} & \hat{B} & & & 0 \\ \hat{B}^\top & \hat{A} & \hat{B} & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{B}^\top & \hat{A} & \hat{B} \\ 0 & & & \hat{B}^\top & \hat{A} \end{pmatrix}$$

$$= \; I_n \otimes \hat{A} + \Sigma_n \otimes \hat{B} + \Sigma_n^\top \otimes \hat{B}^\top,$$

where $\Sigma_n$ and $\Sigma_n^\top$ are the *backward shift* and the *forward shift*, respectively (see [45]):

$$\Sigma_n \;=\; \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This kind of matrix often appears when one works with numerical methods of partial differential equations (for instance, in the eigenvalue problem of the Laplace operator defined on a square).

When $B = B^\top$ we can diagonalize $H_{0,n+1}$ as follows. First of all, we know [11] that the tridiagonal $n$-matrix $T_n = \Sigma_n + \Sigma_n^\top$ can be diagonalized as

$$T_n S_n \;=\; 2\, S_n C_n,$$

where

$$C_n \;=\; \mathrm{diag}(c_1, c_2, \dots, c_n),$$

with $c_j = \cos\left(\frac{j\pi}{n+1}\right)$ $(j = 1 \div n)$, and the entries of $S_n$ are

$$s_{ij} = \sin\left(\frac{ij\pi}{n+1}\right).$$

Then the eigenvalues of $H_{0,n+1}$ are the same that those of

$$\begin{aligned}(S_n \otimes I_d)^{-1} H_{0,n+1} (S_n \otimes I_d) &= (S_n^{-1} \otimes I_d)(I_n \otimes \hat{A} + T_n \otimes \hat{B})(S_n \otimes I_d) \\ &= I_n \otimes \hat{A} + 2\, C_n \otimes \hat{B},\end{aligned}$$

and finally, we obtain that

$$\sigma(H_{0,n+1}) = \bigcup_{j=1}^{n} \sigma\left(\hat{A} + 2c_j \hat{B}\right).$$

### 5.5.1 Area preserving maps

As an easy example, we shall consider the 2D case $(d = 1)$. Hence, let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ be a symplectomorphism, whose primitive function is $S : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We shall write $F = (f, g)$ and

$$\mathrm{D}F(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $z_0 = (x_0, y_0)$ be a fixed point. Its dynamical type is defined by the *trace* $(\tau = a + d)$. In fact, some people prefers to use the *residue* $R = \frac{2-\tau}{4}$. We distinguish the next types (see [63, 76]):

- $\tau > 2$: *regular hyperbolic* or *non-reflection hyperbolic*;
- $\tau = 2$: *regular parabolic* or *non-reflection parabolic*;
- $-2 < \tau < 2$: *elliptic*;
- $\tau = -2$: *inversion parabolic* or *reflection hyperbolic*:
- $\tau < -2$: *inversion hyperbolic* or *reflection hyperbolic*.

In order to study the extremal type we distinguish the monotone and the non monotone case.

**Monotone case.-** If the fixed point is monotone, that is, $b \neq 0$, then the extremal character as fixed point is given by

$$h = \frac{a + d - 2}{b}.$$

We shall call $\tau = a + d$, the trace of the matrix. Hence, if we suppose $b > 0$ (opposite case being similar):

- $x_0$ is non degenerate minimum $\Leftrightarrow \tau > 2$.

- $x_0$ is degenerate $\Leftrightarrow \tau = 2$.
- $x_0$ is non degenerate maximum $\Leftrightarrow \tau < 2$.

As we have seen, the character of a fixed point does not only depend on its index as critical point of the action, but also on the index of its torsion.

A natural question arises:

given a minimizing fixed point, is its orbit minimizing?

The second derivative of the $(n+1)$-orbital action $W_{0,n+1}$ is given by the $n \times n$ matrix

$$H_{0,n+1} = \begin{pmatrix} \hat{a} & \hat{b} & & & \\ \hat{b} & \hat{a} & \hat{b} & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{b} & \hat{a} & \hat{b} \\ & & & \hat{b} & \hat{a} \end{pmatrix}.$$

Its eigenvalues are

$$\sigma(H_{0,n+1}) = \left\{ \lambda_j = \hat{a} + 2c_j\hat{b} = \frac{\tau - 2c_j}{b} \right\}_{j=1 \div n},$$

where $c_j = \cos\left(\frac{j\pi}{n+1}\right)$.

Suppose its torsion is positive: $b > 0$. Then, the eigenvalues are disposed in increasing order by $j$:

$$h = \frac{\tau - 2}{b} < \lambda_1 < \ldots < \lambda_n < \frac{\tau + 2}{b} = h + \frac{4}{b}.$$

Hence,

- if $\tau \geq 2$ the orbit is minimizing;
- if $-2 < \tau < 2$ the orbit is undefinite (or saddle);
- if $\tau \leq -2$ the orbit is maximizing.

## Remarks

i) In the hyperbolic cases, the matrix is strictly diagonal dominant and the eigenvalues are far from zero. In the parabolic cases the eigenvalues are not uniformly away from zero when $n$ increases, and the matrices correponding to the periodic actions are degenerate.

ii) In the elliptic case, we must take $n$ big enough in order to obtain eigenvalues with different sign.

◁

Following with the case $b > 0$, we can define an *extremal index of the fixed point*, being the proportion of negative eigenvalues of the Hessian matrix when $n$ tends to infinity. It is the continuous function of the trace $\tau$

$$\mathrm{ind}(\tau) = \begin{cases} 1 \text{ if } \tau \leq -2, \\ \frac{1}{\pi}\arccos(\frac{\tau}{2}) \text{ if } -2 < \tau < 2, \\ 0 \text{ if } 2 \leq \tau. \end{cases}$$

We note that in the elliptic case, the two eigenvalues are $\exp(\pm\pi\nu\mathbf{i})$ where $\nu$ is the extremal index of that point and $\iota = \pi\nu$ is the average angle of rotation per period [36]. Of course, we can proceed analogously in the case $b < 0$.

**Non monotone case.-** If the fixed point is not monotone, $b = 0$, then it can not be elliptic, because the differential matrix is

$$\begin{pmatrix} a & 0 \\ c & \frac{1}{a} \end{pmatrix}.$$

On one hand, if the point is regular parabolic ($a = 1$) then the vertically transformed set is not regular at that point. On the other hand, if the point is not regular parabolic then the vertically transformed set can be write as a function $x = \nu(y)$ (see Section 5.2.1) and the fixed point is a degenerate critical point of the fixed action (it is generically an inflection point). In all cases, the set of chains in not regular.

Similar considerations can be done using the periodic extremal character. We summarize the previous argumentation in the next table.

| *dynamical character* | *trace* $\tau = a+d$ | *residue* $R = \frac{2-\tau}{4}$ | *multipliers* $\lambda_1, \lambda_2$ | *extremal character* $b > 0$ | $b = 0$ | $b < 0$ |
|---|---|---|---|---|---|---|
| regular hyperbolic | $\tau > 2$ | $R < 0$ | reciprocal pair of positive reals | min. f.p. min. or. | deg. f.p. non. tr. or. | max. f.p. max. or. |
| regular parabolic | $\tau = 2$ | $R = 0$ | pair at $+1$ | deg. f.p. min. or. | non tr. f.p. non. tr. or. | deg. f.p. max. or. |
| elliptic | $-2 < \tau < 2$ | $0 < R < 1$ | complex pair on the unit circle | max. f.p. sad. or. | $\times$ | min. f.p. sad. or. |
| inversion parabolic | $\tau = -2$ | $R = 1$ | pair at $-1$ | max. f.p. max. or. | deg. f.p. non tr. or. | min. f.p. min. or. |
| inversion hyperbolic | $\tau < -2$ | $R > 1$ | reciprocal pair of negative reals | max. f.p. max. or. | deg. f.p. non tr. or. | min. f.p. min. or. |

**Remark**

This table suggest us that the detection of bifurcations of fixed points is related with the study of geometrical changes in the vertical transformed set. ◁

### 5.5.2 The symmetric case

Following with the notation of the beginning of this section, we shall consider now the case $B = B^\top$. We shall obtain similar results to the previous ones. We remember that we must look for $\lambda$ vanishing the determinant of the matrix

$$M_\lambda = H + (1-\lambda^{-1})B^{-\top} + (1-\lambda)B^{-1},$$

where $H$ is the Hessian matrix of the fixed point. In our case, we can write

$$\begin{aligned} M_\lambda &= DB^{-1} + B^{-1}A - 2B^{-1} + 4R(\lambda)\; B^{-1} \\ &= B^{-1}((D^\top + A - 2\; I_d) + 4R(\lambda)\; I_d) \end{aligned}$$

where $R(\lambda) = \frac{2-\lambda-\lambda^{-1}}{4}$ is the residue corresponding to the eigenvalue $\lambda$ (or better, to the pair $\{\lambda, \lambda^{-1}\}$). Hence,

$$\lambda \in \sigma(M) \quad \Leftrightarrow \quad R(\lambda) \in \sigma\left(\frac{1}{4}(2\; I_d - (D^\top + A))\right).$$

In general setting, the residue is real iff the corresponding pair is real hyperbolic, elliptic or parabolic, and in other case the residue, and its conjugate, correspond to a complex hyperbolic quadruplet.

If $B$ is symmetric and positive definite (and so is $B^{-1}$), then we can diagonalize simultaneously the quadratic forms associated to $H$ and $B^{-1}$ by a regular matrix $Q$:

$$Q^\top B^{-1} Q = I_d, \; Q^\top H Q = \Delta,$$

where $\Delta$ is a diagonal matrix. This transformation preserves the *inertia* of the symmetric matrices (that is, the numbers of their negative and positive eigenvalues). Then, since

$$\det M_\lambda = 0 \Leftrightarrow \det(Q^\top M_\lambda Q) = 0 \Leftrightarrow \det(\Delta + 4R(\lambda)I_d) = 0,$$

the residues must be real, and our fixed point can not have complex hyperbolic directions.

Following in the definite positive case, let $n$, $p$ be the numbers of negative and positive eigenvalues of $H$, respectively.

- Hence, since $p$ residues are negative, there exist $p$ regular hyperbolic pairs of eigenvalues and $n$ pairs of elliptic or inversion hyperbolic or inversion parabolic eigenvalues. The rest of pairs are regular parabolic.

- The eigenvalues of $H_{0,n+1}$,

  $$\bigcup_{j=1}^{n} \sigma\left(\hat{A} + 2c_j\hat{B}\right) \;=\; \bigcup_{j=1}^{n} \sigma\left(H + 2(1-c_j)B^{-1}\right),$$

  have the same sign that the eigenvalues

  $$\bigcup_{j=1}^{n} \sigma\left(\Delta + 2(1-c_j)I_d\right) \;=\; \bigcup_{j=1}^{n}\bigcup_{i=1}^{d}\{\tau_i - 2c_j\}.$$

Hence, if all the traces are $\geq 2$ (all the pairs are regular hyperbolic or parabolic) then the orbit is minimizing, and if all the traces are $\leq 2$ (all the pairs are inversion hyperbolic or parabolic) then the orbit is maximizing. We can also define an *extremal index* of the orbit, as the average of the all the extremal indices corresponding to the different traces ($\tau = (\tau_1, \dots, \tau_d)$):

$$\mathrm{IND}(\tau) = \frac{1}{d}\sum_{i=1}^{d} \mathrm{ind}(\tau_i).$$

## 5.6 Invariance of the extremal character

We wonder if the extremal character of an orbit is independent of the variables in which we write our symplectomorphism. In fact, we shall see that they do not change under lifts and vertical translations, but it can change under other kinds of symplectomorphisms. This fact is due to the concomitant distinction between $x$ and $y$ variables. For the sake of simplicity, we shall work in the monotone case.

### 5.6.1 Under vertical translations

Let $F$ be our symplectomorphism in $\mathbb{R}^{2d}$, given by

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases},$$

and $G = \tau_{\nabla l}$ be the vertical translation induced by the function $l : \mathbb{R}^d \to \mathbb{R}$, which defines a change of variables

$$\begin{cases} x = \bar{x} \\ y = \bar{y} + \nabla l(\bar{x}) \end{cases}.$$

Our symplectomorphism $F$ written in the new variables if $\bar{F} = G^{-1}{\circ}F{\circ}G$, and it is given by

$$\begin{cases} \bar{x}' = f(\bar{x}, \bar{y} + \nabla l(\bar{x})) \\ \bar{y}' = g(\bar{x}, \bar{y} + \nabla l(\bar{x})) - \nabla l(f(\bar{x}, \bar{y} + \nabla l(\bar{x}))) \end{cases}.$$

We remember that if the primitive function of $F$ is $S$, then the primitive function of $\bar{F}$ is $S{\circ}G + l{\circ}q - l{\circ}f{\circ}G$, where $q$ is the projection on the $x$-variables.

We consider now two corresponding orbits by $F$ and $\bar{F}$. Hence, let $(x_0, y_0)$ the initial point of a $F$-orbit and $(\bar{x}_0, \bar{y}_0) = G^{-1}(x_0, y_0)$ the initial point of the corresponding $\bar{F}$-orbit. We know that the extremal character of a $F$-orbit is given by the recurrence

$$\begin{cases} \hat{D}_1 = \hat{A}_1, \\ \\ \hat{D}_i = \hat{A}_i - \hat{B}_{i-1}^{\top}\hat{D}_{i-1}^{-1}\hat{B}_{i-1} \ (i = 2 \div n-1) \end{cases},$$

where $\hat{A}_i = D_{i-1}B_{i-1}^{-1} + B_i^{-1}A_i$ and $\hat{B}_i = -B_i^{-1}$ (see Section 5.4.1 for the terminology). We should write the same sequence for a $\bar{F}$-orbit putting bars in all the places, but we shall write $\check{A}_i$ rather than $\hat{\bar{A}}_i$, etc. We must relate the two sequences and see that the indexes of the matrices $\hat{D}_i$ and $\check{D}_i$ are the same. First of all, we must relate $\mathrm{D}\bar{F}(\bar{z})$ with $\mathrm{D}F(z)$. By the chain rule we obtain

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} A + BL_0 & B \\ \bar{C} & -L_1 B + D \end{pmatrix},$$

where $L_0 = \mathrm{D}^2 l(\bar{x})$ , $L_1 = \mathrm{D}^2 l(f(\bar{x}, \bar{y}))$, etc. We note that

monotonicity does not change under vertical translations.

Finally, the relationship between the matrices $\hat{A}_i$, $\hat{B}_i$ and $\check{A}_i$, $\check{B}_i$ is given by

$$\check{B}_i = \hat{B}_i$$

and

$$\begin{aligned} \check{A}_i &= \bar{D}_{i-1}\bar{B}_{i-1}^{-1} + \bar{B}_i^{-1}\bar{A}_i^{-1} \\ &= (-L_i B_{i-1} + D_{i-1})B_{i-1}^{-1} + B_i^{-1}(A_i + B_i L_i) \\ &= D_{i-1}B_{i-1}^{-1} + B_i^{-1}A_i \\ &= \hat{A}_i, \end{aligned}$$

and we obtain that they are the same.
**Remarks**

i) The extremal characters of fixed points and periodic orbits do not change by vertical translations.

ii) We note that the monotone positive character of our symplectomorphism can change. In fact: $\bar{D}\bar{B}^{-1} = DB^{-1} - L_1$ and $\bar{B}^{-1}\bar{A}^{-1} = B^{-1}A + L_0$.

◁

### 5.6.2 Under lifts

Let $F$ be our symplectomorphism in $\mathbb{R}^{2d}$, given by

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases},$$

and $G = \hat{\phi}$ be the lift of a certain diffeomorphism $\phi$ on $\mathbb{R}^d$, which defines a change of variables

$$\begin{cases} x = \phi(\bar{x}) \\ y = \mathrm{D}\phi(\bar{x})^{-\top}\bar{y} \end{cases}.$$

Our symplectomorphism $F$,written in the new variables, is $\bar{F} = G^{-1} \circ F \circ G$, and it is given by

$$\begin{cases} \bar{x}' = \phi^{-1}(f(\phi(\bar{x}), \mathrm{D}\phi(\bar{x})^{-\top}\bar{y})) \\ \bar{y}' = \mathrm{D}\phi(\bar{x}'))^{\top} g(\phi(\bar{x}), \mathrm{D}\phi(\bar{x})^{-\top}\bar{y})) \end{cases}.$$

We remember that if the primitive function of $F$ is $S$, then the primitive function of $\bar{F}$ is $S \circ \hat{\phi}$.

We consider now two corresponding orbits by $F$ and $\bar{F}$. Hence, let $(x_0, y_0)$ the initial point of a $F$-orbit and $(\bar{x}_0, \bar{y}_0) = G^{-1}(x_0, y_0)$ the initial point of the corresponding $\bar{F}$-orbit. We use the same notation than in the previous subsection.

By the chain rule we obtain

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} F_1^{-1} A F_0 + F_1^{-1} B G_0 & F_1^{-1} B F_0^{-\top} \\ \bar{C} & -G_1^\top B F_0^{-\top} + F_1^\top D F_0^{-\top} \end{pmatrix},$$

where $F_0 = \mathrm{D}\phi(\bar{x})$, $F_1 = \mathrm{D}\phi(\bar{x}')$, etc, and $G_0 = \frac{\partial y}{\partial \bar{x}}(\bar{x}, \bar{y})$, $G_1 = \frac{\partial y}{\partial \bar{x}}(\bar{x}', \bar{y}')$, etc. We note that the matrix $G_0^\top F_0$ is symmetric. We also note that

monotonicity does not change under vertical translations.

Thus, the relationship between the matrices $\hat{A}_i$, $\hat{B}_i$ and $\check{A}_i$, $\check{B}_i$ is given by

$$\check{B}_i = F_i^\top \hat{B}_i F_{i+1}$$

and

$$\begin{aligned} \check{A}_i &= \bar{D}_{i-1} \bar{B}_{i-1}^{-1} + \bar{B}_i^{-1} \bar{A}_i^{-1} \\ &= (-G_i^\top B_{i-1} F_i^{-\top} + F_i^\top D_{i-1} F_{i-1}^{-\top}) F_{i-1}^\top B_{i-1}^{-1} F_i + \\ &\quad F_i^\top B_i^{-1} F_{i+1} (F_{i+1}^{-1} A_i F_i + F_{i+1}^{-1} B_i G_i) \\ &= -G_i^\top F_i + F_i^\top G_i + F_i^\top (D_{i-1} B_{i-1}^{-1} + B_i^{-1} A_i) F_i \\ &= F_i^\top \hat{A}_i F_i. \end{aligned}$$

Finally, we obtain by induction that $\check{D}_i = F_i^\top \hat{D}_i F_i$, and, the extremal characters are the same.

**Remarks**

i) The extremal characters of fixed points and periodic orbits do not change by lifts.

ii) We note that the monotone positive character of our symplectomorphism can change. In fact: $\bar{D}\bar{B}^{-1} = F_1^\top D B^{-1} F_1 - G_1^\top F_1$ and $\bar{B}^{-1}\bar{A}^{-1} = F_0^\top B^{-1} A F_0 + F_0^\top G_0$.

◁

### 5.6.3 Statement of the result

The previous argumentation are summarized in the following.

**Proposition 5.6** *:*

*Given a symplectomorphism $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, the extremal character of monotone*

- *fixed points,*

- *periodic orbits,*
- *orbits,*

*do not change by*

- *vertical translations,*
- *lifts.*

A physical interpretation of this result is that the laws of the *discrete mechanics* are independent from the coordinates on our configuration space and certain privileged observers. This fact is geometrically connected with the choice of a certain 1-form $\alpha = y\,\mathrm{d}x$ in our phase space, and the distinction between $x$ and $y$ coordinates that it produces.

# Chapter 6

# Invariant Lagrangian graphs

A first step in order to understand the properties of invariant Lagrangian manifolds is to study the easier ones: the invariant Lagrangian graphs.

This chapter is devoted to extend some results due to Mather [73], Herman [40] and MacKay, Meiss and Stark [68], obtained by them by means of the use of a (global) Lagrangian generating function. In some sense our results are more local, because they do not use the existence of this global function, and they will let us to study different regions in our phase space where some positiveness condition will be satisfied.

This chapter will be completed in Chapter 9, where we deal with more general phase spaces, and in Appendix B, where we relate the BHM theory with Converse KAM theory and we obtain some non-existence criteria of invariant Lagrangian graphs when the configuration space is a torus.

## 6.1 Characterization

Given an open set $\mathcal{U} \subset \mathbb{R}^d$ and a function $l : \mathcal{U} \to \mathbb{R}$, we know that the immersion

$$\begin{array}{rccl} \nu : & \mathcal{U} & \longrightarrow & \mathbb{R}^d \times \mathbb{R}^d \\ & x & \longrightarrow & (x, \nabla l(x)) \end{array}$$

defines a Lagrangian embedding of $\mathcal{U}$ into $\mathbb{R}^d \times \mathbb{R}^d$, and its primitive function is $l$. We also know that if $\nu$ is invariant for a certain symplectomorphism $F = (f, g)$, we have a conserved quantity, given by the function

$$\begin{array}{rccl} \Phi : & \mathcal{U} & \longrightarrow & \mathbb{R} \\ & x & \longrightarrow & S(x, \nabla l(x)) - (l(f(x, \nabla l(x))) - l(x)). \end{array}$$

We want to obtain more information.

First of all, we extend the function $\Phi$ to the function $\hat{\Phi} : \mathcal{U} \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$\hat{\Phi}(x, y) = S(x, y) - (l(f(x, y)) - l(x)),$$

so that $\Phi(x) = \hat{\Phi}(x, \nabla l(x))$. We have the next proposition of characterization of invariant Lagrangian graphs (in short, *i.L.g.*).

**Proposition 6.1** *:*

*Let $F = (f, g)$ be a symplectomorphism on $\mathbb{R}^d \times \mathbb{R}^d$, with primitive function $S$,and $l : \mathcal{U} \to \mathbb{R}$ be a generating function of the exact Lagrangian graph $\mathcal{L}$, being $\mathcal{U}$ a connected open set $\mathcal{U} \subset \mathbb{R}^d$ into $\mathbb{R}^{2d}$.*
*We define the functions $\Phi$ and $\hat{\Phi}$ as above.*
*Then:*

1. *(Conserved quantity on an i.L.g.)*

$$\mathcal{L} \text{ is } F\text{-invariant} \Rightarrow \Phi \text{ is constant.}$$

2. *(Characterization of i.L.g.)*

$$\mathcal{L} \text{ is } F\text{-invariant} \Leftrightarrow \forall x \in \mathcal{U} \ \frac{\partial \hat{\Phi}}{\partial (x, y)}(x, \nabla l(x)) = 0.$$

3. *(Characterization of the points of an i.L.g.)*
*If $F$ is monotone, and $\mathcal{L}$ is $F$-invariant:*

$$y = \nabla l(x) \Leftrightarrow \frac{\partial \hat{\Phi}}{\partial (x, y)}(x, y) = 0.$$

*Moreover, if $\mathcal{L}$ is $F^{-1}$-invariant:*

$$y = \nabla l(x) \Leftrightarrow \frac{\partial \hat{\Phi}}{\partial y}(x, y) = 0.$$

*Proof*:

We write the *invariance condition* as

$$\forall x \in \mathcal{U} \ \ g(x, \nabla l(x)) = \nabla l(f(x, \nabla l(x))).$$

1. First point is an immediate consequence of the second.
2. The derivatives of $\hat{\Phi}$ are:

$$\begin{aligned} \frac{\partial \hat{\Phi}}{\partial x}(x, y) &= \left( g(x, y)^\top - \frac{\partial l}{\partial x}(f(x, y)) \right) \cdot \frac{\partial f}{\partial x}(x, y) - y^\top + \frac{\partial l}{\partial x}(x), \\ \frac{\partial \hat{\Phi}}{\partial y}(x, y) &= \left( g(x, y)^\top - \frac{\partial l}{\partial x}(f(x, y)) \right) \cdot \frac{\partial f}{\partial y}(x, y). \end{aligned}$$

So if $l$ gives a $F$-invariant graph the two derivatives vanish (the points of the invariant graph are critical for the function $\hat{\Phi}$) and, in particular, the function $\Phi$ is constant:

$$\frac{\partial \Phi}{\partial x}(x) = \frac{\partial \hat{\Phi}}{\partial x}(x, \nabla l(x)) + \frac{\partial \hat{\Phi}}{\partial y}(x, \nabla l(x)) \cdot \frac{\partial^2 l}{\partial x^2}(x) = 0.$$

Conversely, if the derivatives vanish at a point $(x, \nabla l(x))$, then we obtain

$$g(x, \nabla l(x)) \quad = \quad \nabla l(f(x, \nabla l(x))),$$

because the rank of the matrix $(A(x,y) \;\; B(x,y))$ is maximal at all points.

3. Suppose $F$ be monotone, that is to say, $|B(x,y)| \neq 0$, $\forall (x,y) \in \mathcal{U} \times \mathbb{R}^d$. Then

$$\frac{\partial \hat{\Phi}}{\partial(x,y)}(x,y) = 0 \quad \Rightarrow \quad y = \nabla l(x)$$

(the points of the $F$-invariant graph correspond with the critical points of a certain function). If, moreover, the graph is $F^{-1}$-invariant, then

$$\begin{aligned} \frac{\partial \hat{\Phi}}{\partial y}(x,y) = 0 \quad &\Rightarrow \quad g(x,y) = \nabla l(f(x,y)) \\ &\Rightarrow \quad F(x,y) \in \mathcal{L} \\ &\Rightarrow \quad (x,y) \in \mathcal{L} \\ &\Rightarrow \quad y = \nabla l(x) \end{aligned}$$

(the *fibered critical points* of $\hat{\Phi}$ correspond with the points of the invariant graph).

□

## 6.2 Extremal character of an i.L.g.

As we have seen, a point of an i.L.g. $\mathcal{L} = \mathcal{L}_{\nabla l}$ is a fibered critical point of the function $\hat{\Phi}$, that is to say, for all point $x \in \mathcal{U}$

$$\frac{\partial \hat{\Phi}}{\partial y}(x, \nabla l(x)) \quad = \quad 0.$$

The extremal character of the graph in each point $(x, \nabla l(x))$ is given by

$$\frac{\partial^2 \hat{\Phi}}{\partial y^2}(x, \nabla l(x)) \quad = \quad (D^\top(x) - B(x)^\top \, \mathrm{D}^2 l(f(x)))B(x),$$

where we write $f(x) = f(x, \nabla l(x))$, $A(x) = A(x, \nabla l(x))$, etc. If all the points have the same character as critical points of the 'fiber' function $\hat{\Phi}$, and then all the corresponding Hessian matrices have non vanishing eigenvalues, we shall say that our graph is *non degenerate*. In such a case, the graph have to be *monotone* (i.e., it have to be included in a monotone region). Then, if all these matrices are positive definite we shall say that our graph is *minimizing*, and if all of them are negative definite we shall say that it is *maximizing*. Otherwise we shall say that it is *undefinite*.

As we shall see, the extremal character of our graph does not change under vertical translations and lifts. In fact, we shall see a little less than this, but enough for us. We shall perform two steps of normal form in order to simplify the dynamics around an i.L.g.. For the sake of simplicity, we shall suppose $\mathcal{U} = \mathbb{R}^d$.

**Proposition 6.2** *:*

*Let $F = (f, g)$ be a symplectomorphism on $\mathbb{R}^d \times \mathbb{R}^d$, with primitive function $S$, and $\mathcal{L}$ be an i.L.g. generated by $l : \mathbb{R}^d \to \mathbb{R}$.*
*Hence, the extremal character of the graph does not change after the next two steps of normal form:*

1. *projection of the zero-section,*
2. *simplification of the dynamics on that zero-section, via conjugation by a lift.*

*Proof*:

1. Let $\mathcal{L}$ be an invariant graph given by a generating function $l : \mathbb{R}^d \to \mathbb{R}$. Then its character is given by the indexes of the symmetric matrices

$$(D^\top(x) - B(x)^\top \, \mathrm{D}^2 l(f(x)))B(x).$$

If we make a change of variables, by means of the vertical translation

$$\begin{cases} x = \bar{x} \\ y = \bar{y} + \nabla l(\bar{x}) \end{cases},$$

then in the new variables $\bar{x}, \bar{y}$ the zero-section $\{y = 0\}$ is fixed. After this projection, the character of the graph (the zero-section) is given by

$$\begin{aligned} \bar{D}(\bar{x})^\top \bar{B}(\bar{x}) &= (D(x) - L_1(\bar{x}, 0)B(x))^\top B(x) \\ &= (D(x)^\top - B(x)^\top \mathrm{D}^2 l(f(x)))B(x), \end{aligned}$$

and the character does not change (see the notation in Section 5.6). In fact, while

$$\hat{\Phi}(x, y) = S(x, y) + l(x) - l(f(x, y))$$

we have

$$\begin{aligned} \check{\Phi}(\bar{x}, \bar{y}) &= \bar{S}(\bar{x}, \bar{y}) \\ &= S(\bar{x}, \bar{y} + \nabla l(\bar{x})) + l(\bar{x}) - l(f(\bar{x}, \bar{y} + \nabla l(\bar{x}))). \end{aligned}$$

2. Suppose the zero-section is fixed. If we conjugate by a lift $\hat{\phi}$

$$\begin{cases} x = \phi(\bar{x}) \\ y = \mathrm{D}\phi(\bar{x})^{-\top}\bar{y} \end{cases},$$

then in the new variables $\bar{x}, \bar{y}$ the zero-section $\{y = 0\}$ is also fixed. While the character of the zero-section for $F$ is given by

$$\hat{\Phi}(x, y) = S(x, y),$$

for $\bar{F}$ is given by

$$\check{\Phi}(\bar{x}, \bar{y}) = S(\phi(\bar{x}), \mathrm{D}\phi(\bar{x})^{-\top}\bar{y}).$$

Hence, since $\frac{\partial S}{\partial y}(x, 0) = 0,\ \forall x \in \mathbb{R}^d$, then

$$\frac{\partial^2\check{\Phi}}{\partial\bar{y}^2}(\bar{x}, 0) = \mathrm{D}\phi(\bar{x})^{-1}\frac{\partial^2\hat{\Phi}}{\partial y^2}(\phi(\bar{x}), 0)\mathrm{D}\phi(\bar{x})^{-\top}.$$

and the extremal characters coincide.

□

## 6.3 Minimizing invariant Lagrangian graphs

For minimizing invariant Lagrangian graphs we have the following theorem. It asserts that the orbits on a minimizing i.L.g. are also minimizing, and it will be a key result in order to perform non existence criteria of i.L.g. (see Appendix B).

**Theorem 6.1** *:*

*Let $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be a symplectomorphism, with primitive function $S$, and $\mathcal{L} = \mathcal{L}_{\nabla l}$ be a minimizing i.L.g., generated by the function $l : \mathbb{R}^d \to \mathbb{R}$. Then:*

*All the orbits on the graph are minimizing.*

*Proof*:

Before doing a complete proof we shall see what is the key of our methodology. For the sake of simplicity, we shall suppose that our graph is globally minimizing, i.e., if $C \in \mathbb{R}$ is the conserved quantity associated to the graph then $\forall (x, y) \in \mathbb{R}^{2d}$ $\hat{\Phi}(x, y) \geq C$. Hence, let $(x, \nabla l(x))$ be any point on $\mathcal{L}_{\nabla l}$. First, we fix $m, n \in \mathbb{Z}$, with the condition $m + 1 < n$ and take $x_m = q\circ F^m(x, \nabla l(x))$ and $x_n = q\circ F^n(x, \nabla l(x))$. Now, let $\boldsymbol{\rho} = (x_i, y_i)_{i=m\div n-1}$ be any $F$-chain connecting $x_m$ with $x_n$, and $\boldsymbol{o}$ be the corresponding segment of orbit (i.e. $\boldsymbol{o} = (F^i(x, \nabla l(x)))_{i=m\div n-1}$). Hence, the corresponding actions verify:

$$\begin{aligned}
\boldsymbol{S}_{mn}(\boldsymbol{\rho}) &= \sum_{i=m}^{n-1} S(x_i, y_i) \\
&= \sum_{i=m}^{n-1} (S(x_i, y_i) - l(f(x_i, y_i)) + l(x_i)) \ + l(x_n) - l(x_m) \\
&= \sum_{i=m}^{n-1} \hat{\Phi}(x_i, y_i) \ + l(x_n) - l(x_m) \\
&\geq (n-m)C + l(x_n) - l(x_m) \\
&= \sum_{i=m}^{n-1} \hat{\Phi}(F^i(x, \nabla l(x))) \ + l(x_n) - l(x_m) \\
&= \boldsymbol{S}_{mn}(\boldsymbol{o}).
\end{aligned}$$

Hence, the connecting orbit minimizes the action on the chains. Notice that the restriction of the action to the set of $F$-chains is fundamental.

We have to improve this result and obtain that any segment of orbit is, in fact, a non degenerate minimum of the corresponding action. We do not need global conditions. Thanks to the invariance of the extremal character of orbits and graphs under vertical translations, we can restrict our attention to the case in which our graph is the zero-section. In such a case, our symplectomorphism is given by

$$\begin{cases} x' = \phi(x) + B(x)y + \dots \\ y' = \mathrm{D}\phi(x)^{-\top} y + \dots \end{cases},$$

where '$\dots$' means terms in upper orders in $y$. Hence we have

$$\mathrm{D}F(x,0) = \begin{pmatrix} A(x) & B(x) \\ 0 & D(x) \end{pmatrix},$$

where $A(x) = \mathrm{D}\phi(x)$ , $D(x) = A(x)^{-\top}$, and $A(x)B(x)^\top = B(x)A(x)^\top$. Therefore

$$\begin{aligned}
\frac{\partial^2 \hat{\Phi}}{\partial y^2}(x,0) &= \frac{\partial^2 S}{\partial y^2}(x,0) \\
&= D(x)^\top B(x) = A(x)^{-1}B(x)
\end{aligned}$$

Hence, our graph is minimizing iff [1]

$$A(x)^{-1}B(x) \succ 0,$$

[1] For any symmetric matrix $S$, $S \succ 0$ means that $S$ is positive definite.

for all the points $x \in \mathbb{R}^d$. In fact, the symmetric matrices $A^{-1}B = B^\top A^{-\top}$ and $B^{-\top}A^{-1} = A^{-\top}B^{-1}$ have the same inertia (recall that 'minimizing' implies 'monotone'). Moreover [2]:

$$A^{-1}B \succ 0 \Leftrightarrow B^{-1}A \succ 0 \Leftrightarrow A^{-\top}B^{-1} \succ 0 \Leftrightarrow AB^\top \succ 0.$$

Now, let $x$ be giving any point on the graph. As always, we shall write $A_i = A(\phi^i(x))$, $B_i = B(\phi^i(x))$ and $D_i = D(\phi^i(x)) = A_i^{-\top}$. We shall use the MMS iteration. In this case we have

$$\hat{A}_i \;=\; A_{i-1}^{-\top}B_{i-1}^{-1} + B_i^{-1}A_i \succ 0.$$

We shall prove by induction that $\hat{D}_i = B_i^{-1}A_i + K_i$, where $K_i \succ 0$, for any $i \geq 1$.

- For $i = 1$ we have $\hat{D}_1 = B_1^{-1}A_1 + K_1$, where $K_1 = A_0^{-\top}B_0^{-1} \succ 0$, and the property holds.
- Suppose that the property is true for $i-1$. Hence

$$\hat{D}_i \;=\; B_i^{-1}A_i + A_{i-1}^{-\top}B_{i-1}^{-1} - B_{i-1}^{-\top}\hat{D}_{i-1}^{-1}B_{i-1}^{-1}$$

  and then

$$\begin{aligned} K_i &= A_{i-1}^{-\top}B_{i-1}^{-1} - B_{i-1}^{-\top}\hat{D}_{i-1}^{-1}B_{i-1}^{-1} \\ &= A_{i-1}^{-\top}B_{i-1}^{-1} - (B_{i-1}\hat{D}_{i-1}B_{i-1}^\top)^{-1}. \end{aligned}$$

  Therefore, since

$$\begin{aligned} \hat{D}_{i-1} &= B_{i-1}^{-1}A_{i-1} + K_{i-1} \\ &\succ B_{i-1}^{-1}A_{i-1} \succ 0 \end{aligned}$$

  then

$$B_{i-1}\hat{D}_{i-1}B_{i-1}^\top \;\succ\; A_{i-1}B_{i-1}^\top \succ 0,$$

  and, finally

$$K_i \;\succ\; A_{i-1}^{-\top}B_{i-1}^{-1} - B_{i-1}^{-\top}A_{i-1}^{-1} = 0.$$

In summary, all the matrices $\hat{D}_i$ are positive definite. □

**Remark**

Of course, we obtain a similar result for maximizing i.L.g.. ◁

---

[2] $A \succ 0 \Rightarrow A^{-1} \succ 0$, for any symmetric matrix $A$.

# Part III

# ON THE COTANGENT BUNDLE

# Chapter 7

# Symplectic geometry on the cotangent bundle

We recall and introduce some basic results related with the canonical symplectic structure on the cotangent bundle of a certain manifold $\mathcal{M}$, $T^*\mathcal{M}$. At the first section, we recall basic facts about the Liouville form and the Liouville vector field on $T^*\mathcal{M}$, and introduce the Liouville derivative, that is the derivation $\mathbf{\Lambda}$ in this context (see Section 2.3). Secondly, we apply the definition of exact symplectomorphism and, finally, we recall some examples of Lagrangian manifolds.

## 7.1 Liouville objects

Let $\mathcal{M}$ be a $d$-dimensional manifold and $T^*\mathcal{M}$ its cotangent bundle. The *zero-section* is

$$\begin{array}{rccc} z: & \mathcal{M} & \to & T^*\mathcal{M} \\ & x & \to & 0_x \end{array}$$

and $\mathcal{M}_0 = z(\mathcal{M})$, and the *projection* is

$$\begin{array}{rccc} q: & T^*\mathcal{M} & \to & \mathcal{M} \\ & \rho_x & \to & x. \end{array}$$

We know that we can define a differentiable structure on $T^*\mathcal{M}$ by means of the *cotangent charts* $\mathcal{U}\times\mathbb{R}^d$, where each $\mathcal{U}$ is a local chart of $\mathcal{M}$. We write the corresponding *cotangent coordinates* as $(x, y) = (x_1, \ldots, x_d, y_1 \ldots, y_d)$.

Given a map $F : \mathcal{P} \to T^*\mathcal{M}$ from a manifold $\mathcal{P}$ to the cotangent bundle $T^*\mathcal{M}$, we shall refer to $f = q{\circ}F$ as *its basic component.*

### 7.1.1 The Liouville form

We recall that the *Liouville form* is the Pfaffian form on $T^*\mathcal{M}$ whose value at a point $\rho \in T^*\mathcal{M}$ is given by

$$\alpha_\rho \;=\; \rho{\circ}q_*(\rho).$$

Moreover, $\alpha$ is the unique Pfaffian form on $T^*\mathcal{M}$ which satisfies

$$\rho^*\alpha = \rho,$$

for any Pfaffian form on $\mathcal{M}$, $\rho \in \Omega^1(\mathcal{M})$.

Then, $\boldsymbol{\omega} = \mathrm{d}\alpha$ is the *canonical* symplectic structure on $T^*\mathcal{M}$, and it is exact. In cotangent coordinates on $T^*\mathcal{M}$, $(x, y) \in U \times \mathbb{R}^d$, these forms are:

$$\alpha = \sum_{i=1}^{d} y_i \mathrm{d}x_i, \; \boldsymbol{\omega} = \sum_{i=1}^{d} \mathrm{d}y_i \wedge \mathrm{d}x_i$$

($\boldsymbol{\alpha} = y\mathrm{d}x$ and $\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x$ for short).

**Remark**

> We can define other symplectic structures on the cotangent bundle by means of closed 2-forms on $\mathcal{M}$, $\mu \in \Omega^2(\mathcal{M})$, by
>
> $$\boldsymbol{\omega}_\mu = \mathrm{d}\boldsymbol{\alpha} + q^*\mu.$$
>
> ◁

### 7.1.2 The Liouville vector field

We shall denote by $\boldsymbol{Z}^*$ the *Liouville vector field* on $T^*\mathcal{M}$, which is the only vector field that satisfies the relation

$$\mathrm{i}_{\boldsymbol{Z}^*}\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{\alpha}.$$

Moreover, it satisfies the relations

$$\mathrm{i}_{\boldsymbol{Z}^*}\boldsymbol{\alpha} = 0, \; \mathrm{L}_{\boldsymbol{Z}^*}\boldsymbol{\alpha} = \boldsymbol{\alpha}, \; \mathrm{L}_{\boldsymbol{Z}^*}\mathrm{d}\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\alpha}.$$

(see [61] for further information and generalizations of this subject).

- This vector field is *vertical* ($q_*\boldsymbol{Z}^* = 0$), and it is written in cotangent coordinates as

  $$\boldsymbol{Z}^* = \sum_{i=1}^{d} y_i \frac{\partial}{\partial y_i}$$

  ($\boldsymbol{Z}^* = y\frac{\partial}{\partial y}$ for short).

- It is complete, that is, its flow is defined for all time $t \in \mathbb{R}$. In fact, it is given by the 1-parameter group of positive dilations of each fiber of $T^*\mathcal{M}$:

  $$h_t(\rho_x) = e^t \rho_x.$$

  It gives to $T^*\mathcal{M}$ a principal bundle structure, where the structure group is the additive group $\mathbb{R}$ of real numbers.

### 7.1.3 The Liouville derivative

We remember that we have a derivation on $\mathcal{F} = \mathcal{F}(T^*\mathcal{M})$, endowed with the Poisson bracket, given by the linear operator

$$\begin{array}{rccl} \mathbf{\Lambda}: & \mathcal{F} & \rightarrow & \mathcal{F} \\ & H & \rightarrow & \boldsymbol{\alpha}(X_H) - H. \end{array}$$

(see Section 2.3). In this context, $\mathbf{\Lambda}(H)$ can be written by means of the Liouville vector field on the cotangent bundle. By this reason we shall refer to the $\mathbf{\Lambda}$-derivative as *Liouville derivative*. Furthermore, $\mathbf{\Lambda}(H)$ is also known by the *elementary action* associated with the Hamiltonian $H$, because it is used in order to define a variational principle for its orbits (see [5, 61]). It is used in order to define the Legendre transformation between the tangent and cotangent bundle of the configuration space $\mathcal{M}$.

**Proposition 7.1** *:*

*The derivation* $\Lambda$ *associated to the canonical symplectic structure on* $T^*\mathcal{M}$ *satisfies the next relations:*

- $\mathbf{\Lambda}(H) = \mathrm{d}H(\boldsymbol{Z}^*) - H,$
- $X_{\mathbf{\Lambda}(H)} = [\boldsymbol{Z}^*, X_H]$ .

*Proof*:

- First,

$$\begin{aligned} \boldsymbol{\alpha}(X_H) &= \mathrm{i}_{\boldsymbol{Z}^*}\mathrm{d}\boldsymbol{\alpha}(X_H) = -\mathrm{i}_{X_H}\mathrm{d}\boldsymbol{\alpha}(\boldsymbol{Z}^*) \\ &= \mathrm{d}H(\boldsymbol{Z}^*). \end{aligned}$$

- And

$$\begin{aligned} \mathrm{i}_{[\boldsymbol{Z}^*, X_H]}\,\boldsymbol{\omega} &= \mathrm{L}_{\boldsymbol{Z}^*}\mathrm{i}_{X_H}\boldsymbol{\omega} - \mathrm{i}_{X_H}\mathrm{L}_{\boldsymbol{Z}^*}\boldsymbol{\omega} = -\mathrm{L}_{\boldsymbol{Z}^*}\mathrm{d}H - \mathrm{i}_{X_H}\boldsymbol{\omega} \\ &= -\mathrm{L}_{\boldsymbol{Z}^*}\mathrm{d}H + \mathrm{d}H = \mathrm{d}(H - \mathrm{L}_{\boldsymbol{Z}^*}H) \\ &= -\mathrm{d}(\mathbf{\Lambda}(H)). \end{aligned}$$

□

The expression of $\mathbf{\Lambda}(H)$ in cotangent coordinates $(x, y) \in U \times \mathbb{R}^d$ is:

$$\mathbf{\Lambda}(H)(x, y) = \mathrm{d}H(y\frac{\partial}{\partial y}) - H(x, y) = y \cdot \nabla_y H(x, y) - H(x, y).$$

We see that $\mathbf{\Lambda}$ is a vertical operator, because the value of $\mathbf{\Lambda}(H)$ on a fiber only depends on the value of $H$ on such fiber.

**Remark**

Although we shall not use this in the sequel, we note that we can extend the definition of the Liouville derivative to be applied to forms and vector fields.

- If $X \in \mathcal{X}(T^*\mathcal{M})$ we define $\mathbf{\Lambda}(X) = [\boldsymbol{Z}^*, X]$ , and hence $X_{\mathbf{\Lambda}(H)} = \mathbf{\Lambda}(X_H)$.
- If $\beta \in \Omega(T^*\mathcal{M})$ we define $\mathbf{\Lambda}(\beta) = \mathrm{L}_{\boldsymbol{Z}^*}\beta - \beta$, and hence the Liouville derivative commutes with the exterior derivative

$$\mathbf{\Lambda} \circ \mathrm{d} \;=\; \mathrm{d} \circ \mathbf{\Lambda}.$$

Obviously,

$$\beta \in \ker \mathbf{\Lambda} \;\Rightarrow\; \mathrm{d}\beta \in \ker \mathbf{\Lambda}.$$

The converse is false. For instance, the Liouville form $\boldsymbol{\alpha}$ belongs to $\ker \mathbf{\Lambda}$, but there does not exist any function $H$ such that $\mathrm{d}H = \boldsymbol{\alpha}$. Notice also that $\boldsymbol{Z}^*$ is a *conformal infinitesimal automorphism* of the forms of $\ker \mathbf{\Lambda}$. In fact, if the Liouville derivative of $\beta \in \Omega^k(T^*\mathcal{M})$ vanishes, then

$$h_t^*\eta(\rho) \;=\; e^t \cdot \eta(\rho),$$

that is to say,

$$\eta(\rho)(e^t\hat{X}_1(\rho), \ldots, e^t\hat{X}_k(\rho)) \;=\; e^t \cdot \eta(\rho)(\hat{X}_1(\rho), \ldots, \hat{X}_k(\rho)),$$

for all $t \in \mathbb{R}$ and $\hat{X}_1(\rho), \ldots, \hat{X}_k(\rho) \in T_\rho T^*\mathcal{M}$.

◁

Following with the previous remark, but working with 0-forms, we have that

$$\mathbf{\Lambda}(H) = 0 \;\Leftrightarrow\; H(e^t\rho) = e^t H(\rho),$$

that is, $H$ is positively homogeneous of degree 1 on each fiber. Hence, the functions of $\ker \mathbf{\Lambda}$ are written in cotangent coordinates $(x, y)$ as

$$H(x, y) = a(x) \cdot y.$$

We recall (see Section 2.3) that the flows of these Hamiltonians preserve the Liouville form (in fact, it is enough to have constant Liouville derivative).

**Remarks**

i) If $\mathbf{\Lambda}(H) = C$, being $C$ a certain constant, then we can consider the Hamiltonian $H' = H + C$. Hence, $\mathbf{\Lambda}(H') = 0$ and we can apply the previous results. Of course, the corresponding flows to $H$ and $H'$ coincide.

ii) Let $\varphi_t$ be the flow of $X_H$. Then:

$$\begin{aligned}
\boldsymbol{\alpha} \text{ is invariant under } X_H \;&\Leftrightarrow\; \phi_t^*\alpha = \alpha \\
&\Leftrightarrow\; \mathrm{L}_{X_H}\boldsymbol{\alpha} = 0 \\
&\Leftrightarrow\; \mathrm{d}(\mathbf{\Lambda}(H)) = 0 \\
&\Leftrightarrow\; \mathbf{\Lambda}(H) = C,\; C \in \mathbb{R} \\
&\Leftrightarrow\; [\boldsymbol{Z}^*, X_H] \;= 0 \\
&\Leftrightarrow\; \text{the flows of } \boldsymbol{Z}^* \text{ and } X_H \text{ commute}
\end{aligned}$$

(that is to say, $e^s\varphi_t(\rho) = \varphi_t(e^s\rho)$ when the times have sense). In particular, $\varphi_t(0_x) = 0_{q\circ\varphi_t(x)}$, that is to say, the zero-section is invariant under $X_H$.

◁

In Chapter 10, we shall do a more intensive study of the Liouville derivative.

## 7.2 Exact symplectomorphisms

### 7.2.1 Exactness formulae

Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, and $S : T^*\mathcal{M} \to \mathbb{R}$ be its primitive function. Since

$$F^*\alpha - \alpha = \mathrm{d}S,$$

then

$$\begin{aligned} \mathrm{d}S(\rho_x) &= (F^*\alpha)_{\rho_x} - \alpha_{\rho_x} \\ &= \alpha_{F(\rho_x)}\circ F_*(\rho_x) - \alpha_{\rho_x} \\ &= F(\rho_x)\circ q_*(F(\rho_x))\circ F_*(\rho_x) - \rho_x\circ q_*(\rho_x) \\ &= F(\rho_x)\circ(q\circ F)_*(\rho_x) - \rho_x\circ q_*(\rho_x). \end{aligned}$$

Let $\rho \in \Omega^1(\mathcal{M})$ be an 1-form on $\mathcal{M}$. Then

$$\rho^*(\mathrm{d}S) = \mathrm{d}(\rho^*S) = \mathrm{d}(S\circ\rho)$$

and

$$\rho^*(F^*\alpha - \alpha) = (F\circ\rho)^*\alpha - \rho.$$

So, we have

$$\mathrm{d}(S\circ\rho) = (F\circ\rho)^*\alpha - \rho.$$

Hence, we have obtained the following proposition.

**Proposition 7.2** :

*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with $pf(F) = S$, and let $f = q\circ F$ be its basic component. Then:*

- $\forall\rho_x \in T^*_x\mathcal{M}$

$$\mathrm{d}S(\rho_x) = F(\rho_x)\circ f_*(\rho_x) - \rho_x\circ q_*(\rho_x);$$

- $\forall\rho \in \Omega^1(\mathcal{M})$

$$\mathrm{d}(S\circ\rho) = (F\circ\rho)^*\alpha - \rho.$$

### 7.2.2 Lifts

By means of the Liouville vector field (and thanks to its completeness), it can be proved that the unique actionmorphisms on *all* $T^*\mathcal{M}$ are the lifts of diffeomorphisms on $\mathcal{M}$ (see [61], p.66). If $f:\mathcal{M}\to\mathcal{M}$ is a diffeomorphisms, its *lift* (or *lifting*) is $\hat{f}:T^*\mathcal{M}\to T^*\mathcal{M}$, defined by:

$$\hat{f}(\rho_x) = (f^{-1})^*_{f(x)}\rho_x \in T^*_{f(x)}\mathcal{M}.$$

Obviously: $f\circ q = q\circ\hat{f}$. In cotangent coordinates we write the lift as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} f(x) \\ \mathrm{D}f(x)^{-\top}y \end{pmatrix}.$$

From this, an exact symplectomorphism on $T^*\mathcal{M}$ is determined by its primitive function up to diffeomorphisms on the base.

Let $X\in\mathcal{X}(M)$ be a vector field on $M$. While its lift to the tangent bundle is given by the variational equations, we can lift it to the cotangent bundle following two procedures.

- By defining the Hamiltonian vector field corresponding to the function

$$\begin{array}{rccc} H: & T^*\mathcal{M} & \to & \mathbb{R} \\ & \rho_x & \to & \rho_x(X_x) \end{array}.$$

  We write $\hat{X}=X_H$. In cotangent coordinates, the Hamiltonian function is

$$H(x,y) = y\cdot f(x),$$

  and the corresponding vector field is

$$\left\{\begin{array}{l} \dot{x} = f(x) \\ \dot{y} = -\mathrm{D}f(x)^\top y \end{array}\right. .$$

- By defining the vector field as the velocity of the continuous group given by the lift of the flow $\varphi_t$ of the initial vector field:

$$\hat{X}(\rho_x) = \frac{d}{dt}\left(\hat{\varphi}_t(\rho_x)\right)\Big|_{t=0} = \frac{d}{dt}\left(\varphi^*_{-t}(\varphi_t(x))\rho_x\right)\Big|_{t=0}.$$

In both cases we obtain the same vector field, which verifies $\hat{X}^*\circ z = z_*X$ (see [61]). Moreover, we note that these lifts of 'configurational' vector fields belong to ker $\mathbf{\Lambda}$.

**Remark**

In fact, we can lift the vector field to the fiber product $T\mathcal{M}\otimes_M T^*M$. Moreover, this product is a symplectic vector bundle. ◁

### 7.2.3 Fiberwise translations

Given a 1-form $\beta$ on $\mathcal{M}$, a *fiberwise translation* by $\beta$ is the map $\tau_\beta : T^*\mathcal{M} \to T^*\mathcal{M}$ defined by

$$\tau_\beta = id_{T^*\mathcal{M}} + \beta\circ q.$$

If $\beta$ is closed, then $\tau_\beta$ is a symplectomorphism. If it is exact, with $\beta = \mathrm{d}l$, then we have an exact symplectomorphism, and its primitive function is

$$pf(\tau_{dl}) = l\circ q.$$

### 7.2.4 Monotonicity

Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be a diffeomorphism, and $f : T^*\mathcal{M} \to \mathcal{M}$ be its basic component. In this context, we also are able to define the monotony condition, which is an important property that $F$ can verify, as we shall see later. We shall say that $F$ is *monotone* iff

$$\forall x \in \mathcal{M}\ f : T^*_x\mathcal{M} \to \mathcal{M} \text{ is a local diffeomorphism,}$$

or, equivalently, iff

$$\forall \rho_x \in T^*\mathcal{M}\ f_*(\rho_x) : V_{\rho_x}T^*\mathcal{M} \to T_{q\circ F(\rho_x)}\mathcal{M} \text{ is an isomorphism.}$$

Here, $V\,T^*\mathcal{M}$ means the *vertical tangent bundle* of the cotangent bundle: $V\,T^*\mathcal{M} = \ker q_*$ (we can define this for all fibration). Geometrically speaking, $F$ is monotone iff it is transversal to the leaves of the standard foliation of the cotangent bundle.
**Remark**

> We note that the lifts and the fiberwise translations are not monotone. In general, the composition of two monotone maps is not a monotone map. $\triangleleft$

## 7.3 Exact Lagrangian graphs

Let $\mathcal{L}$ be a submanifold of dimension $d$ of the symplectic manifold $(\mathcal{N}, \omega)$ of dimension $2d$. We recall that $\mathcal{L}$ is a *Lagrangian manifold* iff $\omega_{|\mathcal{L}} = 0$, i.e.:

$$\omega_z(X_z, Y_z) = 0,\ \forall z \in \mathcal{L},\ \forall X_z, Y_z \in T_z\mathcal{L}.$$

The following result furnishes an important example of an exact Lagrangian manifold in a cotangent bundle (see [61], p. 92).

> *Let $\mathcal{M}$ be a manifold and $\mathcal{N} = T^*\mathcal{M}$ its cotangent bundle equipped with its canonical symplectic form* $\mathrm{d}\boldsymbol{\alpha}$.
> *Let $\beta : \mathcal{M} \to T^*\mathcal{M}$ be a Pfaffian form on the manifold $\mathcal{M}$ and $\mathcal{L}_\beta = \{\beta_x \mid x \in \mathcal{M}\}$ its graph.*
> *Then:*
>
> $\mathcal{L}_\beta$ *is a Lagrangian manifold of* $\mathcal{N} \Leftrightarrow \beta$ *is closed.*

Then, we say that $\mathcal{L}_\beta$ is a *Lagrangian graph*. If $\beta$ is exact, with $\beta = \mathrm{d}l$ (where $l : \mathcal{M} \to \mathbb{R}$), we say that $l$ is the *generating function* of the *exact Lagrangian graph* $\mathcal{L}_{dl}$. In particular, the image of the zero-section, $\mathcal{L}_z$, is an exact Lagrangian graph (it is often identified with $\mathcal{M}$) that admits the zero-function as a generating function.

It is interesting to notice that the problem of finding intersections of two exact Lagrangian graphs, $\mathcal{L}_{\mathrm{d}l_1}$ and $\mathcal{L}_{\mathrm{d}l_2}$, is reduced to finding critical points of a real-valued function, $l_1 - l_2$. Hence, the theory of intersections between exact Lagrangian graphs is rather trivial.

We can transport an invariant (exact) Lagrangian graph to the base space, via a *fiberwise translation* by a closed (exact) 1-form and then obtain a normal form around the zero-section (see Appendix F). In fact, as Weinstein proved [97, 98], a zero-section is the universal model of Lagrangian submanifold, on an open neighborhood of it. A small summary of these results there is in Section G.2.2.

Finally, there are many results about the topological properties of general Lagrangian manifolds defined on a cotangent bundle. A survey of results about exact Lagrangian manifolds is given in [59]. On the other side, we can define exact Lagrangian manifolds with foldings (with respect to the standard foliation) by adding parameters to the generating function. It is the method of the Morse families or phase functions [44, 98].

# Chapter 8

# Variational principles

The purpose of this chapter is to obtain several variational principles associated to any symplectomorphism defined on the cotangent bundle of a manifold (with the natural symplectic structure). In all cases, the variational principles will not depend on the coordinates on the configuration space.

On one side, the idea of associating with a symplectic map $F$ a function $h$ such that the critical points of $h$ are fixed points of $F$ goes back Poincaré [85], and has been used by many authors, as Arnold, Weinstein, Moser, Banyaga, Arnaud, Golé, etc. In many cases, the constructed critical function $h$ is not coordinate-free, and we must work on the standard symplectic manifold $\mathbb{R}^{2d}$. In other cases, we need some type of closeness to the identity.

Here, we work on a certain set of the cotangent bundle where the fixed points of our exact symplectomorphism live, the fiberwise transformed set. Then, the fixed points are critical points of a certain action on this set. Hence, the number of fixed point depends on the topology of this set (due to Schnirelman-Lusternik theory and Morse theory). This idea was already used by Moser [79].

On the other side, the orbits of an exact symplectomorphism also satisfy a variational principle, as the orbits of a mechanical system. As we know, variational principles for orbits of strong monotone symplectomorphisms on the annulus have been very useful in order to study cantori and invariant circles (Aubry-Mather sets, Converse KAM theory), homoclinic orbits, periodic orbits, etc (see the works of Aubry, Mather, Percival, Herman, MacKay, Meiss, Kook, Tabacman, etc).

We think that these variational principles can be interesting for several reasons:

- we can work on any cotangent bundle, not only on the standard symplectic manifold $\mathbb{R}^d \times \mathbb{R}^d$ or on the $d$-annulus;
- we do not need the generating function, which is not always defined, or it is difficult of computing;
- in some sense, they are local, because do not use the existence of this global generating function;
- we could extend these variational principles to neighborhoods of any exact Lagrangian manifolds, thanks to Weinstein's theorems.

## 8.1 Fixed points

Let $\mathcal{M}$ be a $d$-manifold, and $T^*\mathcal{M}$ its cotangent bundle. Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with primitive function $S$, and let $f = q \circ F$ be its basic component.

### 8.1.1 The fiberwise transformed set and the action

We shall obtain the fixed points of our symplectomorphism as critical points of a certain function defined on a certain submanifold of $T^*\mathcal{M}$. Here we state the main definitions.

#### The fiberwise transformed set

We define the *fiberwise transformed set* as the fiber product of $q$ and $f$:

$$K = \{\rho \in T^*\mathcal{M} |\ q(\rho) = f(\rho)\}.$$

Hence, any point of this set goes to the same fiber (it is fiberwise transformed). If $\delta : T^*\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is defined by $\delta = q \times f$, and $\Delta = \{(x, x) |\ x \in \mathcal{M}\} \subset \mathcal{M} \times \mathcal{M}$ is the diagonal of $\mathcal{M} \times \mathcal{M}$, then $K = \delta^{-1}(\Delta)$. We observe that the fixed points of $F$ are in $K$: $Fix(F) \subset K$.

We suppose $K \neq \emptyset$. It is a closed set of the cotangent bundle, and it is a $d$-submanifold provided the map $\delta$ be transversal to $\Delta$, that is to say, the rank of the matrix

$$\begin{pmatrix} I - \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

be maximal in all points of the critical set (using cotangent coordinates). For instance, if $F$ is monotone, the critical set is locally a graph, and the restricted projection $q_{|K}$ is a local diffeomorphism. Of course, $K$ can have many connected components, but we can study everyone.

#### The action

We define the *action* as the primitive function of the symplectomorphism restricted to the fiberwise transformed set: $\boldsymbol{s} = S_{|K}$.

We shall prove that the fixed points of our symplectomorphism are critical points of $\boldsymbol{s}$, and the converse is true provided $F$ verifies the monotony condition. First, we shall show that the topology and the geometry of the fiberwise transformed set is more understanding when $F$ is monotone.

### 8.1.2 Topology of the fiberwise transformed set

**Proposition 8.1** *:*

*Let $f : T^*\mathcal{M} \to \mathcal{M}$ be a monotone map.*
*Let $\delta : T^*\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the map defined by $\delta = id_{\mathcal{M}} \times f$. We define the closed submanifold of dimension $d$ $K = \delta^{-1}(\Delta)$, provided $K \neq \emptyset$.*
*Then, the following holds:*

- *$K$ is a graph, locally;*
- *$q_{|K} : K \to \mathcal{M}$ is a local diffeomorphism;*
- *the fibers $q_{|K}^{-1}(x)$ are discrete.*

*We suppose $\mathcal{M}$ be compact and connected. Let $k$ be a compact and connected component of $K$. Then:*

- *$(k, q_{|k})$ is a (smooth) covering space of $\mathcal{M}$, with a finite number of leaves. Hence, all the fibers in $k$ have the same number of elements: $\forall x, y \in \mathcal{M}, \sharp q_{|k}^{-1}(x) = \sharp q_{|k}^{-1}(y)$.*

*Proof*:

The first three points are an immediate consequence of the implicit function theorem. The fourth point comes from the adaptation of the next topological result [1] (see [70]).

> Let $X$ and $Y$ path connected and locally path connected spaces, being $X$ compact and $Y$ Hausdorff.
> Let $f : X \to Y$ be a local homeomorphism.
> Then:
>
> > $(X, f)$ is a covering space of $Y$, with a finite number of leaves.

□

**Remark**

Of course, we can apply the same ideas in order to study the topology of a component of fiberwise transformed set included into a monotone region of our symplectomorphism. ◁

[1] We recall some definitions of Topology:

- A continuous map $f : X \to Y$ between two topological spaces $X$ and $Y$ is a *local homeomorphism* iff each point $x \in X$ has a open neighborhood $V_x$ that is mapped homeomorphically by $f$ onto its image $f(V_x)$, which is open, too.
- Let $X$ and $\tilde{X}$ be two path connected and locally path connected spaces and let $p : \tilde{X} \to X$ be continuous and surjective. We shall say that the pair $(\tilde{X}, p)$ is a *covering space* of $X$ iff every point $x \in X$ has a path connected open neighborhood $U$ such that every path connected component of $p^{-1}(U)$ is mapped homeomorphically by $p$ onto $U$.

  The map $p$ is called a *covering map* or *projection*. It can be proved that all the sets $p^{-1}(x)$ have the same cardinality, for all $x \in X$, which is called *number of leaves* (or *folds*) of the covering space.

In our case, we must substitute continuous by smooth, homeomorphism by diffeomorphism, etc.

### 8.1.3 Geometry of the fiberwise transformed set

We know that the fiberwise transformed set is a $d$-submanifold of $T^*\mathcal{M}$, provided a certain non-degeneracy condition be satisfied (Section 8.1.1). We define a map

$$\begin{array}{rccc} \beta : & K & \longrightarrow & T^*\mathcal{M} \\ & \rho & \longrightarrow & F(\rho) - \rho \end{array},$$

which is well defined because $F(\rho)$ and $\rho$ have the same point basis. This map is an immersion, provided the rank of the matrix

$$\left( I - \frac{\partial g}{\partial y} \, , \, \frac{\partial f}{\partial y} \right)$$

is maximal in all points of $K$ (using cotangent coordinates). Again, this condition is automatically satisfied when $F$ is monotone.

We note that the fixed points of our exact symplectomorphism $F$ are in correspondence with the intersection

$$\beta(K) \cap \mathcal{M}_0.$$

Furthermore, since the immersion is exact Lagrangian, as we shall see in the next proposition, this relates the theory of fixed points of exact symplectomorphisms with the theory of Lagrangian intersections, that is, the theory of intersections between Lagrangian manifolds [2].

**Proposition 8.2** *:*

*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with primitive function $S$, and let $f : T^*\mathcal{M} \to \mathcal{M}$ be its basic component: $f = q \circ F$.*
*Let $K$ be its fiberwise transformed set, and suppose that the rank of the matrices*

$$\left( I - \frac{\partial f}{\partial x} \, , \, \frac{\partial f}{\partial y} \right) \, , \, \left( I - \frac{\partial g}{\partial y} \, , \, \frac{\partial f}{\partial y} \right)$$

*are maximal in all of its points (written using cotangent coordinates). We consider the map $\beta$ defined above. Then:*

> *$\beta$ is an exact Lagrangian immersion of $K$ in $T^*\mathcal{M}$, and its primitive function is the action $\boldsymbol{s} = S_{|K}$.*

---

[2] There is another way of relating both theories. For instance, given a symplectomorphis $F : \mathcal{N} \to \mathcal{N}$ on a symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$, then is easy to prove that its graph

$$\Gamma \;=\; \{(z, F(z)) \mid z \in \mathcal{N}\} \subset \mathcal{N}^2$$

is a Lagrangian submanifold of $(\mathcal{N}^2, \pi_2^*\boldsymbol{\omega} - \pi_1^*\boldsymbol{\omega})$. On the other side, the diagonal

$$\Delta \;=\; \{(z, z) \mid z \in \mathcal{N}\} \subset \mathcal{N}^2$$

is also a Lagrangian submanifold of $\mathcal{N}^2$. Finally, fixed points of $F$ are in correspondence with the intersection of both Lagrangian submanifolds.

*Proof*:

We only must prove that $\beta^*\alpha = \mathrm{d}\boldsymbol{s}$. As we know, if $\nu : K \to T^*\mathcal{M}$ is the inclusion of $K$ into $T^*\mathcal{M}$, then the function $\boldsymbol{s} : K \to \mathbb{R}$ is defined as $\boldsymbol{s} = S \circ \nu$. Hence, we must prove that

$$(\beta^*\alpha)(\rho)\ \xi_\rho \ = \ (S\circ\nu)_*(\rho)\ \xi_\rho,$$

for any $\rho \in K$ and $\xi_\rho \in T_\rho K$. This last assertion is equivalent to the condition $f_*(\rho)\ \xi_\rho = q_*(\rho)\ \xi_\rho$ (identifying $\xi_\rho$ with $\nu_*(\rho)\xi_\rho$ and $\rho$ with $\nu(\rho)$.

Then,

- on one side

$$\begin{aligned}(\beta^*\alpha)(\rho) &= \alpha(\beta(\rho))\ \beta_*(\rho)\\ &= \beta(\rho)\ q_*(\beta(\rho))\ \beta_*(\rho)\\ &= \beta(\rho)\ (q\circ\beta)_*(\rho)\\ &= \beta(\rho)\ q_*(\rho),\end{aligned}$$

- and on the other side, using the exactness formulae,

$$\begin{aligned}(S\circ\nu)_*(\rho) &= S_*(\nu(\rho))\ \nu_*(\rho)\\ &= F(\rho)\circ f_*(\rho) - \rho\circ q_*(\rho)\\ &= (F(\rho) - \rho)\ q_*(\rho) + F(\rho)\ (f_*(\rho) - q_*(\rho))\\ &= \beta(\rho)\ q_*(\rho) + F(\rho)\ (f_*(\rho) - q_*(\rho)).\end{aligned}$$

Finally, applying both of formulas to the vector $\xi_\rho$, we arrive to the desired result. □

### 8.1.4 Fixed points as critical points of the action

We shall prove that fixed points are critical points of the action $w$, which is the primitive function restricted to the fiberwise transformed set.

**Theorem 8.1** :

*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with primitive function $S$, and let $f : T^*\mathcal{M} \to \mathcal{M}$ be its basic component: $f = q\circ F$.*
*Let $K$ be its fiberwise transformed set. We suppose $K \neq \emptyset$ and $\delta$ transversal to $\Delta$ (so, $K$ is a d-submanifold of $T^*\mathcal{M}$).*
*Then:*

- *The fixed points of $F$ are critical points of $\boldsymbol{s} = S_{|K}$.*
- *If $F$ is monotone, the fixed points of $F$ correspond with the critical points of $\boldsymbol{s}$.*

*Proof*:

In order to obtain the proof, we need the next result which generalizes the *Lagrange multipliers* in classical calculus (see [2], p. 177):

> Let $\mathcal{M}$ and $\mathcal{P}$ be two manifolds.
> Let $g : \mathcal{M} \to \mathcal{P}$ be transversal to the submanifold $W$ of $\mathcal{P}$, $\mathcal{N} = g^{-1}(W)$, and let $f : \mathcal{M} \to \mathbb{R}$ be $\mathcal{C}^r$, $r \geq 1$.
> Let $E_{g(n)}$ be a closed component to $T_{g(n)}W$ in $T_{g(n)}\mathcal{P}$ so $T_{g(n)}\mathcal{P} = T_{g(n)}W \oplus E_{g(n)}$ and let $p : T_{g(n)}\mathcal{P} \to E_{g(n)}$ be the projection.
> Therefore:
>
> > A point $n \in \mathcal{N}$ is a critical point of $f_{|\mathcal{N}}$ iff there exists $\lambda \in E^*_{g(n)}$ called a *Lagrange multiplier* such that $f_*(n) = \lambda \circ p \circ g_*(n)$.

In our case, we have a map $\delta : T^*\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ transversal to the submanifold $\Delta$, $K = \delta^{-1}(\Delta)$ and $S : T^*\mathcal{M} \to \mathbb{R}$ a function. Given $\rho_x \in K$ (i.e.: $\delta(\rho_x) = (x, x)$), a complement to $T_{(x,x)}\Delta$ in $T_{(x,x)}(\mathcal{M} \times \mathcal{M}) = T_x\mathcal{M} \times T_x\mathcal{M}$ is $E_{(x,x)} = \{0_x\} \times T_x\mathcal{M}$. In fact, we have:

$$\begin{array}{ccccc} T_{(x,x)}(\mathcal{M} \times \mathcal{M}) & = & T_{(x,x)}\Delta & \oplus & \{0_x\} \times T_x\mathcal{M} \\ \begin{pmatrix} \zeta \\ \nu \end{pmatrix} & = & \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} & + & \begin{pmatrix} 0 \\ \nu - \zeta \end{pmatrix} \end{array}.$$

Since $E_{(x,x)} \simeq T_x\mathcal{M}$, we define the projection

$$\begin{array}{rccc} p: & T_x\mathcal{M} \times T_x\mathcal{M} & \to & T_x\mathcal{M} \\ & \begin{pmatrix} \zeta \\ \nu \end{pmatrix} & \to & \nu - \zeta \end{array}.$$

So $\rho_x \in K$ is a critical point of $\boldsymbol{s} = S_{|K}$ iff there exists $\lambda \in T^*_x\mathcal{M} \simeq E^*_{(x,x)}$ such that

$$S_*(\rho_x) = \lambda \circ p \circ \delta_*(\rho_x).$$

Using the exactness formulae and the previous definitions, this is translated to

$$F(\rho_x) \circ f_*(\rho_x) - \rho_x \circ q_*(\rho_x) = \lambda \circ (f_*(\rho_x) - q_*(\rho_x)).$$

The proof follows now from this formula.

- If $\rho_x$ is a fixed point of $F$, i.e. $F(\rho_x) = \rho_x$, then it is enough to choose $\lambda = \rho_x = F(\rho_x)$.
- We suppose now that $F$ is monotone. If we apply the formula to a vertical vector $\xi_{\rho_x} \in V_{\rho_x}T^*\mathcal{M}$ (i.e. $q_*(\rho_x)\xi_{\rho_x} = 0$), we have:

$$F(\rho_x)(f_*(\rho_x)\xi_{\rho_x}) = \lambda(f_*(\rho_x)\xi_{\rho_x}).$$

Since $F$ is monotone, i.e., $f_*(\rho_x)$ is an isomorphism between $V_{\rho_x}T^*\mathcal{M}$ and $T_{f(\rho_x)}\mathcal{M}$, then $\lambda = F(\rho_x)$. Hence, we have:

$$\rho_x \circ q_*(\rho_x) = \lambda \circ q_*(\rho_x).$$

Finally, since $q_*(\rho_x)$ is an epimorphism, we reach $\rho_x = \lambda = F(\rho_x)$.

□

**Remarks**

i) Suppose $\mathcal{M}$ be compact and connected and $F$ be monotone. On every compact connected component $k$ of $K$ we have at least two fixed points. In fact, the number of fixed points is bounded from below by the Schnirelman-Lusternik category of $k$. If all of they are non degenerate (as critical points of $\boldsymbol{s}$), then we have at least the sum of the Betti numbers of $k$. That is to say, the number of fixed points depends on the topology of the fiberwise transformed set.

ii) We can reduce the study of the $q$-periodic orbits of an exact symplectomorphism to the study of the fixed points of another one. Note that if we consider a power of a monotone map $F$, it cannot be monotone. In order to preserve the monotonicity, it is better to work on the symplectic product.

◁

### 8.1.5 An example

Let $F = (f, g) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ be an exact symplectomorphism with primitive function $S$, such that we can write $f(x, y) = x + \bar{f}(x, y) \pmod 1$ and $g(x, y) = \bar{g}(x, y)$, where $\bar{f}$ and $\bar{g}$ are 1-periodic in all their $x$-variables. We suppose that $F$ verifies the strong monotone condition:

$$\forall x \in \mathbb{T}^d, \bar{f}(x, .) : \mathbb{R}^d \to \mathbb{R}^d \text{ is a diffeomorphism.}$$

We can decompose the critical set in this way:

$$K = \{(x, y) \in \mathbb{T}^d \times \mathbb{R}^d |\ f(x, y) = x \pmod 1\} = \bigcup_{p \in \mathbb{Z}^d} K_p,$$

where, $\forall p \in \mathbb{Z}^d$:

$$K_p = \{(x, y) \in \mathbb{T}^d \times \mathbb{R}^d |\ \bar{f}(x, y) = p\}.$$

We say that a point $(x, y)$ is a fixed point of type $p \in \mathbb{Z}^d$, or that $p$ is its *rotation number* is $p$, iff $F(x, y) = (x + p, y))$. Hence, on every component $K_p$ the fixed points of type $p$ live. In this case, everyone of this components is homeomorphic to the $d$-dimensional torus: $K_p \simeq \mathbb{T}^d, \forall p \in \mathbb{Z}^d$. Each torus $K_p$ is a *radially transformed torus* (see [79, 3]), and it is given by a map $\eta_p : \mathbb{T}^d \to \mathbb{R}^d$ (i.e. $\bar{f}(x, \eta_p(x)) = p$). Hence, $\forall p \in \mathbb{Z}^d$:

there exists $d+1$ fixed points of type $p$, and $2^d$ if all of them are non degenerate, as critical points of the critical function $\boldsymbol{s}_p(x) = S(x, \eta_p(x))$ [3].

In this case we obtain that

$$g(x, \eta_p(x)) - \eta_p(x) \quad = \quad \frac{\partial \boldsymbol{s}_p}{\partial x}(x).$$

**Remark**

While here the fixed points are classified by their rotation number, in the general case, when we work on any cotangent bundle, the fixed points are classified by the different connected components of the fiberwise transformed set. ◁

## 8.2 Variational construction of orbits

Let $\mathcal{M}$ be a $d$-manifold and $\mathcal{N} = T^*\mathcal{M}$ its cotangent bundle. We shall obtain a variational principle for the orbits of an exact symplectomorphism $F : \mathcal{N} \to \mathcal{N}$ with primitive function $S$.

### 8.2.1 The set of chains and the action

Given two basic points $\mathbf{x}_m, \mathbf{x}_n \in \mathcal{M}$, where $m, n \in \mathbb{Z}|\ m+1 < n$, we ask for the *connecting orbits* between $\mathbf{x}_m$ and $\mathbf{x}_n$, that is to say, finite sequences $(\rho_i)_{i=m\div n-1}$ such that $q(\rho_m) = \mathbf{x}_m, q\circ F(\rho_{n-1}) = \mathbf{x}_n$ and $\rho_{i+1} = F(\rho_i), \forall i = m \div n-2$.

First, we shall define the set where the action will act. This set is the set of $F$-chains connecting $\mathbf{x}_m, \mathbf{x}_n$.

#### The set of $F$-chains connecting $\mathbf{x}_m$ and $\mathbf{x}_n$

It is the set $K_{m,n} = K_{\mathbf{x}_m, \mathbf{x}_n}$ of finite sequences

$$\boldsymbol{\rho} = (\rho_i)_{i=m\div n-1} \in \prod_{i=m}^{n-1} T^*\mathcal{M}$$

verifying the following conditions:

- $q(\rho_m) = \mathbf{x}_m$,
- $f(\rho_i) = q(\rho_{i+1}), \forall i = m \div n-2$,
- $f(\rho_{n-1}) = \mathbf{x}_n$.

[3]While $d+1$ is the *cup length* of $\mathbb{T}^d$, $2^d$ is the sum of its Betti numbers.

Suppose $K_{m,n}$ is not empty. If the map

$$\delta_{m,n} : \prod_{i=m}^{n-1} T^*\mathcal{M} \to \prod_{i=m}^{n-1} (\mathcal{M} \times \mathcal{M}),$$

defined by

$$\delta_{m,n}((\rho_i)_{i=m \div n-1}) = ((q(\rho_i), f(\rho_i)))_{i=m\div n-1},$$

is transversal to the $d(n-m-1)$-submanifold of the $2d(n-m)$-manifold $\prod_{i=m}^{n-1}(\mathcal{M}\times\mathcal{M})$ defined by

$$\Delta_{m,n} = \Delta_{\mathbf{x}_m,\mathbf{x}_n} = \{((\mathbf{x}_m, x_{m+1}), (x_{m+1}, x_{m+2}), \ldots, (x_{n-1}, \mathbf{x}_n))|\ x_{m+1}, \ldots, x_{n-1} \in \mathcal{M}\},$$

the set of chains $K_{m,n} = \delta_{m,n}^{-1}(\Delta_{m,n})$ is a $d(n-m-1)$-submanifold of the $2d(n-m)$-manifold $\prod_{i=m}^{n-1} T^*\mathcal{M}$. For instance, this is the case when $F$ is monotone.

Secondly, we must define the action on the previous set.

#### The action

The action on the set of $F$-chains will be

$$\boldsymbol{S}_{m,n}((\rho_i)_{i=m\div n-1}) = \sum_{i=m}^{n-1} S(\rho_i).$$

We shall see that the connecting orbits are critical chains of the action. The converse is true if $F$ is monotone.

### 8.2.2 Connecting orbits as extremal chains

Orbits extremize the orbital action.

**Proposition 8.3** *:*

*Let $\mathcal{M}$ be a d-manifold.*
*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with pf$(F) = S$, and let $f = q\circ F$ be its basic component.*
*Let $m, n \in \mathbb{Z}$ be integers such that $m+1 < n$, $\mathbf{x}_m, \mathbf{x}_n \in \mathcal{M}$ be basic points, and $K_{m,n} = \delta_{m,n}^{-1}(\Delta_{m,n})$ be the set of $F$-chains connecting $\mathbf{x}_m$ with $\mathbf{x}_n$. We suppose $K_{m,n} \neq \emptyset$ and $\delta_{m,n}$ transversal to $\Delta_{m,n}$.*
*Let $\boldsymbol{S}_{m,n} : \prod_{i=m}^{n-1} T^*\mathcal{M} \to \mathbb{R}$ be the function defined by*

$$\boldsymbol{S}_{m,n}((\rho_i)_{i=m\div n-1}) \quad = \quad \sum_{i=m}^{n-1} S(\rho_i),$$

*i.e. $\boldsymbol{S}_{m,n} = \sum_{i=m}^{n-1} S\circ\pi_i$, where the $\pi_i$'s are the projections. Let $\boldsymbol{S}_{m,n}$ be the restriction of $\boldsymbol{S}_{m,n}$ to the set of $F$-chains.*
*Then:*

- *The connecting orbits between* $\mathbf{x}_m$ *and* $\mathbf{x}_n$ *are critical chains of* $\boldsymbol{S}_{m,n}$.
- *If* $F$ *is monotone, the connecting orbits correspond with the critical chains.*

*Proof*:

Given $\boldsymbol{\rho} = (\rho_i)_{i=m\div n-1} \in K_{m,n}$, with

$$\delta_{m,n}(\boldsymbol{\rho}) = X = ((\mathbf{x}_m, x_{m+1}), (x_{m+1}, x_{m+2}), \ldots, (x_{n-1}, \mathbf{x}_n)),$$

we have the decomposition

$$\begin{array}{ccccc} T_X \prod_{i=m}^{n-1}(\mathcal{M}\times\mathcal{M}) & = & T_X\Delta_{m,n} & \oplus & E_X \\ \begin{pmatrix} \nu_m \\ \zeta_{m+1} \\ \nu_{m+1} \\ \zeta_{m+2} \\ \nu_{m+2} \\ \vdots \\ \zeta_{n-1} \\ \nu_{n-1} \\ \zeta_n \end{pmatrix} & = & \begin{pmatrix} 0 \\ \zeta_{m+1} \\ \zeta_{m+1} \\ \zeta_{m+2} \\ \zeta_{m+2} \\ \vdots \\ \zeta_{n-1} \\ \zeta_{n-1} \\ 0 \end{pmatrix} & + & \begin{pmatrix} \nu_m \\ 0 \\ \nu_{m+1}-\zeta_{m+1} \\ 0 \\ \nu_{m+2}-\zeta_{m+2} \\ \vdots \\ 0 \\ \nu_{n-1}-\zeta_{n-1} \\ \zeta_n \end{pmatrix} \end{array}.$$

Since $E_X \simeq \prod_{i=m}^{n} T_{x_i}\mathcal{M}$, we will take as projection the map

$$\begin{array}{rccc} p: & T_X \prod_{i=m}^{n-1}(\mathcal{M}\times\mathcal{M}) & \to & \prod_{i=m}^{n} T_{x_i}\mathcal{M} \\ & \begin{pmatrix} \nu_m \\ \zeta_{m+1} \\ \nu_{m+1} \\ \zeta_{m+2} \\ \nu_{m+2} \\ \vdots \\ \zeta_{n-1} \\ \nu_{n-1} \\ \zeta_n \end{pmatrix} & \to & \begin{pmatrix} \nu_m \\ \nu_{m+1}-\zeta_{m+1} \\ \nu_{m+2}-\zeta_{m+2} \\ \vdots \\ \nu_{n-1}-\zeta_{n-1} \\ \zeta_n \end{pmatrix} \end{array}.$$

Hence $\boldsymbol{\rho} \in K_{m,n}$ is a critical point of $\boldsymbol{S}_{m,n}$ iff there exists $\lambda \in \prod_{i=m}^{n} T^*_{x_i}\mathcal{M}$ such that

$$\boldsymbol{S}_{mn*}(\boldsymbol{\rho}) = \lambda\circ p\circ\delta_{mn*}(\boldsymbol{\rho}),$$

that is to say, iff there exist $n-m+1$ forms $\lambda_i \in T^*_{x_i}\mathcal{M}$ $(i = m \div n)$ such that, $\forall \xi = (\xi_m, \xi_{m+1}, \dots, \xi_{n-1}) \in \prod_{i=m}^{n-1} T_{\rho_i} T^*\mathcal{M}$,

$$\sum_{i=m}^{n-1} (F(\rho_i)\circ f_*(\rho_i) - \rho_i \circ q_*(\rho_i))\xi_i =$$

$$-\lambda_m \circ q_*(\rho_m)\xi_m + \sum_{i=m+1}^{n-1} \lambda_i \left(f_*(\rho_{i-1})\xi_{i-1} - q_*(\rho_i)\xi_i\right) +$$

$$\lambda_n \circ f_*(\rho_{n-1})\xi_{n-1}.$$

The proof follows now from this formula.

- If $\boldsymbol{\rho}$ is a segment of orbit connecting $\mathbf{x}_m$ with $\mathbf{x}_n$, then

$$\sum_{i=m}^{n-1} (F(\rho_i)\circ f_*(\rho_i) - \rho_i \circ q_*(\rho_i))\xi_i =$$

$$-\rho_m \circ q_*(\rho_m)\xi_m + \sum_{i=m+1}^{n-1} \rho_i \left(f_*(\rho_{i-1})\xi_{i-1} - q_*(\rho_i)\xi_i\right) +$$

$$F(\rho_{n-1})\circ f_*(\rho_{n-1})\xi_{n-1}.$$

  Hence, it is enough to choose $\lambda_i = \rho_i$, $\forall i = m \div n-1$ and $\lambda_n = F(\rho_{n-1})$.

- We suppose now that $F$ is monotone. If we apply the formula to a vector $\xi = (0, \dots, \xi_i, \dots, 0)$, with $\xi_i \in V_{\rho_i} T^*\mathcal{M}$ $(i = m \div n-1)$, then we obtain

$$F(\rho_i)(f_*(\rho_i)\xi_i) = \lambda_{i+1}(f_*(\rho_i)\xi_i).$$

  Since this fact is true $\forall \xi_i \in V_{\rho_i} T^*\mathcal{M}$ and $F$ is monotone, we reach to

$$\lambda_{i+1} = F(\rho_i), \ \forall i = m \div n-1.$$

  Hence, $\forall \xi = (\xi_m, \xi_{m+1}, \dots, \xi_{n-1}) \in \prod_{i=m}^{n-1} T_{\rho_i} T^*\mathcal{M}$,

$$\sum_{i=m}^{n-1} \rho_i(q_*(\rho_i)\xi_i) = \sum_{i=m}^{n-1} \lambda_i(q_*(\rho_i)\xi_i),$$

  and we reach

$$\lambda_i = \rho_i, \ \forall i = m \div n-1.$$

  Finally, we obtain that

$$\rho_{i+1} = \lambda_{i+1} = F(\rho_i), \ \forall i = m \div n-2.$$

□

**Remark**

The transversality condition on the hypothesis of the proposition is satisfied when our symplectomorphism is monotone. ◁

### 8.2.3 Minimizing orbits

An orbit of our diffeomorphism $F$ is a bisequence

$$(\rho_i = F^i(\rho))_{i\in\mathbb{Z}},$$

where $\rho \in T^*\mathcal{M}$. We have seen that each finite segment

$$(\rho_i)_{i=m\div n-1}$$

is a critical chain for the corresponding $m,n$-action (fixing $\mathbf{x}_m = q(F^m(\rho)), \mathbf{x}_n = q(F^n(\rho))$. If each finite segment is a (global or local) minimum chain for the corresponding action, we shall say that the orbit is *(globally or locally) minimizing*. The same definitions can be applied to (global or local) maximizing orbits.

**Remark**

This property is invariant under lifts and fiberwise translations. ◁

# Chapter 9

# Invariant exact Lagrangian graphs

To any invariant exact isotropic submanifold of a certain exact symplectomorphism we can associate a conserved quantity, with the aid of the corresponding primitive functions. If the invariant manifold is an exact Lagrangian graph, we can obtain more information. We use the primitive function in order to characterize it. The results of this section extend the results of Mather and Herman appearing, for instance, in the last appendix of [68], and those seen in Chapter 6.

## 9.1 Characterization

Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be a symplectomorphism, and let $\beta : \mathcal{M} \to T^*\mathcal{M}$ be a closed Pfaffian form, inducing a Lagrangian graph $\mathcal{L}_\beta$. We say that $\mathcal{L}_\beta$ is $F$*-invariant* iff:

$$F\circ\beta = \beta\circ f\circ\beta,$$

where $f = q\circ F$, $q$ being the projection of $T^*M$ onto $M$. The dynamics on the invariant manifold is given by the injective immersion $\bar{f} = f\circ\beta : \mathcal{M} \to \mathcal{M}$. It is a diffeomorphism iff $\mathcal{L}_\beta$ is, moreover, $F^{-1}$-invariant.

If $\beta = dl$, for some function $l : \mathcal{M} \to \mathbb{R}$, we know (see Section 3.1) that the function $\Phi : \mathcal{M} \to \mathbb{R}$, given by

$$\Phi = S\circ dl - (l\circ\bar{f} - l),$$

is constant (because $\mathcal{M}$ is connected). We can improve this result for invariant exact Lagrangian graphs (in short, *i.e.L.g.*).

**Proposition 9.1** *:*

*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with $pf(F) = S$, and let $f = q\circ F$ be its basic component.*
*From the function $l : \mathcal{M} \to \mathbb{R}$, we define the functions $\hat{\Phi} : T^*\mathcal{M} \to \mathbb{R}$ and $\Phi : \mathcal{M} \to \mathbb{R}$ by*

$$\hat{\Phi} = S - (l\circ f - l\circ q)$$

*and*

$$\Phi = \hat{\Phi}\circ dl.$$

*Then:*

1. *(Conserved quantity on an i.e.L.g.)*

$$\mathcal{L}_{dl} \text{ is } F\text{-invariant} \Rightarrow \Phi \text{is constant.}$$

2. *(Characterization of i.e.L.g.)*

$$\mathcal{L}_{dl} \text{ is } F\text{-invariant} \Leftrightarrow \forall x \in \mathcal{M}\ \hat{\Phi}_*(dl_x) = 0.$$

3. *(Characterization of the points of an i.e.L.g.)* *If $F$ is monotone, and $\mathcal{L}_{dl}$ is $F$-invariant:*

$$\rho_x = dl_x \Leftrightarrow \hat{\Phi}_*(\rho_x) = 0.$$

*Moreover, if $\mathcal{L}_{dl}$ is $F^{-1}$-invariant:*

$$\rho_x = dl_x \Leftrightarrow \hat{\Phi}_*(\rho_x)_{|V_{\rho_x}T^*\mathcal{M}} = 0.$$

*Proof*:

1. First point is an immediate consequence of the second.
2. We have to compute $\hat{\Phi}_*(\rho_x), \forall \rho_x \in T^*\mathcal{M}$:

$$\begin{aligned} \hat{\Phi}_*(\rho_x) &= S_*(\rho_x) - (l_*(f(\rho_x))\circ f_*(\rho_x) - l_*(x)\circ q_*(\rho_x)) = \\ &= (F(\rho_x) - dl(f(\rho_x)))\circ f_*(\rho_x) - (\rho_x - dl(x))\circ q_*(\rho_x). \end{aligned}$$

So, $\forall x \in \mathcal{M}$:

$$\hat{\Phi}_*(dl(x)) = (F(dl(x)) - dl(f(dl(x))))\circ f_*(dl(x)).$$

The proposition follows directly from this formula, because $f_*(\rho_x)$ is an epimorphism $\forall \rho_x \in T^*\mathcal{M}$.

3. Suppose that $F$ is monotone and $\mathcal{L}_{dl}$ is $F$-invariant. The $\Rightarrow$ is the previous point, so then we must prove the $\Leftarrow$. Hence, we assume that $\rho_x \in T^*\mathcal{M}$ verifies $\hat{\Phi}_*(\rho_x) = 0$, i.e.:

$$(F(\rho_x) - dl(f(\rho_x)))\circ f_*(\rho_x) - (\rho_x - dl(x))\circ q_*(\rho_x) = 0.$$

In particular, if we apply this formula to vertical tangent vectors $\xi_{\rho_x} \in V_{\rho_x}T^*\mathcal{M} = \ker q_*(\rho_x)$, we obtain

$$(F(\rho_x) - dl(f(\rho_x)))\circ f_*(\rho_x)\xi_{\rho_x} = 0.$$

Since $F$ is monotone, i.e., $f_*(\rho_x)$ is an isomorphism between $V_{\rho_x}T^*\mathcal{M}$ and $T_{f(\rho_x)}\mathcal{M}$, then

$$F(\rho_x) - dl(f(\rho_x)) = 0.$$

Therefore, we return to the first formula:

$$(\rho_x - dl(x))\circ q_*(\rho_x) = 0,$$

Finally, applying that $q_*(\rho_x)$ is an epimorphism, we reach $\rho_x = dl_x$. If, moreover, $\mathcal{L}_{dl}$ is $F^{-1}$-invariant, we proceed in the same way, and we obtain that $F(\rho_x) \in \mathcal{L}_{dl}$. Finally, by $F^{-1}$-invariance, $\rho_x \in \mathcal{L}_{dl}$.

□

## 9.2 Minimizing invariant exact Lagrangian graphs

Let $\mathcal{L}_{dl}$ be an invariant graph of an exact symplectomorphism $F : T^*\mathcal{M} \to T^*\mathcal{M}$ with primitive function $S$. Let $\hat{\Phi}$ the function associated to the graph. We know that this function is constant on the graph (because $\mathcal{M}$ is connected):

$$\exists C \in \mathbb{R} | \ \forall x \in \mathcal{M} \ \hat{\Phi}(dl(x)) = C.$$

We shall say that the graph is (global or local) *minimizing* or *minimal* iff each point of the graph $dl(x)$ is (global or local) minimum of the function $\hat{\Phi}$ restricted to the corresponding fiber. For instance, $\mathcal{L}_{dl}$ is global minimizing iff

$$\hat{\Phi}(\rho_x) \geq \hat{\Phi}(dl(x)) = C, \ \forall x \in \mathcal{M}, \ \forall \rho_x \in T^*_x\mathcal{M}.$$

The next proposition shows that orbits on a minimizing graph are minimizing.

**Theorem 9.1** :

*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an exact symplectomorphism, with primitive function $S$, and $\mathcal{L}_{dl}$ be a minimizing i.e.L.g., generated by the function $l : \mathcal{M} \to \mathbb{R}$.*
*Then:*

*All the orbits on $\mathcal{L}_{dl}$ are minimizing.*

*Proof*:

We shall suppose that $\mathcal{M}$ is connected, but this is not important. Hence, let $C \in \mathbb{R}$ be the conserved quantity.

Let $dl(x)$ be any point on the minimizing graph. First, we fix $m, n \in \mathbb{Z}$, with the condition $m + 1 < n$. Then, we take $x_m = q\circ F^m(dl(x))$ and $x_n = q\circ F^n(dl(x))$. Now, let $\boldsymbol{\rho} = (\rho_i)_{i=m\div n-1}$ be any $F$-chain connecting $x_m$ with $x_n$, and $\boldsymbol{o}$ be the corresponding segment of orbit (i.e. $\boldsymbol{o} = (F^i(dl(x)))_{i=m\div n-1}$). Hence, the corresponding actions verify:

$$\begin{aligned}
\boldsymbol{S}_{mn}(\boldsymbol{\rho}) &= \sum_{i=m}^{n-1} S(\rho_i) \\
&= \sum_{i=m}^{n-1} (S(\rho_i) - l(f(\rho_i)) + l(q(\rho_i))) \ + l(x_n) - l(x_m) \\
&= \sum_{i=m}^{n-1} \hat{\Phi}(\rho_i) \ + l(x_n) - l(x_m) \\
&\geq (n-m)C + l(x_n) - l(x_m) \\
&= \sum_{i=m}^{n-1} \hat{\Phi}(F^i(dl(x))) \ + l(x_n) - l(x_m) \\
&= \boldsymbol{S}_{mn}(\boldsymbol{o}).
\end{aligned}$$

□

**Remarks**

i) The same proof works in the local minimizing case, choosing chains close enough to the orbit.

ii) This proposition is more geometrical than the analogous proposition in Section 6.3. Notice that the key point is restrict the action to the set of chains.

iii) These properties are invariant under lifts and fiberwise translations.

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# Chapter 10

# Interpolation of an exact symplectomorphism

We consider the exact symplectic manifold $\mathcal{N} = T^*\mathcal{M}$, and an exact symplectomorphism $F : \mathcal{N} \to \mathcal{N}$ with primitive function $S$. We wonder if we can obtain a time-dependent Hamiltonian whose time-1 flow be $F$: $F = \varphi_1$. In this case, we shall say that $F$ *is homologous to the identity.*

We shall study the case in which the zero-section is $F$-invariant. Therefore, applying the results in Chapter 9, we have $\mathrm{d}S \circ z = 0$ and $S \circ z = 0$ (without loss of generality). Following Section 2.4, this interpolation problem is related with the properties of the Liouville derivative. We shall work in analytic set up, and the differentiable case will remain open (cf. [16]).

The previous results will be enough for many cases, due to Weinstein's theorems.

## 10.1 A fiber p.d.e.

To integrate with respect to the Liouville derivative is to solve the problem

> given a function $S \in \mathcal{F}(T^*\mathcal{M})$, what are the functions $H \in \mathcal{F}(T^*\mathcal{M})$ such that $\mathbf{\Lambda}(H) = S$?

Since $\mathbf{\Lambda}$ is a vertical operator, we can restrict our attention to each fiber, where it is easy to work. On each fiber (fixing $x$) we have a linear operator $\boldsymbol{\lambda}$, which transforms $y$-valued functions. Its properties will be inherited by our Liouville derivative.

### 10.1.1 Solving the p.d.e. $y \cdot \nabla_y H - H = S$

Let $\mathcal{U} \subset \mathbb{R}^d$ be an open neighborhood of the origin in $\mathbb{R}^d$, with coordinates $y = (y_1, \ldots, y_d)$, and let $S : \mathcal{U} \to \mathbb{R}$ be a function. We want to solve the p.d.e.

$$y \cdot \nabla_y H(y) - H(y) \quad = \quad S(y).$$

If we derive the previous equation we get a necessary condition to solve it:

$$\nabla_y S(0) \quad = \quad 0.$$

(i.e., the origin must be a critical point of $S$). We shall consider the case $S(0) = 0$. If not, we define $\bar{S}(y) = S(y) - S(0)$, $\bar{H}$ a solution of the p.d.e with $\bar{S}$ and, finally, $H(y) = \bar{H}(y) - S(0)$ will be a solution of the original p.d.e.. We note that the case $S \simeq 0$ is the well known *Euler's equation* for the homogeneous functions of degree 1.

We can define a linear operator $\boldsymbol{\lambda}$ on the space of smooth functions defined on $\mathcal{U}$, $\mathcal{F} = \mathcal{F}(\mathcal{U})$:

$$\begin{aligned} \boldsymbol{\lambda}:\ \mathcal{F} &\rightarrow \mathcal{F} \\ H &\rightarrow y \cdot \nabla_y H - H. \end{aligned}$$

The problem reduces to solve the linear equation $\boldsymbol{\lambda}(H) = S$.

First, we begin remembering that if we have a function $F : \mathcal{U} \rightarrow \mathbb{R}$, defined on an star-shaped open set $\mathcal{U} \subset \mathbb{R}^d$, centered in the origin, then we can write

$$F(y) = F(0) + \sum_{i=1}^{d} y_i f_i(y),$$

where the $d$ functions $f_i$ are given by

$$f_i(y) = \int_0^1 \frac{\partial F}{\partial y_i}(ty)\ dt.$$

Since $H(0) = 0$ (by the equation), then we shall decompose $H$ as

$$H(y) = \sum_{i=1}^{d} y_i h_i(y).$$

Moreover, we can apply that result to the function $S$ and its derivatives and obtain

$$S(y) = \sum_{i,j=1 \div d} y_i y_j s_{ij}(y),$$

with

$$s_{ij}(y) = \int_{[0,1]^2} t \cdot \frac{\partial^2 S}{\partial y_i \partial y_j}(sty)\ d(s,t)$$

(note that $s_{ij} = s_{ji}$).

Then, we impose that $\forall i, j = 1 \div d$

$$\frac{\partial h_i}{\partial y_j}(y) = s_{ij}(y).$$

Since $s_{ij} = s_{ji}$, we know, by Poincaré's lemma, that there exists a function $u : \mathcal{U} \rightarrow \mathbb{R}$ function such that $h = \nabla u$. We must find this function.

Since

$$s_{ij}(y) = \frac{\partial^2 u}{\partial y_i \partial y_j}(y) = \int_{[0,1]^2} t\ \frac{\partial^2 S}{\partial y_i \partial y_j}(sty) d(s,t)$$

then

$$u(y) \;=\; a \cdot y + \int_{[0,1]^2} \frac{1}{s^2 t} S(sty)\; d(s,t)$$

where $a \in \mathbb{R}^d$. Finally, we have $\forall i = 1 \div d$

$$h_i(y) = \frac{\partial u}{\partial y_i} \;=\; a_i + \int_{[0,1]^2} \frac{1}{s} \frac{\partial S}{\partial y_i}(sty)\; d(s,t)$$

and, particularly, $h_i(0) = a_i$.

Summarizing, we have the next proposition.

**Lemma 10.1** *:*

*Let $\mathcal{U} \subset \mathbb{R}^d$ be a star-shaped open set centered in the origin and $S : \mathcal{U} \to \mathbb{R}$ be a function satisfying:*

$$S(0) = 0, \nabla_y S(0) = 0.$$

*Then:*

*The solutions of the p.d.e.*

$$y \cdot \nabla_y H(y) - H(y) \;=\; S(y)$$

*are*

$$H(y) \;=\; a \cdot y + \int_0^1 \frac{1}{t^2} S(ty)\; dt$$

*where $a \in \mathbb{R}^d$.*

*Proof*:

It is enough to substitute in the equation, but we shall recover the solution from the previous formulae. We have

$$\begin{aligned} H(y) &= \sum_{i=1}^{d} y_i h_i(y) \\ &= \sum_{i=1}^{d} y_i \left( a_i + \int_{[0,1]^2} \frac{1}{s} \frac{\partial S}{\partial y_i}(sty)\; d(s,t) \right) \\ &= a \cdot y + \int_{[0,1]^2} \frac{1}{s} y \cdot \nabla_y S(sty)\; d(s,t) \\ &= a \cdot y + \int_0^1 \frac{1}{s^2} S(sy)\; ds. \end{aligned}$$

□

**Remarks**

i) We do not have problems with the integrals, thanks to the conditions satisfied by $S$.

ii) The function $H$ is defined up a constant vector $a$, and: $\nabla_y H(0) = a$. So then, there is an unique solution with the origin being a critical point.

iii) The functions $H(y) = a \cdot y$ belong to $\ker \boldsymbol{\lambda}$, that is, they are the homogeneous functions of degree 1. In fact, the eigenfunctions of $\boldsymbol{\lambda}$ are the homogeneous functions. That is, if $\boldsymbol{\lambda}(H) = \alpha H$ iff $H$ is homogeneous of degree $\alpha + 1$.

iv) If $S$ has the form $S(y) = y^k s_k(y)$, with $|k| \geq 2$, then $H$ has the same form $H(y) = y^k h_k(y)$ (we choose $a = 0$), with:

$$h_k(y) = \int_0^1 t^{|k|-2} s_k(ty)\, dt.$$

◁

Finally, we shall obtain formal results of the problem using the previous formula and imposing directly the condition. We shall compare the results.

- Using the previous formula, we have

$$\begin{aligned} S(y) = \sum_n s_n y^n \;\Rightarrow\;& S(ty) = \sum_n s_n t^{|n|} y^n \\ \Rightarrow\;& \frac{1}{t^2} S(sty) = \sum_n s_n t^{|n|-2} y^n \\ \Rightarrow\;& H(y) = a \cdot y + \sum_n \frac{s_n}{|n|-1} y^n; \end{aligned}$$

- and imposing directly the conditions, since

$$\begin{aligned} S(y) = \sum_n s_n y^n &= \sum_i y_i \frac{\partial H}{\partial y_i}(y) - H(y) \\ &= \sum_i \left( y_i \sum_n n_i h_n y^{n-e_i} \right) - \sum_n h_n y^n \\ &= \sum_n (|n|-1) h_n x^n \end{aligned}$$

then

- $\forall i = 1 \div d,\ s_{e_i} = 0$ (necessary condition),
- $\forall |n| \neq 1,\ h_n = \frac{s_n}{|n|-1}$,
- $\forall i = 1 \div d,\ h_{e_i}$ is undetermined.

We see that if $S$ is an analytic function in a certain polydisk, so $H$ is . We have seen that if we work on

$$\mathcal{F}_k = \{H \in \mathcal{F} \mid j_0^{k-1} H = 0\},$$

with $k \geq 2$ (i.e., the space of functions with zero $(k-1)$-order Taylor's polynomial in the origin), the operator $\boldsymbol{\lambda}$ is invertible (at least if we work on an open star-shaped set).

In summary, we have obtained that

$$\mathcal{F} = \ker \boldsymbol{\lambda} \oplus \boldsymbol{\lambda}(\mathcal{F}),$$

because

- $\ker \lambda = \{h_a(y) = ay \mid a \in \mathbb{R}^d\}$, and
- $\boldsymbol{\lambda}(\mathcal{F}) = \{S \in \mathcal{F} \mid \frac{\partial S}{\partial y}(0) = 0\}$.

Hence, $\boldsymbol{\lambda}_| = \boldsymbol{\lambda}_{|\boldsymbol{\lambda}(\mathcal{F})}$ is isomorphism onto $\boldsymbol{\lambda}(\mathcal{F})$.

### 10.1.2 A splitting lemma

As a corollary of the previous results we obtain the next lemma.

**Lemma 10.2** *:*

*The space of functions defined on $T^*\mathcal{M}$ (or in a tubular neighborhood of its zero-section), $\mathcal{F}$, splits as*

$$\mathcal{F} \quad = \quad \ker \mathbf{\Lambda} \oplus \mathbf{\Lambda}(\mathcal{F}).$$

*Moreover, the vertical derivatives of the functions of $\mathbf{\Lambda}(\mathcal{F})$ vanish on the zero-section, and the functions of* $\ker \mathbf{\Lambda}$ *are the fiberwise homogeneous functions of degree* 1.

We shall write $\mathbf{\Lambda}_| = \mathbf{\Lambda}_{|\mathbf{\Lambda}(\mathcal{F})}$. Hence $\mathbf{\Lambda}_|$ is an isomorphism in $\mathbf{\Lambda}(\mathcal{F})$.

## 10.2 An evolution problem

We recall that we have to solve the evolution problem:

$$\begin{cases} \dfrac{\mathrm{d}S_t}{dt} = -\{\mathbf{\Lambda}_|^{-1}(S_t), S_t\}, \\ \\ \text{Cauchy's data: } S_0 = S. \end{cases}$$

We want to solve the evolution problem using expansions in powers of $t$. If $S_t = \sum_{k\geq 0} S_k t^k$ is the expansion of $S_t$ (where $S_0 = S$), then we can compute all the terms by the recurrence

$$S_{k+1} = \frac{-1}{k+1} \sum_{u+v=k} \{\mathbf{\Lambda}_|^{-1}(S_u), S_v\},$$

thanks to the next two properties (bearing in mind that $\mathrm{d}S \circ z = 0$).

**Lemma 10.3** *:*

*Let $S, T$ be two functions defined in $T^*\mathcal{M}$ (or in a neighborhood of its zero-section). Then:*

- $\mathrm{d}S \circ z = 0 \Rightarrow \mathrm{d}(\mathbf{\Lambda}_{|}^{-1}(S)) \circ z = 0.$
- $\mathrm{d}S \circ z = 0,\ \mathrm{d}T \circ z = 0 \Rightarrow \mathrm{d}\{S, T\} \circ z = 0.$

This lemma can be easily proved using cotangent coordinates. Hence, all the terms of the expansion verify $\mathrm{d}S_k \circ z = 0$ (and, in particular, belong to $\mathbf{\Lambda}(T^*\mathcal{M})$). In fact, since the function $S_0 = S$ has $y$-order 2, the $y$-orders of the $S_k$ increase: the $y$-order of $S_k$ is $k+2$. This is the key point in order to prove the convergence of the expansions.

We see that if our manifold is analytic (differentiable) and the initial term $S$ is analytic (differentiable), so are all the terms in the expansion. The problem is to obtain the analyticity (differentiability) of the expansions in the 'spatial' variables, at least until a time $t > 1$ in a neighborhood of the zero-section. Now, the analysis is local, and we shall prove the analytic case using majorant estimates.

### 10.2.1 Majorant estimates

Recall that for any two functions $f(z)$, $g(z)$ $(z = (z_1, \ldots, z_m))$ analytic at $z = 0$:

$$f(z) = \sum_n f_n z^n, \quad g(z) = \sum_n g_n z^n$$

(using multi-index notation), we say that $g$ is a majorant for $f$ ($f \ll g$) iff $\forall n\ |f_n| \leq g_n$.

A very close lemma to the next can be found in [86].

**Lemma 10.4** *:*

*The relation $\ll$ satisfies the following properties:*

1. $f_1 \ll g_1,\ f_2 \ll g_2 \ \Rightarrow\ f_1 + f_2 \ll g_1 + g_2,\ f_1 g_1 \ll g_1 g_2;$
2. $f \ll g \ \Rightarrow\ \dfrac{\partial f}{\partial z_i} \ll \dfrac{\partial g}{\partial z_i}\ (i = 1 \div m);$
3. $f_t \ll g_t\ \forall t \in [a, b] \ \Rightarrow\ \displaystyle\int_a^b f_t(z)dt \ll \int_a^b g_t(z)dt.$

*Let $w_b$ be the product $w_b(z) = \prod_{i=1}^m (b - z_i)$, where $b > 0$. Hence:*

4. $\forall i, k = 1 \div m$

$$1 \ll \frac{b}{b - z_i}, \quad \frac{z_i}{w_b} \ll \frac{b}{w_b}, \quad \frac{1}{(b - z_1)\ldots(b - z_k)} \ll \frac{b^{m-k}}{w_b}.$$

5. $|f(z)| < c, \forall z \mid \|z\|_\infty < b \ \Rightarrow\ f \ll \dfrac{cb^m}{w_b}.$

We complete the previous lemma.

**Lemma 10.5** *:*

*We suppose now that:* $m = 2d, z = (x, y), w(z) = \prod_{i=1}^{d}(b - x_i)(b - y_i)$. *Then:*

6. $f \ll w^{-k}, g \ll w^{-l} \Rightarrow \{f, g\} \ll 2db^{2d-2}klw^{-(1+k+l)}$,
7. $f \ll \prod_{t=1}^{u} y_{i_t} w^{-k}, g \ll \prod_{t=1}^{v} y_{j_t} w^{-l}$ $(u > 0, v > 0) \Rightarrow$
$$\{f, g\} \ll b^{2d-1} \left( \begin{array}{c} c_{uvkl} \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \prod_{t=1, t\neq s}^{u} y_{i_t} \end{array} \right) w^{-(1+k+l)}.$$
*where* $c_{uvkl} = k + \frac{2dkl}{u+v}$.
8. $f \ll \prod_{t=1}^{u} y_{i_t} w^{-k}$ $(u > 1) \Rightarrow \mathbf{\Lambda}_{|}^{-1}(f) \ll \frac{1}{u-1} \prod_{t=1}^{u} y_{i_t} w^{-k}$.

*Proof*:

6. (See [86])
$$\begin{aligned} \{f, g\} &= \sum_{i=1}^{d} \left( \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} \right) \\ &\ll 2kl \sum_{i=1}^{d} \frac{1}{(b - x_i)(b - y_i)} w^{-(k+l)} \\ &\ll 2db^{2(d-1)} klw^{-(1+k+l)}. \end{aligned}$$

7. Since
$$\sum_{j=1}^{d} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} \ll \sum_{j=1}^{d} \left( \prod_{t=1}^{u} y_{i_t} \frac{kw^{-k}}{b - x_j} \left( \frac{\partial \prod_{t=1}^{v} y_{j_t}}{\partial y_j} w^{-l} + \prod_{t=1}^{v} y_{j_t} \frac{lw^{-l}}{b - y_j} \right) \right)$$
and
$$\sum_{j=1}^{d} \frac{\partial}{\partial y_j} \left( \prod_{t=1}^{v} y_{j_t} \right) = \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t},$$

then

$$\{f,g\} \ll \begin{pmatrix} l\prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \prod_{t=1,t\neq s}^{u} y_{i_t} \\ + \\ k\prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \prod_{t=1,t\neq s}^{v} y_{j_t} \end{pmatrix} b^{2d-1} w^{-(1+k+l)}$$
$$+2db^{2d-2}kl\prod_{t=1}^{u} y_{i_t} \prod_{t=1}^{v} y_{j_t} w^{-(1+k+l)}.$$

Using that

$$\prod_{t=1}^{u} y_{i_t} \prod_{t=1}^{v} y_{j_t} = \frac{1}{u+v} \begin{pmatrix} \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \left( \prod_{t=1,t\neq s}^{v} y_{j_t} \cdot y_{j_s} \right) \\ + \\ \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \left( \prod_{t=1,t\neq s}^{u} y_{i_t} \cdot y_{i_s} \right) \end{pmatrix},$$

we obtain, finally:

$$\{f,g\} \ll b^{2d-1} \begin{pmatrix} c_{uvkl} \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \prod_{t=1,t\neq s}^{v} y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \prod_{t=1,t\neq s}^{u} y_{i_t} \end{pmatrix} w^{-(1+k+l)}.$$

8. Since

$$\mathbf{\Lambda}_{|}^{-1}(f)(x,y) = \int_0^1 t^{-2} f(x,ty)dt,$$

and, $\forall t \in [0,1]$

$$f(x,ty) \ll \prod_{s=1}^{u} y_{i_s} t^u w^{-k}(x,ty),$$
$$w^{-k}(x,ty) \ll w^{-k}(x,y),$$

then

$$\mathbf{\Lambda}_{|}^{-1}(f) \ll \frac{1}{u-1} \prod_{s=1}^{u} y_{i_s} w^{-k}.$$

□

### 10.2.2 Solving the problem in the analytic case

**Proposition 10.1** :

*Let $\mathcal{M}$ be an analytic d-manifold, and $\mathcal{N} = T^*\mathcal{M}$ its cotangent bundle (or a tubular neighborhood of its zero-section).*
*Let $S : \mathcal{N} \to \mathbb{R}$ be an analytic function, with $\mathrm{d}S\circ z = 0$.*
*Then:*

> *There exists a tubular neighborhood of the zero-section where the solution of the evolution problem $S_t$ is defined until some time $t > 1$.*

*Proof*:

We can use cotangent coordinates $(x, y)$ in a neighborhood of every point of the zero-section. It is sufficient to prove that we can get an small neighborhood of zero where the series $\sum_{k\geq 0} S_k(x, y)t^k$ is defined for $t < T$ and $T \geq 1$.

$S(x, 0)$ is constant, and we have supposed that this constant is 0. Moreover, $\frac{\partial S}{\partial y}(x, 0) = 0\ \forall x$, and we can write:

$$S(x, y) = \sum_{i,j} y_i y_j s_{ij}(x, y),$$

where de functions $s_{ij}$ are analytic (and $s_{ij} = s_{ji}$). Fixing a radius $b > 0$, let $c$ be the maximum of the sup-norms of the functions $s_{ij}$ on "$\|(x, y)\|_\infty < b$". So then, $\forall i, j = 1 \div d$

$$s_{ij} \ll cb^{2d} w^{-1},$$

where $w(x, y) = \prod_{i=1}^{d}(b - x_i)(b - y_i)$. Hence

$$S_0 \ll \gamma_0 \sum_{i_1, i_2} y_{i_1} y_{i_2}\ w^{-1},$$

where $\gamma_0 = cb^{2d}$.

Suppose that $\forall u \leq n$

$$S_u \ll \gamma_u \sum_{i_1,\ldots,i_{u+2}} \prod_{t=1}^{u+2} y_{i_t}\ w^{-(2u+1)}.$$

We want to estimate $S_{n+1}$. So then, applying the previous majorant estimates

$$S_{n+1} = \frac{-1}{n+1} \sum_{u+v=n} \{\boldsymbol{\Lambda}_{|}^{-1}(S_u), S_v\}$$

$$\ll \frac{b^{2d-1}}{n+1} \sum_{\substack{u+v=n \\ i_1,\ldots,i_{u+2} \\ j_1,\ldots,j_{v+2}}} \frac{\gamma_u\gamma_v}{u+1} \left( \begin{array}{c} \hat{c}_{uv} \prod_{t=1}^{u+2} y_{i_t} \cdot \sum_{s=1}^{v+2} \prod_{t=1,t\neq s}^{v+2} y_{j_t} \\ + \\ \hat{c}_{vu} \prod_{t=1}^{v+2} y_{j_t} \cdot \sum_{s=1}^{u+2} \prod_{t=1,t\neq s}^{u+2} y_{i_t} \end{array} \right) w^{-(2n+3)},$$

where $\hat{c}_{uv} = c_{(u+2),(v+2),(2u+1),(2v+1)} = (2u+1) + \frac{2d(2u+1)(2v+1)}{u+v+4}$. Applying that

$$\begin{aligned} \sum_{\substack{i_1,\ldots,i_{u+2} \\ j_1,\ldots,j_{v+2}}} \left( \prod_{t=1}^{u+2} y_{i_t} \cdot \sum_{s=1}^{v+2} \prod_{t=1,t\neq s}^{v+2} y_{j_t} \right) &= \sum_{s=1}^{v+2} \sum_{\substack{i_1,\ldots,i_{u+2} \\ j_1,\ldots,j_{v+2}}} \left( \prod_{t=1}^{u+2} y_{i_t} \prod_{t=1,t\neq s}^{v+2} y_{j_t} \right) \\ &= d(v+2) \sum_{k_1,\ldots,k_{n+3}} \prod_{t=1}^{n+3} y_{k_t}, \end{aligned}$$

we reach to

$$S_{n+1} \ll \gamma_{n+1} \sum_{k_1,\ldots,k_{n+3}} \prod_{t=1}^{n+3} y_{k_t} \; w^{-(2n+3)},$$

where

$$\gamma_{n+1} = \frac{db^{2d-1}}{n+1} \sum_{u+v=n} \frac{\gamma_u\gamma_v}{u+1} C_{uv}$$

and

$$\begin{aligned} C_{uv} &= \hat{c}_{uv}(v+2) + \hat{c}_{vu}(u+2) \\ &= (2u+1)(v+2) + (2v+1)(u+2) + 2d(2u+1)(2v+1) \\ &= (4+8d)uv + (5+4d)u + (5+4d)v + (4+2d) \\ &\leq 4(1+2d)(u+1)(v+1). \end{aligned}$$

Thus

$$\begin{aligned} \gamma_{n+1} &\leq \frac{4(1+2d)db^{2d-1}}{n+1} \sum_{u+v=n} (v+1)\gamma_u\gamma_v \\ &\leq 4d(1+2d)b^{2d-1} \sum_{u+v=n} \gamma_u\gamma_v. \end{aligned}$$

Hence, we have majorated the $S_n$s by

$$S_n \ll \gamma_n \sum_{i_1,\ldots,i_{n+2}} \prod_{t=1}^{n+2} y_{i_t} \; w^{-(2n+1)},$$

where the new sequence $\gamma_n$ verifies the recurrence

$$\begin{cases} \gamma_0 = cb^{2d} \\ \gamma_{n+1} = K \sum_{u+v=n} \gamma_u \gamma_v \end{cases},$$

where $K = 4d(1+2d)b^{2d-1}$.

Let $\rho \in [0,1[$ be a ratio that we shall choose later. If $\|x\|_\infty \leq \rho b$ and $\|y\|_\infty \leq \rho b$, then

$$|S_n(x,y)| \leq \gamma_n (d\rho b)^{n+2} (b(1-\rho))^{-2d(2n+1)}.$$

We call the right term in this formula $\beta_n$. Therefore, we have bounded all the terms of the expansion in a certain domain of $x, y$:

$$\sum_{n\geq 0} |S_n(x,y)| t^n \leq \sum_{n\geq 0} \beta_n t^n.$$

We want the convergence radius of this series to be greater than 1. Since

$$\lim_n \frac{\beta_{n+1}}{\beta_n} = \frac{d\rho}{b^{4d-1}(1-\rho)^{4d}} \lim_n \frac{\gamma_{n+1}}{\gamma_n}.$$

we have to compute the convergence radius of $\sum_{n\geq 0} \gamma_n t^n$. We can write:

$$\gamma_n = K^n \gamma_0^{n+1} a_n,$$

where $a_n$ is the sequence of natural numbers given by

$$\begin{cases} a_0 = 1 \\ a_{n+1} = \sum_{u+v=n} a_u a_v \end{cases}.$$

The elements of this sequence are the coefficients of the Taylor series of the function

$$f(t) = \frac{1-\sqrt{1-4t}}{2t}.$$

and, hence

$$\lim_n \frac{a_{n+1}}{a_n} = 4.$$

Finally:

$$\lim_n \frac{\beta_{n+1}}{\beta_n} = \frac{d\rho}{b^{4d-1}(1-\rho)^{4d}} \lim_n \frac{\gamma_{n+1}}{\gamma_n} = \frac{4K\gamma_0 d\rho}{b^{4d-1}(1-\rho)^{4d}}$$

$$= 16d^2(1+2d)c\frac{\rho}{(1-\rho)^{4d}} < 1,$$

if $\rho$ is sufficiently small. □

**Remarks**

i) Of course, if we use a better sequence $\gamma_n$, we can improve the factor 16 in the last formula. For example, we can change it by $4e$ if we consider the constants $\gamma_n$ satisfying the recurrence

$$\gamma_{n+1} = \frac{K}{n+1} \sum_{u+v=n} (v+1)\gamma_u \gamma_v$$

ii) If our basic manifold is a torus, we can use a more adapted method. We can use suitable norms, taking account that the functions $s_n$ are 1-periodic in all their variables. We should use Fourier-Taylor expansions.

$\triangleleft$

## 10.3 Solving the interpolation problem

Suppose that we have an exact symplectomorphism $F$ defined on a certain neighborhood of the zero-section on $T^*\mathcal{M}$, being $S$ its primitive function, and that we have the solution of the evolution problem: $S_t$. Hence, if $H_t$ is a time-dependent hamiltonian verifying $\mathbf{\Lambda}(H_t) = S_t$, then the corresponding Hamiltonian vector field is tangent to the zero-section, of course, because:

$$d(\mathbf{\Lambda}(H)) \circ z = 0 \Leftrightarrow d(H \circ z) = 0,$$

as it can be easily checked. We want to recover $H_t$ from the solution $S_t$, and that the flow of $H_t$ interpolates $F$.

If all the points of the zero-section are fixed, then there is an only possibility: $H_t = \mathbf{\Lambda}_|^{-1}(S_t)$. If the dynamics on the zero-section is given by a certain diffeomorphism $f$, we need it to be interpolable by the flow of a time-dependent vector field on the zero-section. Then, we need this vector field to be extended to a neighborhood of the zero-section by a time-dependent Hamiltonian vector field, whose Hamiltonian function $h_t$ belongs to $\ker \mathbf{\Lambda}$, $\forall t$. But this is easy, we just have to lift the basic vector field to the cotangent bundle.

As a summary we have the next theorem, which says that dynamics around and exact Lagrangian manifold is homologous to the identity.

**Theorem 10.1** *:*

*Let $\mathcal{M}$ be an analytic manifold, and $\mathcal{N}$ be its cotangent bundle $T^*\mathcal{M}$ (or a tubular neighborhood of its zero-section).*
*Let $F : \mathcal{N} \to \mathcal{N}$ be an analytic exact symplectomorphism, such that the zero-section is invariant, whose dynamics is given by $f : \mathcal{M} \to \mathcal{M}$. Suppose that $f$ is interpolated by the flow $f_t = f_{t,0}$ of an analytic time-dependent vector field $X_t \in \mathcal{X}(\mathcal{M})$: $f = f_1$*
*Then:*

*$F$ is (analytically) homologous to the identity (at least in a tubular neighborhood of the zero-section).*

*Proof*:

Let $S_t$ the solution of the evolution problem. Then, it is sufficient to choose

$$H_t = h_t + \mathbf{\Lambda}_|^{-1}(S_t)\circ\hat{f}_t^{-1},$$

where $h_t(\rho_x) = \rho_x(X_t(x))$, because $\mathrm{d}H_t\circ z = \mathrm{d}h_t\circ z$ and $h_t \in \ker \mathbf{\Lambda}$. □

**Remarks**

i) In fact, this let us to prove the analyticity of the expansions given in the determination problem, and solve it in analytic set up.

ii) As a particular case, the dynamics of a symplectomorphism around an invariant torus whose dynamics is conjugated to an ergodic translation is homologous to the identity, and the time-dependent Hamiltonian can be chosen periodic, thanks to the results in [86].

◁

## 10.4 Dynamics around an i.e.L.g.

As a summary of the results of this chapter, we are going to obtain a theorem about the dynamics around an invariant exact Lagrangian graph.

**Theorem 10.2** :

*Let $\mathcal{M}$ be an analytic manifold.*
*Let $F : T^*\mathcal{M} \to T^*\mathcal{M}$ be an analytic exact symplectomorphism, having and invariant exact Lagrangian graph $\mathcal{L}$ given by an analytic generating function $l : \mathcal{M} \to \mathbb{R}$, whose dynamics is analytically conjugated to the time-1 flow of a certain analytic time-dependent vector field.*
*Then:*

*$F$ is (analytically) homologous to the identity (at least in a tubular neighborhood of the Lagrangian graph $\mathcal{L}$).*

*Proof*:

By conjugation of our symplectomorphism by the fiberwise translation associated to $l$ and, after, by the lift of the basic conjugation, we can transport the Lagrangian graph to the zero-section and obtain that its dynamics is the time-1 flow of the basic vector field. Let $G$ be the composition of these two conjugations and $\bar{F} = G\circ F\circ G^{-1}$ the new exact symplectomorphism.

By the previous result, $\bar{F}$ can be interpolated by the time-1 flow of a certain Hamiltonian vector field (at least in a neighborhood of the zero-section). Let

$\bar{H}_t$ be such a Hamiltonian and $\bar{\varphi}_t$ be its corresponding flow (from $t_0 = 0$): $\bar{F} = \bar{\varphi}_1$.

Now, we apply the flow of $H_t := \bar{H}_t \circ G^{-1}$ is given by $\varphi_t = G \circ \bar{\varphi}_t \circ G^{-1}$ and, finally

$$F = G \circ \bar{F} \circ G^{-1} = G \circ \bar{\varphi}_1 \circ G^{-1} = \varphi_1.$$

□

**Remarks**

i) If the Lagrangian manifold $\mathcal{L}$ is not a graph, then we must use the Weinstein's theorems to transport this manifold to the zero-section of its cotangent bundle, via a symplectomorphism defined from a tubular neighborhood of $\mathcal{L}$ onto a neighborhood of the zero-section in $T^*\mathcal{L}$. Moreover, using a generalized Poincaré's lemma, he also proved that if our Lagrangian manifold is exact then the symplectomorphism is also exact (between two different manifolds, of course). For these results and their application to the construction of Morse families see [98, 61].

ii) We recall that although our symplectic objects may be non exact, sometimes it is possible that they turn into exact ones by lifting to suitable covering spaces.

iii) If the basic manifold is compact, say a torus, then we can get a time-periodic Hamiltonian $H_t$, of period 1, at least in a relatively compact open neighborhood of the graph. This can be done thanks to the results due to Pronin and Treschev [86], in analytic set up.

iv) These results can be applied for rather far of integrable symplectomorphisms. So, the dynamics around an invariant torus whose dynamics is conjugated to an ergodic translation is homologous to the identity, and the time-dependent Hamiltonian can be chosen 1-periodic.

◁

# Part IV

# APPLICATIONS

# Appendix A

# Some examples

Although the theory that we have introduced can be applied to a wide quantity of dynamical systems, it is advantageous to apply it to simple models. For instance, it would be hard, from a computational point of view, to experiment with a time-periodic Hamiltonian. The computation of the differential of the time-period map must be done by means of the variational equations. Notice that we do not worry about if that map has generating function.

This chapter is devoted to give different examples of exact symplectomorphisms defined on the annulus, which will be used in the sequel. They are an extension of the generalized standard-like maps introduced by MacKay [63], and provide examples of twist symplectomorphisms, monotone positive symplectomorphisms but not twist, symplectomorphisms whose monotonicity changes its sign, monotone undefinite symplectomorphisms, etc.

## A.1 Definitions

### The annulus

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the *d-torus*, and let $\mathbb{A}^d = \mathbb{T}^d \times \mathbb{R}^d$ be the *d-annulus* (or *d-cylinder*). We recall that $T^*\mathbb{T}^d \simeq \mathbb{A}^d$. The coordinates on $\mathbb{A}^d$ are the *angle-action coordinates* $z = (x \pmod 1, y)$.

We consider the symplectic structure on $\mathbb{A}^d$ inherited from its universal covering $\tilde{\mathbb{A}}^d = \mathbb{R}^d \times \mathbb{R}^d$ (or the symplectic structure as cotangent bundle of $\mathbb{T}^d$). Let $\pi : \tilde{\mathbb{A}}^d \to \mathbb{A}^d$ be the projection: $\pi(\tilde{z}) = z$ (and we shall write $\tilde{z} = (x, y)$).

Let $F : \mathbb{A}^d \to \mathbb{A}^d$ be a diffeomorphism, and let $\tilde{F} : \tilde{\mathbb{A}}^d \to \tilde{\mathbb{A}}^d$ be its lift: $\pi \circ \tilde{F} = F \circ \pi$. If $F$ is a symplectomorphism then $\tilde{F}$ is an exact symplectomorphism. If the primitive function of $\tilde{F}$ is 1-periodic in all its $x$-variables, then $F$ is exact symplectic.

### Integrability

We shall say that a symplectomorphism $L : \mathbb{A}^d \to \mathbb{A}^d$ is *completely integrable* iff it is given by

$$L(x, y) = (x + \nabla l(y), y),$$

for some function $l : \mathbb{R}^d \to \mathbb{R}$.

In such a case, the primitive function is given by

$$S(x, y) = y \cdot \nabla l(y) - l(y).$$

Each tori $\{y = y_0\}$ is invariant and the motion on it is given by a shift by $\omega = l(y_0)$. If it is rational, then all the orbits in such a torus are periodic, while if it is irrational the orbits are dense, and the dynamics is topologically transitive.

**Rotation vector of an orbit**

Given the lift of an orbit $\{(x_k, y_k)\}_{k\in\mathbb{Z}} \subset \mathbb{R}^{2d}$, its *rotation vector* (or frequency vector) is defined as the following limit, if it exists:

$$\lim_{k\to\infty} \frac{x_k}{k}.$$

In particular, the rotation vector of a periodic point of period $n$, $(x, y) \in Per_n(F)$, is rational. It is given by $\frac{p}{n} \in \mathbb{Q}^d$, where $p \in \mathbb{Z}^d$ satisfies

$$\tilde{F}^n(x, y) = (x, y) + (p, 0).$$

This is the equation to look for periodic orbits of rotation vector $\frac{p}{n}$.

## A.2 Generalized standard-like maps

A *generalized standard-like map* is a diffeomorphism on the d-cylinder

$$F : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$$

given by

$$\begin{cases} y' = y - \nabla V(x) \\ x' = x + \nabla W(y') \pmod 1 \end{cases},$$

where the *potentials* $V$ and $W$ are functions $V, W : \mathbb{R}^d \to \mathbb{R}$, being $V$ 1-periodic in all its variables. Its inverse is given by

$$\begin{cases} y = y' + \nabla V(x) \\ x = x' - \nabla W(y') \pmod 1 \end{cases},$$

$F$ is an exact symplectomorphism, and its primitive function is

$$S(x, y) = (y - \nabla V(x)) \cdot \nabla W(y - \nabla V(x)) - W(y - \nabla V(x)) - V(x).$$

The Jacobian matrix is

$$\mathrm{D}F(x, y) = \begin{pmatrix} I_d - \mathrm{D}^2W(y')\,\mathrm{D}^2V(x) & \mathrm{D}^2W(y') \\ -\mathrm{D}^2V(x) & I_d \end{pmatrix}.$$

We note that

$$\hat{A}(x, y) = \mathrm{D}^2W(y)^{-1} + \mathrm{D}^2W(y')^{-1} - \mathrm{D}^2V(x),$$

and, if $\nabla V(x_0) = 0$, then

$$\hat{A}(x_0, y) = 2\,\mathrm{D}^2W(y)^{-1} - \mathrm{D}^2V(x_0).$$

**Remark**

Notice that if we only ask $V$ to have periodic gradient, then we obtain a symplectomorphism in the annulus, but not necessarily exact. Its lift is, of course, exact. ◁

**Examples**

1) If we choose $V(x) = 0$ we get an integrable map, and all the torus $\{y = y_0\}$ are invariant. The corresponding frequency vectors are $\omega(y_0) = \nabla W(y_0)$. The extremal character of the orbits depends on $\mathrm{D}^2 W(y_0)$. If $\mathrm{D}^2 W(y_0) \succ 0$ the orbits are minimizing and if $\mathrm{D}^2 W(y_0) \prec 0$ the orbits are maximizing. Undefinite orbits appear when $\mathrm{D}^2 W(y_0)$ is undefinite.

2) If we take $W(y) = \frac{1}{2} y^2$ we obtain a *standard-like map*, which has a Lagrangian generating function

$$L(x, x') \;=\; \frac{1}{2}(x' - x)^2 - V(x)$$

and it is given by

$$\begin{cases} y' = y - \nabla V(x) \\ x' = x + y - \nabla V(x) \pmod 1 \end{cases}$$

The standard-like maps are monotone ($+_d$), and twist. They are a discrete model of the second Newton's law, because

$$x'' - 2x' + x \;=\; -\nabla V(x').$$

Hence, an orbit is determined by its sequence of angles.

Moreover, these maps are models of Poincaré maps of Hamiltonians defined on the $(d+1)$-cylinder (in angle-action coordinates: $\tilde{x} = (x_0, x)$ and $\tilde{y} = (y_0, y)$) with a double resonance:

$$H(x_0, x; y_0, y) = y_0 + \frac{1}{2}\|y\|_2^2 + \epsilon \sum_{k \in \mathbb{Z}^d} c_k(y) e^{2\pi <k,x> i} + \dots \,.$$

(The section through $x_0 = 0$ is equivalent to the time-unit map).

◁

### A.2.1 Fixed points

The fixed points of our generalized standard map are given by

$$\mathrm{Fix}(F) \;=\; \{(x_0, y_0) \in \mathbb{A}^d \mid \nabla V(x_0) = 0, \nabla W(y_0) \in \mathbb{Z}^d\}.$$

The stability of a fixed point is given by the $2d$ eigenvalues of the matrix

$$\mathrm{D}F(x_0, y_0) \;=\; \begin{pmatrix} I_d - \mathrm{D}^2 W(y_0)\, \mathrm{D}^2 V(x_0) & \mathrm{D}^2 W(y_0) \\ -\mathrm{D}^2 V(x_0) & I_d \end{pmatrix}.$$

These eigenvalues are paired off in the $d$ residues. The residues are the eigenvalues of the matrix

$$\frac{1}{4}\mathrm{D}^2W(y_0)\mathrm{D}^2V(x_0).$$

In particular, if the symmetric matrix $B(x_0, y_0) = \mathrm{D}^2W(y_0)$ is positive definite, then the residues are real, and there are no complex hyperbolic quadruplets.

In particular, if $d = 1$, the residue is

$$\rho = \frac{1}{4}W''(y_0)V''(x_0).$$

### A.2.2 Monotonicity

Our generalized standard map is monotone iff the matrices

$$B(x, y) \;=\; \mathrm{D}^2W(y - \nabla V(x))$$

are regular at all. Anyway, if this is not our case, the non-monotone set is given by the family of graphs

$$y \;=\; y_0 + \nabla V(x),$$

where the actions $y_0$ are those such that $\mathrm{D}^2W(y_0)$ is singular. Generically, the non monotone set is a submanifold of codimension 1, and in this example is foliated by Lagrangian graphs.

The torsion at a point $(x, y)$ coincides with the matrix $B(x, y)$, because it is symmetric. Recall that monotone positiveness is given by the matrices $B^{-1}A$ and $DB^{-1}$. We shall consider monotone positiveness of the second kind, because the condition is easier. Hence, monotone positive regions are given by the points $(x, y)$ satisfying

$$B(x, y) \;\succ\; 0.$$

For $d = 1$, monotonicity is given by

$$B(x, y) \;=\; W''(y - V'(x)).$$

## A.3 Some area preserving maps

Next four examples are generalized standard-like maps on the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$. For these *area preserving maps* we have chosen the potential $V$ as

$$V(x) \;=\; K\, v(x),$$

where $K$ is perturbative parameter and $v$ is any 1-periodic function, for instance

$$v(x) \;=\; -\frac{1}{(2\pi)^2}\cos(2\pi x).$$

We have taken this function $v$ for the examples. For $K = 0$ we have complete integrability. We shall consider $K > 0$.

We have

$$\begin{aligned} V(x) &= Kv(x) &= \tfrac{-K}{(2\pi)^2}\cos(2\pi x), \\ V'(x) &= Kv'(x) &= \tfrac{K}{2\pi}\sin(2\pi x), \\ V''(x) &= Kv''(x) &= K\cos(2\pi x). \end{aligned}$$

The function $V$ has two critical points (in fact, this number is the minimum for a function defined on $\mathbb{S}^1$), and in our case they are $x_0 = 0$ and $x_1 = \frac{1}{2}$. If $W'(y) = p \in \mathbb{Z}$, then the points $(0, y)$ and $(\frac{1}{2}, y)$ are fixed. The corresponding residues are $\rho_0 = \frac{K}{4}W''(y)$ and $\rho_1 = \frac{-K}{4}W''(y)$.

**Remark**

We consider this potential $V$ because it is commonly used, although it could be better to use trigonometric polynomials or even with infinitely many harmonics, which is the generic case.

For instance, we can change the sinus function by the sinus-like function

$$v'(x) = -12\sqrt{3}x(x - \frac{1}{2})(x + \frac{1}{2}), \text{ if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

and extending by periodicity. Then, we obtain $C^1$ area preserving maps which are even cheaper from a computational point of view. ◁

**Examples**

1) The well known *standard map*, or *Taylor-Chirikov map* [24], is given by

$$W(y) = \frac{1}{2}y^2,$$

and it is

$$\begin{cases} y' = y - \dfrac{K}{2\pi}\sin(2\pi x) \\ x' = x + y' \pmod 1 \end{cases}$$

It can also be defined on the torus $\mathbb{T}^2$, because $F(x, y + 1) = F(x, y) + (0, 1)$.

The standard map appears in the study of particle accelerators, for instance in some model which consider an orbiting electron in a cyclotron [76], or in condensed-matter physics, as in the Frenkel-Kontorova model [13].

The standard map is monotone positive and it is given by a Lagrangian generating function.

The fixed points $(\frac{1}{2}, p)$, with $p \in \mathbb{Z}$ have residue $\rho = \frac{-K}{4}$ and they are regular hyperbolic. On the other side, the fixed points $(0, p)$, with $p \in \mathbb{Z}$ have residue $\rho = \frac{K}{4}$, and they are elliptic if $K < 4$ and inversion hyperbolic if $K > 4$.

2) The *exponential standard map* is given by

$$W(y) = \exp(y),$$

It is

$$\begin{cases} y' = y - \dfrac{K}{2\pi}\sin(2\pi x) \\ \\ x' = x + \exp(y') \pmod 1 \end{cases}.$$

The Jacobian matrix is

$$\mathrm{D}F(x,y) = \begin{pmatrix} 1 - \exp(y - V'(x))V''(x) & \exp(y - V'(x)) \\ -V''(x) & 1 \end{pmatrix}.$$

We have introduced this map because it is monotone ($+_d$), has positive torsion but is not twist (because the torsion can be arbitrarily small). Moreover, $F$ has not a global generating function (it must contain logarithms).

The fixed points are given by $p \in \mathbb{N}^*$:

- $(0, \log(p))$: the residue is $\rho = \frac{K}{4}p$, and it is elliptic if $p < \frac{4}{K}$ and inversion hyperbolic if $p > \frac{4}{K}$.
- $(\frac{1}{2}, \log(p))$: the residue is $\rho = \frac{-K}{4}p$, and it is regular hyperbolic.

Note that if $K \geq 4$, there are no elliptic fixed points.

3) If the potential $W$ is given by

$$W(y) = \frac{1}{3}y^3,$$

we get a *quadratic standard map*. It is

$$\begin{cases} y' = y - \dfrac{K}{2\pi}\sin(2\pi x) \\ \\ x' = x + y'^2 \pmod 1 \end{cases}.$$

The Jacobian matrix is

$$\mathrm{D}F(x,y) = \begin{pmatrix} 1 - 2(y - V'(x))V''(x) & 2(y - V'(x)) \\ -V''(x) & 1 \end{pmatrix}.$$

We introduce this a.p.m. because it is not monotone, since the monotonicity condition fails on the curve $\{y = V'(x)\}$. In fact, it is monotone ($+_d$) above this curve and monotone ($-_d$) below it. $F$ has not a global generating function (it must contain square roots).

The fixed points appear in three groups:

- $(0,0)$ and $(\frac{1}{2},0)$: they are regular parabolic;
- $(0,\sqrt{p})$ and $(\frac{1}{2},-\sqrt{p})$, with $p \in \mathbb{N}^*$: since the residue is $\rho = \frac{K}{2}\sqrt{p}$, they are elliptic if $p < \frac{4}{K^2}$ and inversion hyperbolic if $p > \frac{4}{K^2}$;
- $(0,-\sqrt{p})$ and $(\frac{1}{2},\sqrt{p})$, with $p \in \mathbb{N}^*$: since the residue is $\rho = \frac{-K}{2}\sqrt{p}$, they are regular hyperbolic.

Note that, if $K \geq 2$, there is no elliptic fixed points.

We can get many quadratic standard maps taking $W$ as any cubic polynomial.

4) Finally, if our potential $W$ is given by

$$W(y) \;=\; \frac{1}{\pi}\sin(\pi y)$$

then we have a *trigonometric standard map*. It is

$$\begin{cases} y' = y - \dfrac{K}{2\pi}\sin(2\pi x) \\ \\ x' = x + \cos(\pi y') \pmod 1 \end{cases}.$$

We can also consider this map as defined on the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/(2\mathbb{Z})$, because $F(x, y+2) = F(x,y) + (0,2)$.

This a.p.m. is not monotone, because the condition fails on the family of curves $\{y = k + V'(x)\}$, where $k \in \mathbb{Z}$. On these curves the monotonicity changes its sign.

It has three families of fixed points:

- $(0,p)$, $(\frac{1}{2},p)$, with $p \in \mathbb{Z}$: they are regular parabolic;
- $(0,\frac{1}{2}+2p)$, $(\frac{1}{2},\frac{-1}{2}+2p)$, with $p \in \mathbb{Z}$: they are regular hyperbolic;
- $(0,\frac{-1}{2}+2p)$, $(\frac{1}{2},\frac{1}{2}+2p)$, with $p \in \mathbb{Z}$: they are elliptic if $K < \frac{4}{\pi}$ and inversion hyperbolic if $K > \frac{4}{\pi}$.

◁

## A.4 Higher dimensional symplectic maps

On the 2-annulus ($d = 2$), a well known example is due to Froeschlé [33]. The *Froeschlé map* is an standard-like map given by the potential

$$V(x_1,x_2) = -\frac{1}{(2\pi)^2}\left(K_1\cos(2\pi x_1) + K_2\cos(2\pi x_2) + \lambda\cos(2\pi(x_1+x_2))\right)$$

and it is

$$\begin{cases} y'_1 = y_1 - \dfrac{K_1}{2\pi}\sin(2\pi x_1) - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \\ y'_2 = y_2 - \dfrac{K_2}{2\pi}\sin(2\pi x_2) - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \\ x'_1 = x_1 + y'_1 \pmod 1 \\ x'_2 = x_2 + y'_2 \pmod 1 \end{cases}.$$

As we see, it is a product of two standard maps (with parameters $K_1$ and $K_2$) with a coupling parameter $\lambda$. We shall take positive parameters. Moreover, we have the following:

- If $\lambda = 0$, it is the product of two standard maps.
- If $K_1 = K_2 = 0$, it is the product of a rotation and a standard map (of parameter $2\lambda$).

Last claim is seen by using the change of variables:

$$\begin{cases} u_1 = x_1 - x_2, & v_1 = y_1 - y_2, \\ u_2 = x_1 + x_2, & v_2 = y_1 + y_2. \end{cases}$$

Of course, we can obtain many symplectic maps by changing the potential $W$. We can consider combinations of the kind $W(y_1, y_2) = W_1(y_1) + W_2(y_2)$, in order to get a *standard×exponential Froeschlé map*, an *exponential×exponential Froeschlé map*, a *standard×quadratic Froeschlé map*, etc.

# Appendix B

# BHM theory and Converse KAM theory

A fundamental question in symplectic-Hamiltonian dynamics is which parts of the phase space contain invariant tori and which do not. Often one works on the cotangent bundle $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d = \mathbb{A}^d$ of the $d$-torus.

- On one hand, *KAM theory* (by Kolmogorov [52], Arnold [4] and Moser [78]) let us to obtain *many* invariant tori of dimension $d$ for the exact symplectomorphisms which are close enough to completely integrable ones. Some non degeneracy conditions are needed. In fact, this theory proves that the measure of the complement of the invariant tori in any bounded region is arbitrarily small when we make smaller the size of the perturbation. The dynamics on these tori is conjugated to ergodic translations on $\mathbb{T}^d$ satisfying Diophantine conditions. Herman has proven that these tori need to be Lagrangian [40].
- On the other hand, the Birkhoff theory [19] about the invariant curves for area preserving maps in the annulus $\mathbb{A} = \mathbb{T}\times\mathbb{R}$ is a first step to a global (non perturbative) study on the existence of such a curves. This theory gives several Lipschitzian inequalities for the invariant curves and first asserts that they must be graphs (under some non degeneracy conditions). These theorems appear also in the works of Herman [39] and Mather [72]. Herman improved and generalized such results to higher dimensions along several papers [40, 41]. We shall refer to this theory as *BHM theory*.
- Finally, it would be useful to know conditions under which there are no invariant tori though a given point or region in phase space. Following MacKay, Meiss and Stark in [68] we shall refer to the development of such criteria as *Converse KAM theory*.

This chapter is highly inspired in Herman's papers *Existence et non existence de tores invariants par des difféomorphismes symplectiques* [40] and *Inegalités a priori pour des tores lagrangianes invariants par des difféomorphismes symplectiques* [41] and the paper by MacKay, Meiss and Stark *Converse KAM theory for symplectic twist maps* [68].

## B.1 Monotone positiveness

Some non-existence criteria of invariant tori are founded in some kind of positiveness of our symplectomorphism. We have follow [40] rather than [68], that is, we have studied monotone positive cases rather than twist cases.

### B.1.1 Notation

We shall work on the annulus. Let $F : \mathbb{A}^d \to \mathbb{A}^d$ be a symplectomorphism. We shall consider its lift $\tilde{F} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, which is an exact symplectomorphism with primitive function $S : \mathbb{R}^{2d} \to \mathbb{R}$.

Let $\psi : \mathbb{T}^d \to \mathbb{R}^d$ be a differentiable map, whose graph $\mathcal{L}_\psi$ is a $F$-invariant Lagrangian torus. Thus, we can write

$$\psi(x) = a + \nabla l(x),$$

where $a \in \mathbb{R}^d$ and $l : \mathbb{T}^d \to \mathbb{R}$. So, the generating function (on $\mathbb{R}^d$) of $\mathcal{L}_\psi$ is $L(x) = ax + l(x)$.

Let $\bar{f} : \mathbb{T}^d \to \mathbb{T}^d$ be the dynamics on the torus, that is to say, the *diffeomorphism* given by $\bar{f}(x) = f(x, \psi(x))$. We shall also write $\bar{A}(x) = A(x, \psi(x)), \bar{B}(x) = B(x, \psi(x))$, etc [1]. We shall suppose that $\psi$ is monotone ($|\bar{B}(x)| \neq 0,\ \forall x \in \mathbb{T}^d$).

**Remark**

> Along this chapter we suppose that our invariant Lagrangian tori are graphs. There is no an equivalent to higher dimensions of the next theorem due to Birkhoff, for $d = 1$:
>
> Let $F : \mathbb{A} \to \mathbb{A}$ be a $C^1$ monotone symplectomorphism, satisfying
>
> $$\sup_{z\in\mathbb{A}}(|B^{-1}(z)A(z)|, |D(z)B^{-1}(z)|) \ < \ \infty.$$
>
> Then, any $C^0$ $F$-invariant torus homotopic to the circle $\{y = 0\}$ (a rotational invariant curve), is the graph $\mathcal{L}_\psi$ of a certain continuous function $\psi \in C^0(\mathbb{T}^1, \mathbb{R})$.
>
> For the proof of this theorem see [19, 39, 72]. Herman [40] has perturbative generalizations to higher dimension of this theorem. ◁

### B.1.2 Minimizing graphs

Let $\hat{\Phi} : \mathbb{R}^d \to \mathbb{R}$ be the function given by

$$\hat{\Phi}(x, y) = S(x, y) - (L(f(x, y)) - L(x)).$$

[1] Recall the notation in Section 4.1

In order to know if $\mathcal{L}_\psi$ is minimizing we must compute the second derivative of $\hat{\Phi}$ respect to $y$ on the points of the graph. It is

$$\frac{\partial^2 \hat{\Phi}}{\partial y^2}(x, \psi(x)) = (\bar{D}^\top(x) - \bar{B}(x)^\top \, \mathrm{D}\psi(\bar{f}(x)))\bar{B}(x).$$

The character of the second derivative does not change if we multiply it by $\bar{B}^{-1}$ and $\bar{B}^{-\top}$ (recall that the graph is monotone):

$$\bar{B}^{-\top}(x)\frac{\partial^2 \hat{\Phi}}{\partial y^2}(x, \psi(x))\bar{B}^{-1}(x) = \bar{D}(x)\bar{B}(x)^{-1} - \mathrm{D}\psi(\bar{f}(x)).$$

Taking derivatives in the equalities $\bar{f}(x) = f(x, \psi(x))$ and $\psi(\bar{f}(x)) = g(x, \psi(x))$ we obtain

$$\begin{aligned} \mathrm{D}\bar{f} &= \bar{A} + \bar{B}\, \mathrm{D}\psi, \\ \mathrm{D}\psi{\circ}\bar{f}\, \mathrm{D}\bar{f} &= \bar{C} + \bar{D}\, \mathrm{D}\psi. \end{aligned}$$

Hence, as $D^\top A - B^\top C = I_d$, $D^\top B = B^\top D$ and $\mathrm{D}\psi$ is symmetric, we reach

$$\begin{aligned} \bar{B}^{-1}\bar{A} + \mathrm{D}\psi &= \bar{B}^{-1}\mathrm{D}\bar{f}, \\ \bar{D}\bar{B}^{-1} - \mathrm{D}\psi{\circ}\bar{f} &= (\mathrm{D}\bar{f})^{-\top}\bar{B}^{-1}. \end{aligned}$$

Therefore, we define the maps $E_1, E_2 : \mathbb{T}^d \to M_d(\mathbb{R})$ by

$$\begin{aligned} E_1 &= \bar{B}^{-1}\mathrm{D}\bar{f}, \\ E_2 &= (\mathrm{D}\bar{f})^{-\top}\bar{B}^{-1}. \end{aligned}$$

They are symmetric and non-singular matrices, and they are related by the equality

$$E_2 = \bar{B}^{-\top}E_1^{-1}\bar{B}^{-1}.$$

Hence, the positive definiteness of one matrix implies the positive definiteness of the other one. Then, we have obtained next lemma.

**Lemma B.1** *:*

*Let $F : \mathbb{A}^d \to \mathbb{A}^d$ be a symplectomorphism, and $\mathcal{L}_\psi$ be a monotone $F$-invariant Lagrangian torus. Then:*

$$\mathcal{L}_\psi \textit{ is minimizing} \Leftrightarrow E_1(x) \succ 0\ \forall x \in \mathbb{T}^d \Leftrightarrow E_2(x) \succ 0\ \forall x \in \mathbb{T}^d.$$

### B.1.3 BHM theory

The second point of the next theorem is due to Herman, but we have used his proof in order to relate his results with Converse KAM theory. As a summary, we obtain that the orbits on a monotone positive i.L.g. are minimizing.

**Theorem B.1** *:*

*Let $F : \mathbb{A}^d \to \mathbb{A}^d$ be a symplectomorphism, and $\mathcal{L}_\psi$ be a $F$-invariant Lagrangian torus. Suppose that it is monotone positive. Then:*

1. *$\mathcal{L}_\psi$ is minimizing;*
2. *$\|\rho(D\psi)\|_\infty \leq \max(\|\rho(\bar{B}^{-1}\bar{A})\|_\infty, \|\rho(\bar{D}\bar{B}^{-1})\|_\infty)$, where $\|\cdot\|_\infty$ means the sup-norm of a function defined on $\mathbb{T}^d$.*

*Proof*:

1. Suppose that our graph, which is given by $\psi(x) = a + \nabla l(x)$, is monotone $(+_a)$, that is, $\forall x \in \mathbb{T}^d$ $\bar{B}^{-1}(x)\bar{A}(x) \succ 0$. Let $x_0$ be the minimum of the periodic function $l$. Then:

   $$\psi(x_0) = a, \ \mathrm{D}\psi(x_0) = \mathrm{D}^2 l(x_0) \succeq 0.$$

   Hence:
   $$E_1(x_0) = \bar{B}^{-1}(x_0)\bar{A}(x_0) + \mathrm{D}\psi(x_0) \succ 0.$$

   Finally, as $E_1$ is non-singular at all, we deduce that it is always positive definite: $\forall x \in \mathbb{T}^d$ $E_1(x) \succ 0$.
   If we suppose that the graph is monotone $(+_d)$the proof is similar. We must take the antiimage by $\bar{f}$ of the maximum of $l$ in order to prove that $E_2$ is positive definite at all.
2. Second point is an immediate consequence of the inequality

   $$-\bar{B}^{-1}(x)\ \bar{A}(x) \prec \mathrm{D}\psi(x) \prec \bar{D}(\bar{f}^{-1}(x))\ \bar{B}^{-1}(\bar{f}^{-1}(x)),$$

   which is satisfied $\forall x \in \mathbb{T}^d$.

□

**Remarks**

i) We have that $\forall x \in \mathbb{T}^d$

$$\begin{aligned} E_1(x) + E_2(\bar{f}^{-1}(x)) &= \bar{D}(\bar{f}^{-1}(x))\bar{B}^{-1}(\bar{f}^{-1}(x)) + \bar{B}^{-1}(x)\bar{A}(x) \\ &= \hat{A}(x, \psi(x)). \end{aligned}$$

Hence, if our graph is minimizing (for instance, if our graph is monotone positive) then these matrices are all positive definite. These matrices appear on the diagonal of the second derivative of the action, and we had already obtained this result. In particular, if $\hat{A}(x, y) \preceq 0$ then there is no minimizing invariant graph through $(x, y)$. For $d = 1$, this coincides with the first step in the *Lipschitz criterion* for non existence of invariant graphs [67, 76].

ii) The upper bound of the proposition is a bit stronger that the previous one, because we have stronger hypothesis. Herman also proved that if our exact symplectomorphism is $\mathcal{C}^1$ and monotone globally positive and the invariant torus is

$\mathcal{C}^0$-Lagrangian (see [40] or [41] for the definitions and the proofs). It is a generalization to higher dimension of a theorem due to Birkhoff for $d = 1$ [19], which gives us bounds of the slope of invariant rotational curve [2].

Following in the case $d = 1$, such bounds give Lipschitz cones, and inside them there is the i.r.c.. This is the heart of the *cone-crossing* criterion for non-existence of i.r.c. performed by MacKay and Percival [67], and first used by Herman [39] and Mather [72]. On the other side, Newman and Percival [81] and, independently, Aubry and coworkers [12, 13] used criteria connected with action principles. In [67], they prove that both methods are equivalent.

iii) The first point in the proposition was also proved by Herman [40, 41] and MacKay, Meiss and Stark [68] using different assumptions. They need the generating function (and impose twist conditions on the symplectomorphism). We think that many results can be proven without using the existence of a global generating function satisfying some strong conditions of positiveness.

iv) We note that $E_1$ is $B^{-1}A$ and $E_2$ is $DB^{-1}$ *after projection* of the graph on the zero-section. So then, the graph is minimizing iff when we project it on the zero-section then it is monotone positive in the two senses.

◁

### B.1.4 Converse KAM theory

In [68], they derived a *variational criterion* for the non-existence of invariant Lagrangian graphs. We can write it in the next way. We shall use the same notation as in the proposition.

#### Non-existence criterion

> If the orbit by $z$ yields on a monotone positive region, and has a segment which does not have non-degenerate minimal action then it does not lie on any invariant Lagrangian graph included into such a region.

In order to check the minimality of a segment we can use the MMS iteration of Section 5.4.2). Of course, if we take segments of length 1 we obtain rather crude estimates.

In [68], they applied the test to the Froeschlé map, which is a 4D twist map and it is given by a Lagrangian generating function. However, we have seen that we must not be so restrictive, and we can apply their methods to other examples. The idea is that we do not apply global methods because we test if a certain segment of orbit is a local minimum of the corresponding action, and then the existence of a global generating function, which involves global conditions for our symplectomorphism, is not strictly necessary.

In [68], they also performed a generalization to higher dimensions of the cone-crossing criterion, given a geometrical interpretation of the variational criterion.

---

[2] i.r.c. for short

## B.2 Examples

We consider a generalized standard-like map (see Section A.2)

$$\left\{ \begin{array}{l} x' = x + \nabla W(y - \nabla V(x)) \\ y' = y - \nabla V(x) \end{array} \right. .$$

If the potential $W$ is a strictly convex function ($\mathrm{D}^2 W(y) \succ 0$ for any point $y \in \mathbb{R}^d$) then our symplectomorphism is monotone $(+_d)$. If this is not our case, we can study the monotone positive regions. Of course, we can do the same with the monotone negative regions.

Applying the first step in the variational criterion, if there is an invariant Lagrangian graph inside a monotone $(+_d)$region through a point $(x, y)$, then the corresponding orbit must be minimizing and, in particular, the matrix

$$\hat{A}(x,y) \quad = \quad \mathrm{D}^2 W(y)^{-1} + \mathrm{D}^2 W(y')^{-1} - \mathrm{D}^2 V(x)$$

must be positive definite. Since any invariant Lagrangian graph must intersect each fiber $\{x = x_0\}$, we can restrict ourselves to a particular one. We can choose $x_0$ as a critical point of the potential $V$. So then, we have to study the inertia of the matrix

$$\hat{A}(x_0,y) \quad = \quad 2\ \mathrm{D}^2 W(y)^{-1} - \mathrm{D}^2 V(x_0).$$

Hence, if $\mathrm{D}^2 W(y)$ is positive definite we must check if $\hat{A}(x_0, y)$ is positive definite, and if $\mathrm{D}^2 W(y)$ is negative definite, we must see if $\hat{A}(x_0, y)$ is negative definite. Undefinite cases are not considered.

Of course, stronger results may be obtained by iterating, with the aid of a computer. For instance, suppose our map be positive definite. Then, we can throw out the pieces of the phase space where the points are not minimizing after a finite number of iterations. We are sure that in these pieces there are not invariant tori. On the other side, minimizing orbits not only correspond to invariant tori, but also to minimizing periodic orbits, cantori, etc. This is the philosophy in [68].

We shall apply the method to different generalized standard maps.

### B.2.1 Some 2D examples

We shall consider $d = 1$ and the potential $V$ given by

$$V(x) \quad = \quad -\frac{K}{(2\pi)^2}\cos(2\pi x),$$

where $K$ is a positive pertubative parameter.

In all these examples we shall apply:

- the first step in the variational criterion, in order to get rather rude estimates of the critical value of $K$ in which all the i.r.c. have broken;
- the MMS iteration to a region of the phase space, in order to check in which parts of the phase space do not exist invariant tori and in which parts such existence is possible.

In order to show this second point, we have taken different values of $K$ to see the minimizing and maximizing regions. We have taken the same region of the cylinder, $y \in [-1, 1]$, and we have compared:

- the dynamics, taking 1024 points and iterating all of them 1024 times;
- the extremal character of the orbits, applying the MMS iteration to segments of length 128;
- the minimizing and maximizing regions, by choosing the corresponding points of the previous picture.

Moreover, if our map is not monotone, we have drawn in third picture and using white colour the curves where monotonicity fails (and change its sign). We shall see that the i.r.c. which cross these curves can fold, and be no graphs, and they are more robust.

The scale of colors that we have use in order to show the extremal character is


**Examples**

1) **The standard map.**

The standard map has $W(y) = \frac{1}{2}y^2$ and it is monotone $(+_d)$(as all the standard-like maps). Then:

$$\begin{aligned} \hat{A}(x, y) &= 2 - V''(x) \\ &= 2 - K\cos(2\pi x). \end{aligned}$$

As $K > 0$, $A(x, y)$ takes his smallest value at $x = 0$, being $\hat{A}(0, y) = 2 - K$. So then, as in the first step in Mather's calculations ([72]):

If $K \geq 2$, there does not exist any i.L.g..

If we take a segment of length 2, then we must take into account

$$\hat{D}_1(0, y) = 2 - K$$

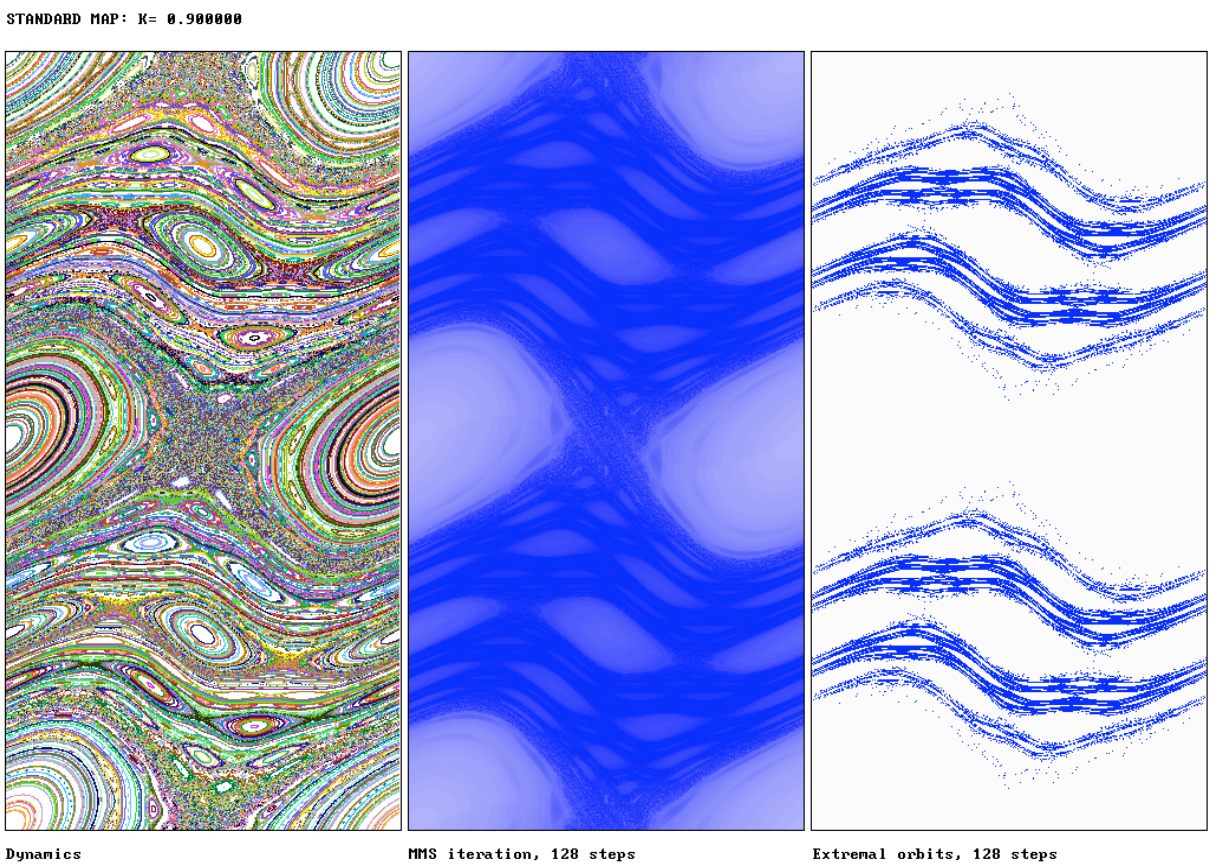
and

$$\hat{D}_2(0, y) = 2 - K\cos(2\pi y) - \frac{1}{2 - K}.$$

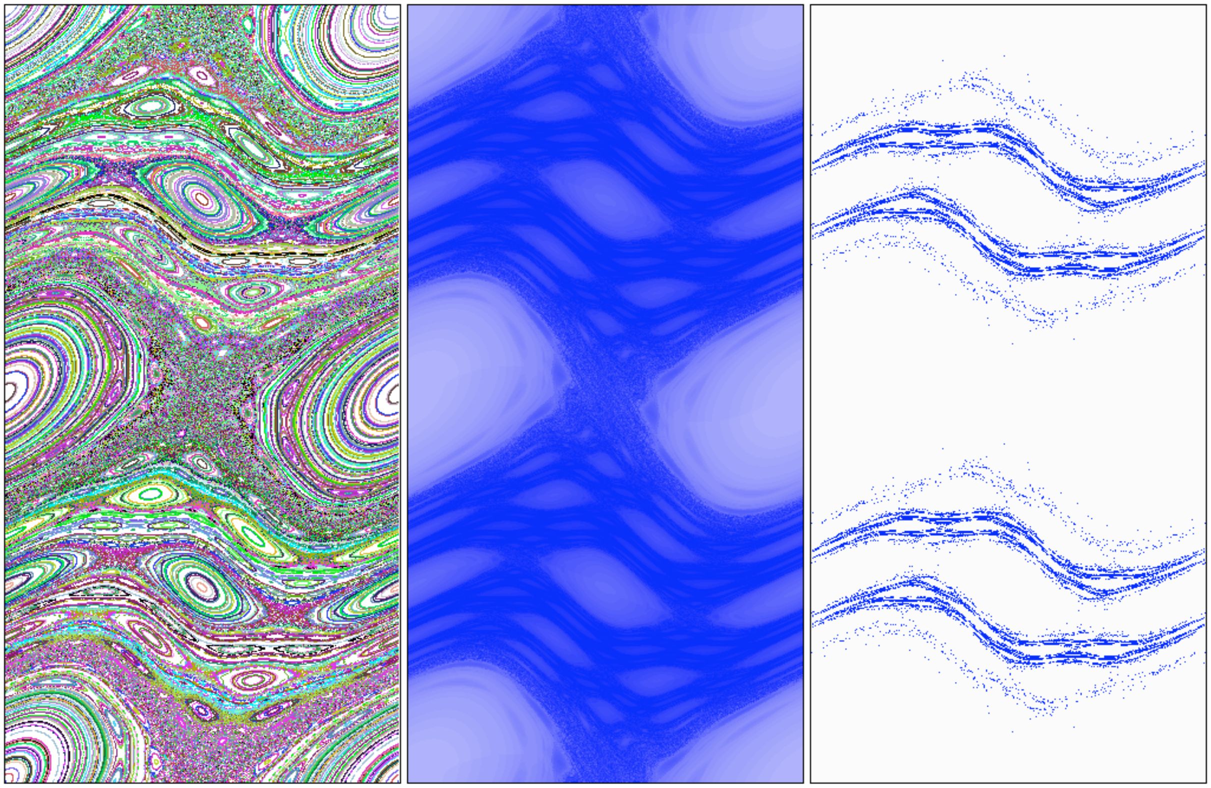
If $0 < K < 2$ then $\hat{D}_1 > 0$ and there does not exist any i.L.g.. if $\hat{D}_2 \leq 0$. The maximum value of $\hat{D}_2$ is taken for $y = \frac{1}{2}$ and it is $2 + K - \frac{1}{2-K}$. Finally, we improve the previous bound and we get that there are not i.L.g.. if $K \geq \sqrt{3}$. We could improve the bounds taking into account segments of higher length.

A better bound is obtained by improving the Lipschitz cone. For instance, Mather [72] obtained the bound $K \geq 4/3$ taking into account segments of length 2 (this kind of bound was generalized by Herman [40] to higher dimensions). Later, MacKay and Percival [67] refined it to obtain $K \geq 63/64 = 0.984375$. This refinement is an example of computer assisted proof. Finally, Jungreis [48] also performed a method for proving (computer assisted) that the standard map has no invariant circles for $K \geq 0.9718$. These bounds are in according with the result of Greene [36], who estimated the bound $K > 0.971635406$, by means of the residue criterion.

Next figures show how the invariant tori disappear when we increase the parameter $K$.

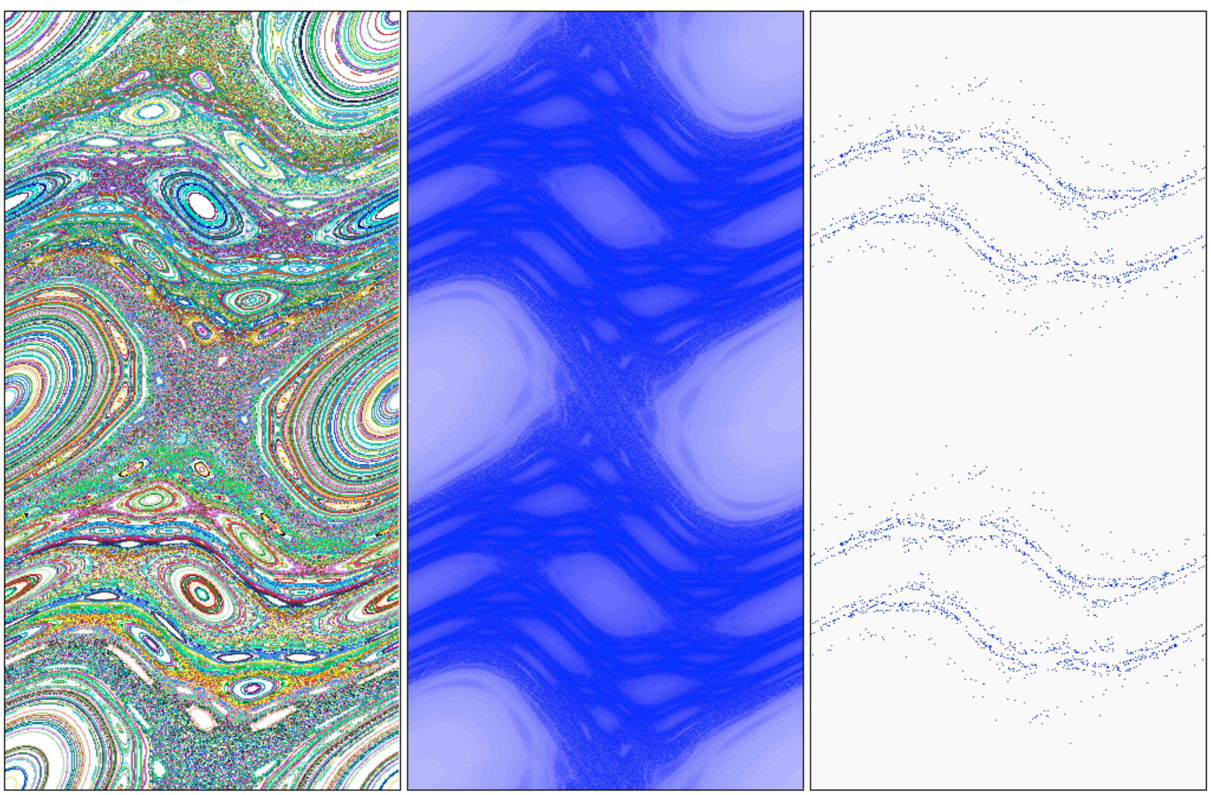


STANDARD MAP: K= 0.950000



Dynamics MMS iteration, 128 steps Extremal orbits, 128 steps

STANDARD MAP: K= 1.000000



Dynamics MMS iteration, 128 steps Extremal orbits, 128 steps

2) **The exponential standard map.**

In this case we take $W(y) = e^y$. Then

$$\hat{A}(x,y) \;=\; e^{-y} + e^{-y+V'(x)} - V''(x),$$

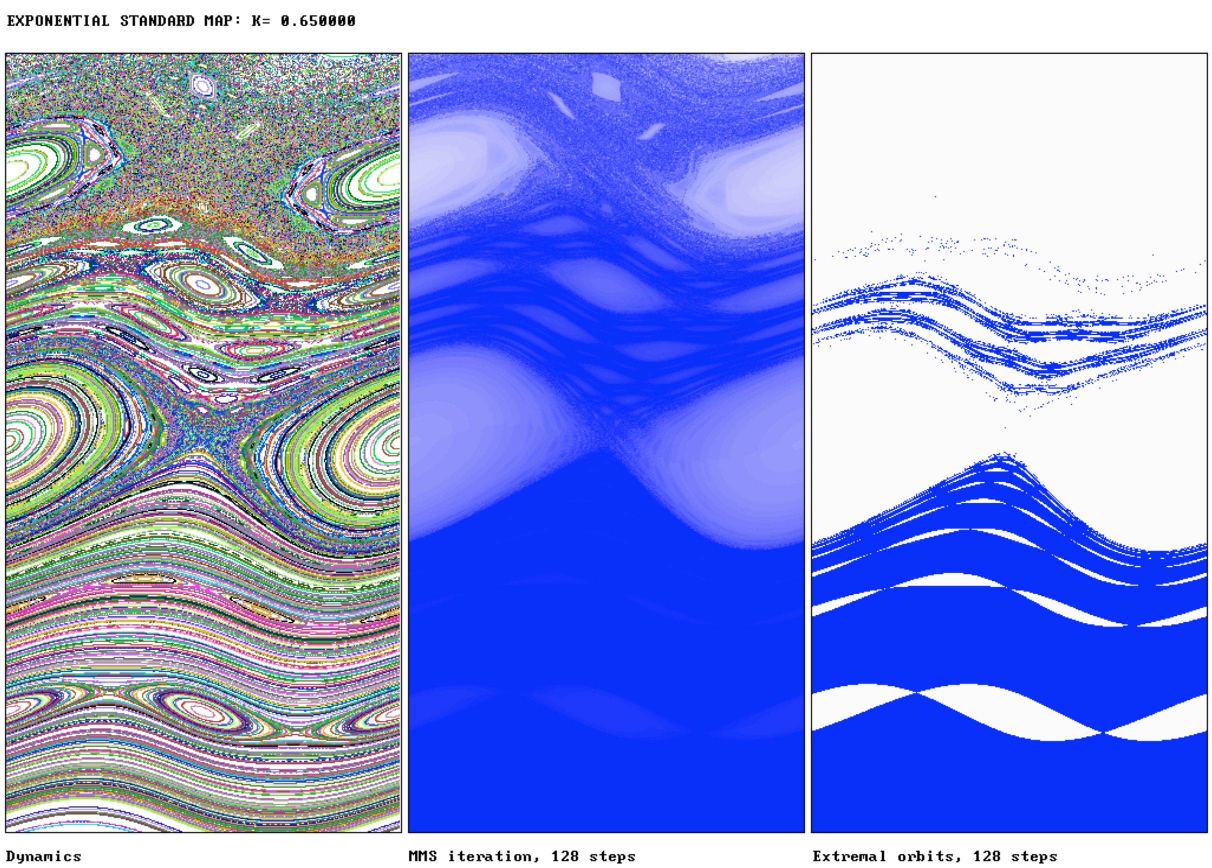
and if we take $x = 0$, the minimum of $V$, then

$$\hat{A}(0,y) \;=\; 2e^{-y} - K.$$

Hence:

> If we fix $K > 0$, there is no i.L.g. through any point $(0, y)$ with $y \geq \log \frac{2}{K}$.

This kind of bound is natural since the dynamics become faster and more chaotic as closer to $+\infty$ we are. The upper invariant curve separates a chaotic region of another which is plenty of invariant curves. A similar situation appears when one studies the boundary of a resonance zone associated to an elliptic fixed point [90]. We have changed this point by the points in $-\infty$, which are fixed.




Note that, although the hypotheses of the Birkhoff theorem are not satisfied, the rotational invariant curves seem to be graphs.

3) **The quadratic standard map.**

We consider $W(y) = \frac{1}{3}y^3$. Then, as $K > 0$, the points over $y = V'(x)$ are monotone $(+_d)$, and the points below $y = V'(x)$ are monotone $(-_d)$. As,

$$\hat{A}(0, y) = \frac{1}{y} - K,\ \hat{A}(\frac{1}{2}, y) = \frac{1}{y} + K$$

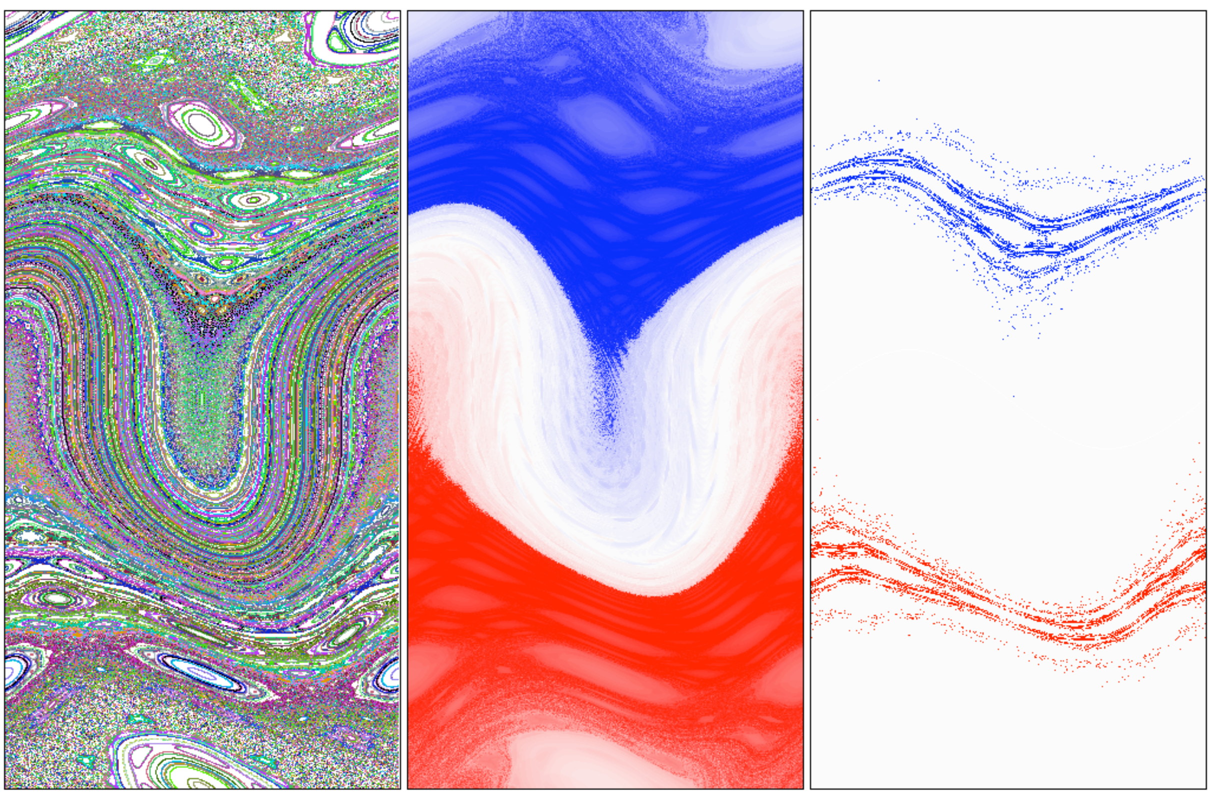
then

> there is no monotone $(+_d)$i.L.g. through any point $(0, y)$ with $y \geq \frac{1}{K}$, and there is no monotone $(-_d)$i.L.g. though any point $(\frac{1}{2}, y)$ with $y \leq -\frac{1}{K}$.

In the next figures, we note that the non-monotone r.i.c. are more robust that the other ones, which seem to be graphs. Moreover, the non-monotone curves, which have folds, forbid the mixing between the monotone positive and monotone negative regions. Then, we see that these curves are minimaximizing, in the sense that 'half' of the eigenvalues of the Hessian matrix are positive. When they break, the mixing is possible. We think that these minimaximizing curves are, in fact, definite (positive or negative), in suitable coordinates (cf. [91]).

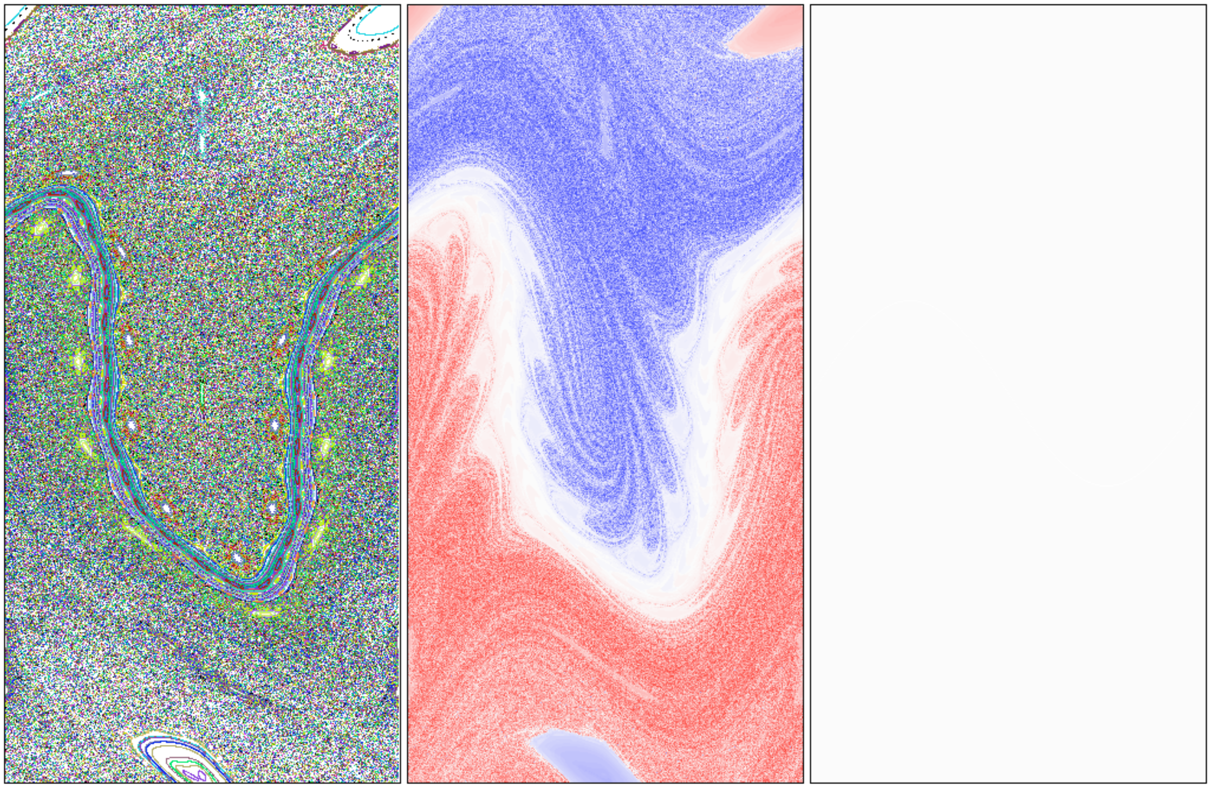
We recall that set of the points which go to a non-monotone one, after iteration, has measure zero.

QUADRATIC STANDARD MAP: K= 0.800000



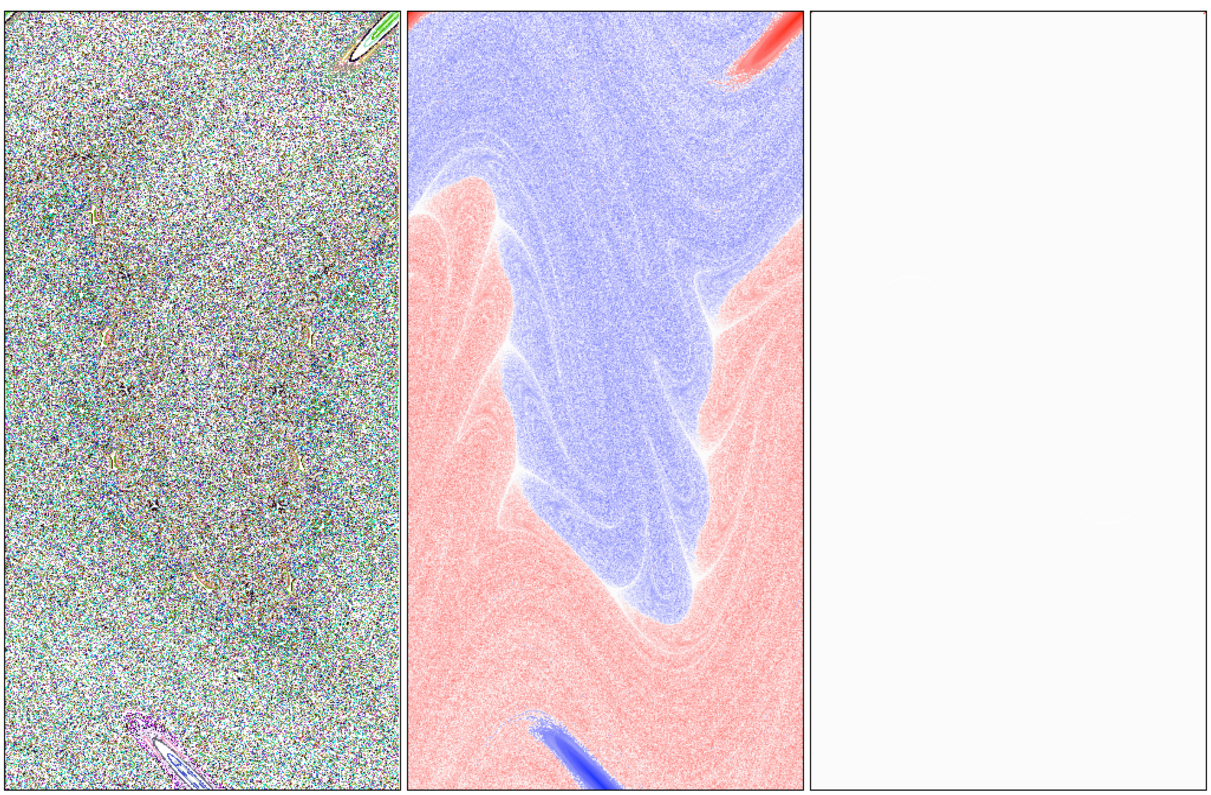
Dynamics MMS iteration, 128 steps Extremal orbits, 128 steps

QUADRATIC STANDARD MAP: K= 1.500000



Dynamics | MMS iteration, 128 steps | Extremal orbits, 128 steps

QUADRATIC STANDARD MAP: K= 2.000000



Dynamics | MMS iteration, 128 steps | Extremal orbits, 128 steps

4) **The trigonometric standard map.**

Finally, we consider $W(y) = \frac{1}{\pi}\sin(\pi y)$. We consider the map defined in the torus $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/(2\mathbb{Z})$, represented by $[0,1]\times[-1,1]$. Then, the region $\{-1 < y - V'(x) < 0\}$ is monotone $(+_d)$ and the region $\{0 < y - V'(x) < 1\}$ is monotone $(-_d)$. We have

$$\hat{A}(\frac{1}{2}, y) = -\frac{2}{\pi\sin(\pi y)} + K,\ \hat{A}(0, y) = -\frac{2}{\pi\sin(\pi y)} - K,$$

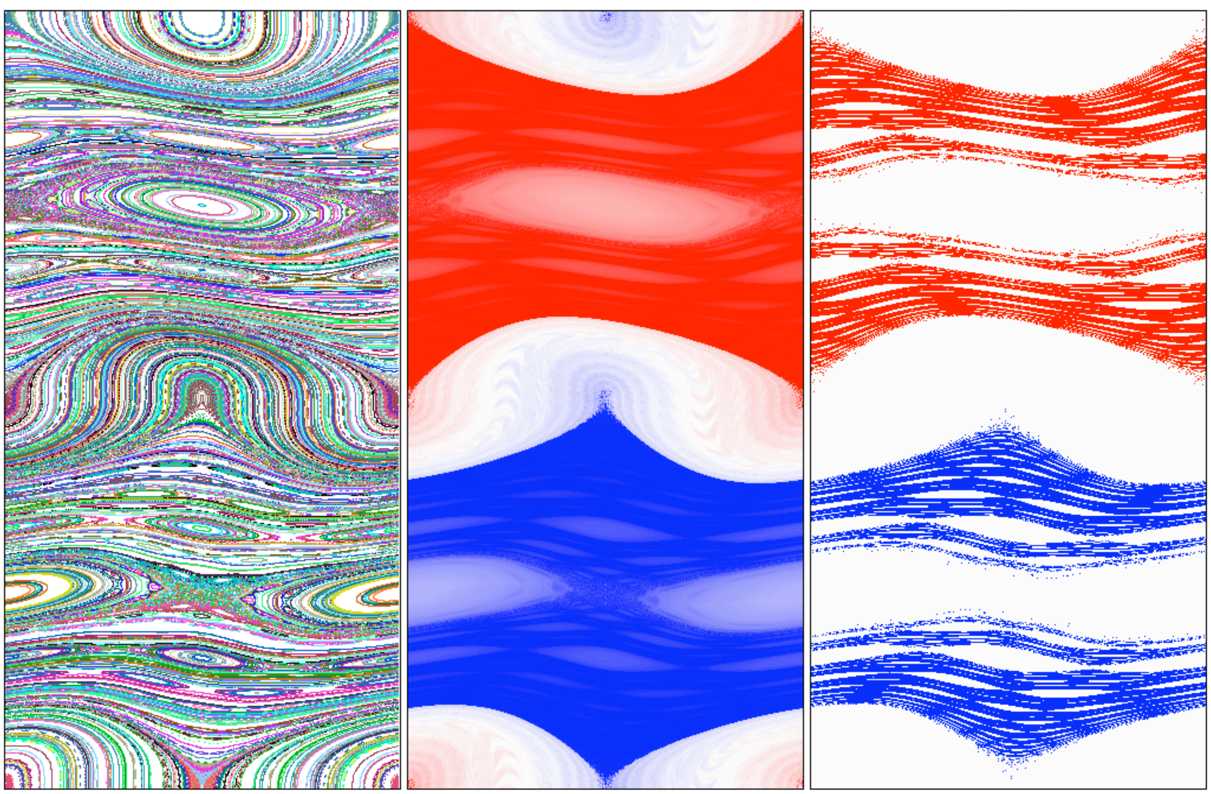
then

> there is no monotone $(-_d)$ i.L.g. through any point $(\frac{1}{2}, y)$ with $y \in ]0,1[$ and $\sin(\pi y) \geq \frac{2}{\pi K}$ and there is no monotone $(+_d)$ i.L.g. through any point $(0, y)$ with $y \in ]-1, 0[$ and $\sin(\pi y) \leq \frac{-2}{\pi K}$.

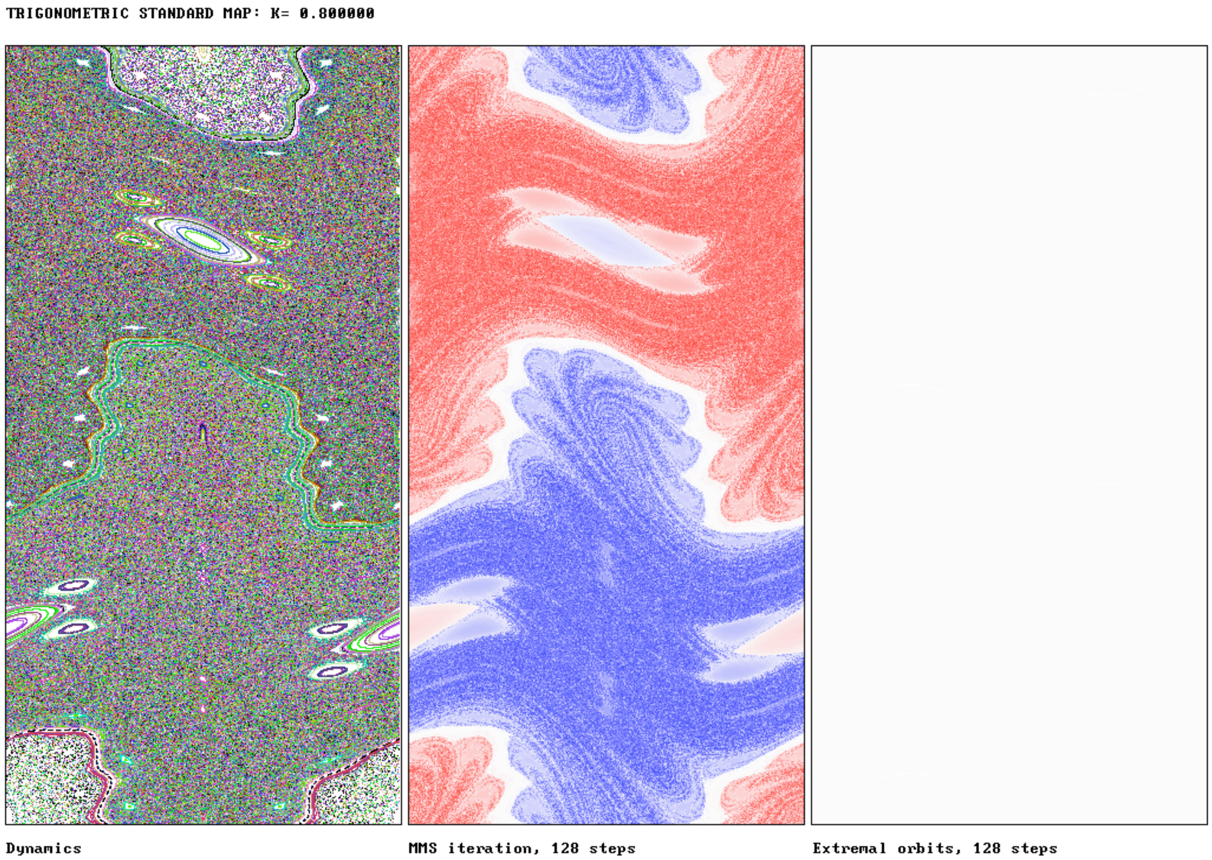
Note that this is dynamically represented by the a resonance zone associated to the fixed points $\left(0, \frac{-1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Anyway, if $K$ is small we do not throw out any piece of phase space, at least is this first step.

In the next figures we also note that the 'last' i.r.c. are non-monotone. The monotone i.r.c. seem to be graphs.

TRIGONOMETRIC STANDARD MAP: K= 0.300000



Dynamics — MMS iteration, 128 steps — Extremal orbits, 128 steps

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### B.2.2 Around an elliptic fixed point

We can also apply these methods to the study of a neighborhood of an elliptic fixed point, by means of suitable changes of variables. For the sake of simplicity we shall consider the 2D case.

Suppose we have a symplectomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$, being the origin an elliptic fixed point. Although it is not strictly necessary, we suppose that the linear part is already reduced:

$$\mathrm{D}F(0,0) \;=\; \begin{pmatrix} c & -s \\ s & c \end{pmatrix},$$

where $c^2 + s^2 = 1$.

We now consider the polar symplectic change of variables

$$\begin{array}{rlcl} P: & \mathbb{S} \times \mathbb{R}^*_+ & \longrightarrow & \mathbb{R}^2 \\ & (\theta, I) & \longrightarrow & (x = \sqrt{2I}\cos\theta, y = \sqrt{2I}\sin\theta). \end{array}$$

In order to do the calculations, we must consider the differential of $P^{-1} \circ F \circ P$. Note that it is not necessary to perform the change of variables and it is enough to consider the matrices

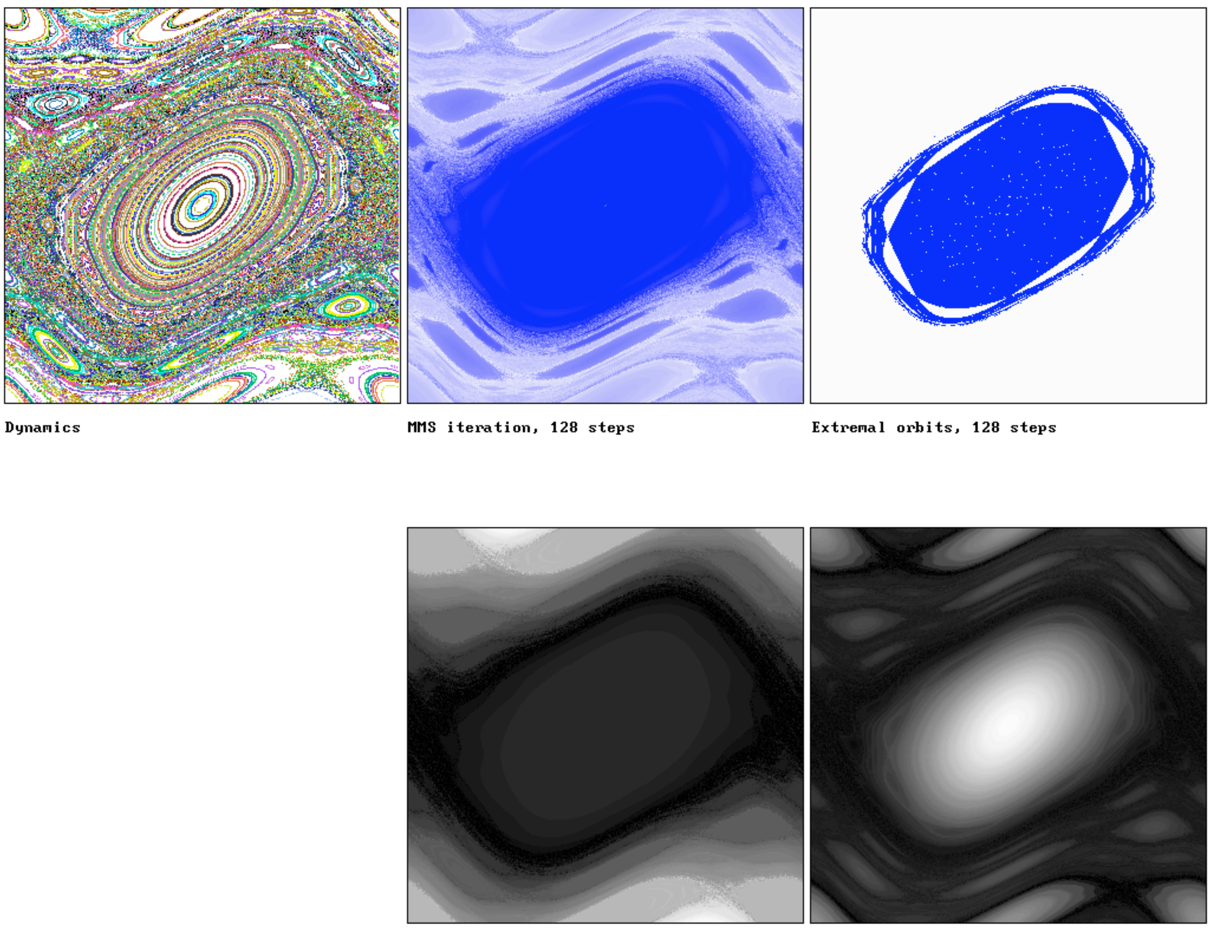
$$M(x,y) \;=\; \begin{pmatrix} \frac{-\bar{y}}{\bar{x}^2+\bar{y}^2} & \frac{\bar{x}}{\bar{x}^2+\bar{y}^2} \\ \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} \begin{pmatrix} -y & \frac{x}{x^2+y^2} \\ x & \frac{y}{x^2+y^2} \end{pmatrix},$$

being $(\bar{x}, \bar{y}) = F(x, y)$.

**Example**

As an example, we consider the standard map with $K = 1$. The square means the box $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, and the elliptic point is in its center (and we must take out it). The pictures in the second line show the averaged action $(A_0)$ and its variation respect to the parameter $K$ $(A_1)$, for the different points of the square (see Section 3.3.1). The level of grays are from black to white, in increasing order respect to the corresponding values of $A_0$ and $A_1$.

STANDARD MAP: K= 1.000000




In these pictures, the resonance zone associated to the elliptic fixed point has been roughly bounded. For a more accurate study of the 'last' invariant curve see the paper by Simó and Treschev [90]. ◁

### B.2.3 Some higher-dimensional examples

Now, we shall consider $d = 2$ and the potential $V$ given by

$$V(x_1, x_2) \quad = \quad -\frac{1}{(2\pi)^2}(K_1 \cos(2\pi x_1) + K_2 \cos(2\pi x_2) + \lambda \cos(2\pi(x_1 + x_2)).$$

We shall consider different potentials $W$, all of them like $W(y_1, y_2) = W_1(y_1) + W_2(y_2)$, mixing the different behaviors appearing in the previous section. In the pictures we have shown the extremal character of the points of a piece of a vertical plane (we have chosen the symmetry plane $\{x_1 = 0, x_2 = 0\}$) and we have extracted those which are minimizing or maximizing.
**Examples**

1) **The Froeschlé map.**

The potential is

$$W(y_1, y_2) \quad = \quad \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2.$$

Following [68], if we consider the symmetry planes $\{x = (0,0)\}$, $\{x = (\frac{1}{2}, \frac{1}{2})\}$, $\{x = (0, \frac{1}{2})\}$ and $\{x = (\frac{1}{2}, 0)\}$. we obtain that there are no i.L.g. outside the parametric region given by

$$K_1 + \lambda < 2 \quad , \quad K_2 + \lambda < 2,$$

$$(2 - \frac{K_1 + K_2}{2} - \lambda)^2 \quad > \quad \frac{(K_1 - K_2)^2}{4} + \lambda^2,$$

$$(2 + \frac{K_1 + K_2}{2} - \lambda)^2 \quad > \quad \frac{(K_1 - K_2)^2}{4} + \lambda^2,$$

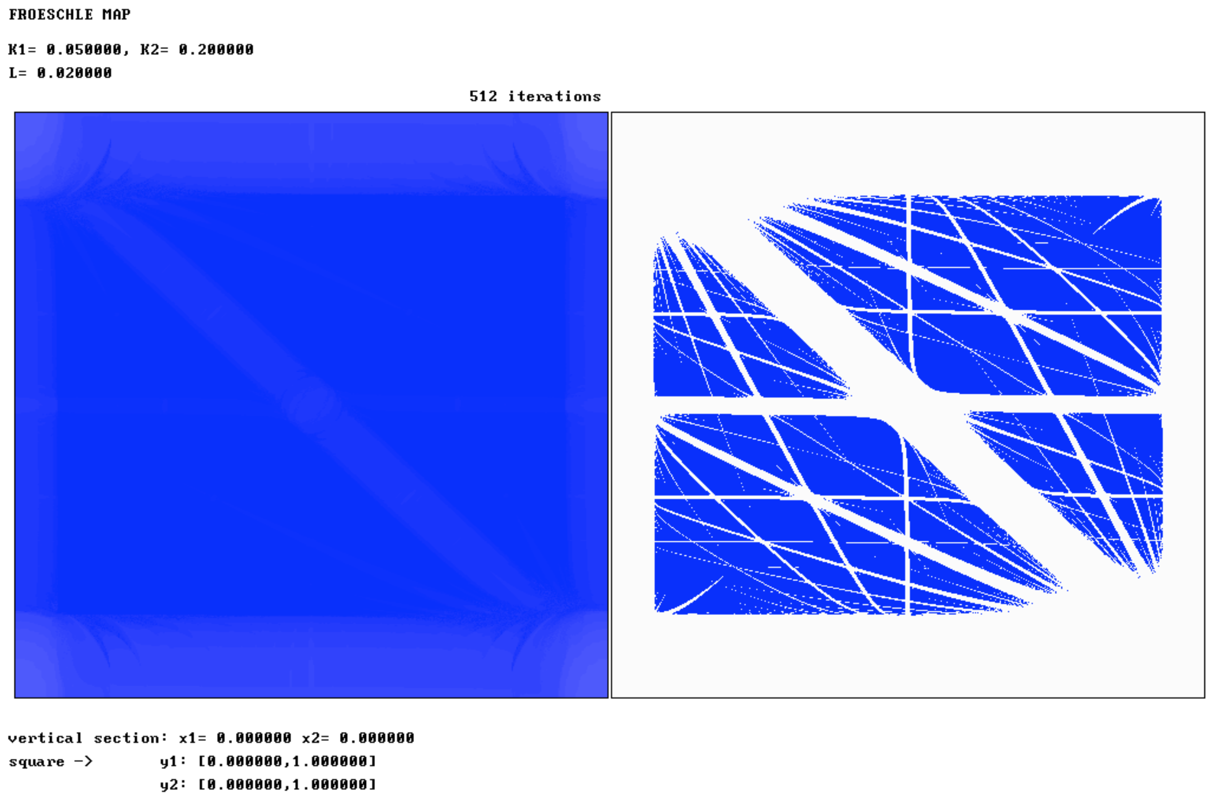
$$(2 - \frac{K_1 + K_2}{2} + \lambda)^2 \quad > \quad \frac{(K_1 + K_2)^2}{4} + \lambda^2,$$

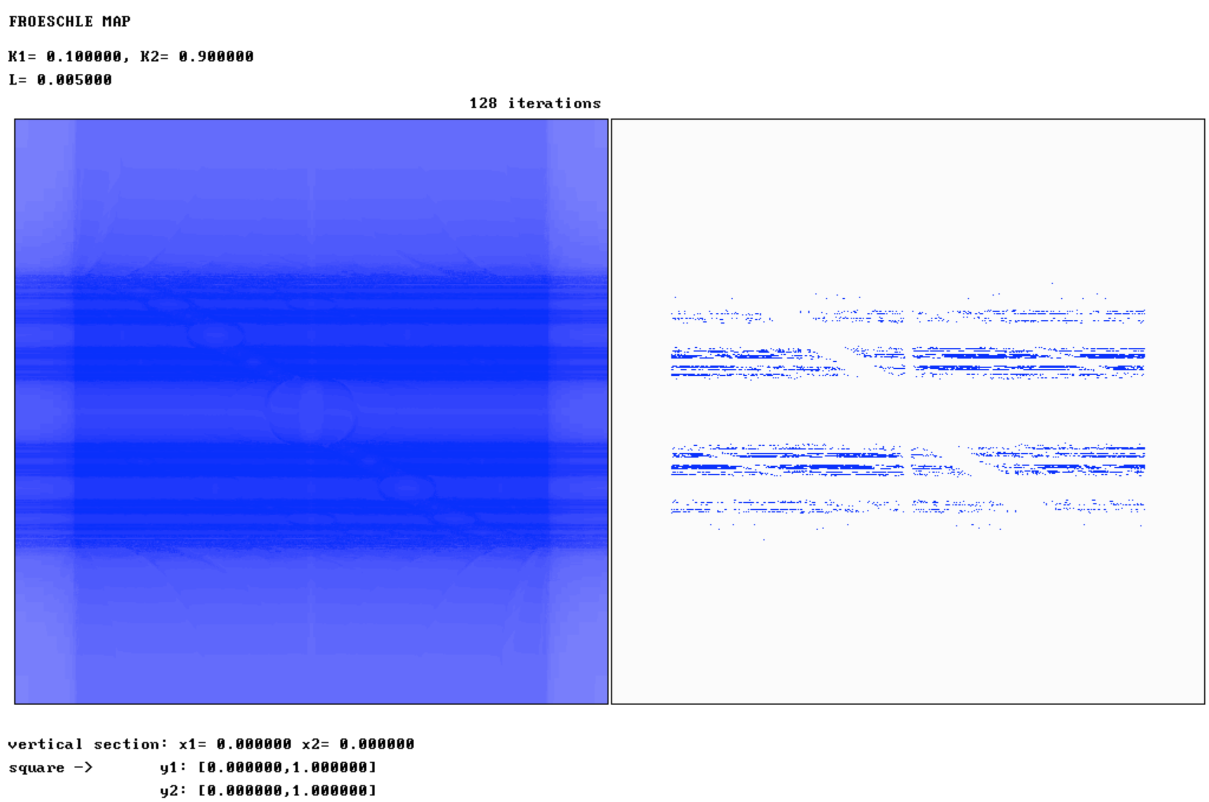
$$(2 + \frac{K_1 + K_2}{2} - \lambda)^2 \quad > \quad \frac{(K_1 + K_2)^2}{4} + \lambda^2.$$

We shall consider two examples, which appear in the next page:

1. The first one also appears in [68], where they relate the channels in the figure with the channels which appear when one look for symmetric periodic orbits [53], or when one takes a thin neighborhood of the symmetry plane and projects the points of a chaotic orbit when they enters into such a neighborhood [49].

2. If we take a small parameter $K_1$ and a parameter $K_2$ rather big then when we increase the coupling parameter $\lambda$ we obtain that the destruction of all the tori is like a dust of the kind Interval$\times$Cantor. This behaviour is shown in the second picture.


FROESCHLE MAP
K1= 0.050000, K2= 0.200000
L= 0.020000
512 iterations
vertical section: x1= 0.000000 x2= 0.000000
square -> y1: [0.000000,1.000000]
y2: [0.000000,1.000000]


FROESCHLE MAP
K1= 0.100000, K2= 0.900000
L= 0.005000
128 iterations
vertical section: x1= 0.000000 x2= 0.000000
square -> y1: [0.000000,1.000000]
y2: [0.000000,1.000000]

2) **The standard×exponential Froeschlé map.**

We take

$$W(y_1, y_2) \;=\; \frac{1}{2}y_1^2 + e^{y_2}.$$

Then

$$\hat{A}(0,0,y_1,y_2) \;=\; \begin{pmatrix} 2-(K_1+\lambda) & -\lambda \\ -\lambda & 2e^{-y_2}-(K_2+\lambda) \end{pmatrix}.$$

Hence, there are not invariant tori if $K_1 + \lambda \geq 2$. Otherwise, there are not invariant tori through any point $(0, 0, y_1, y_2)$ with

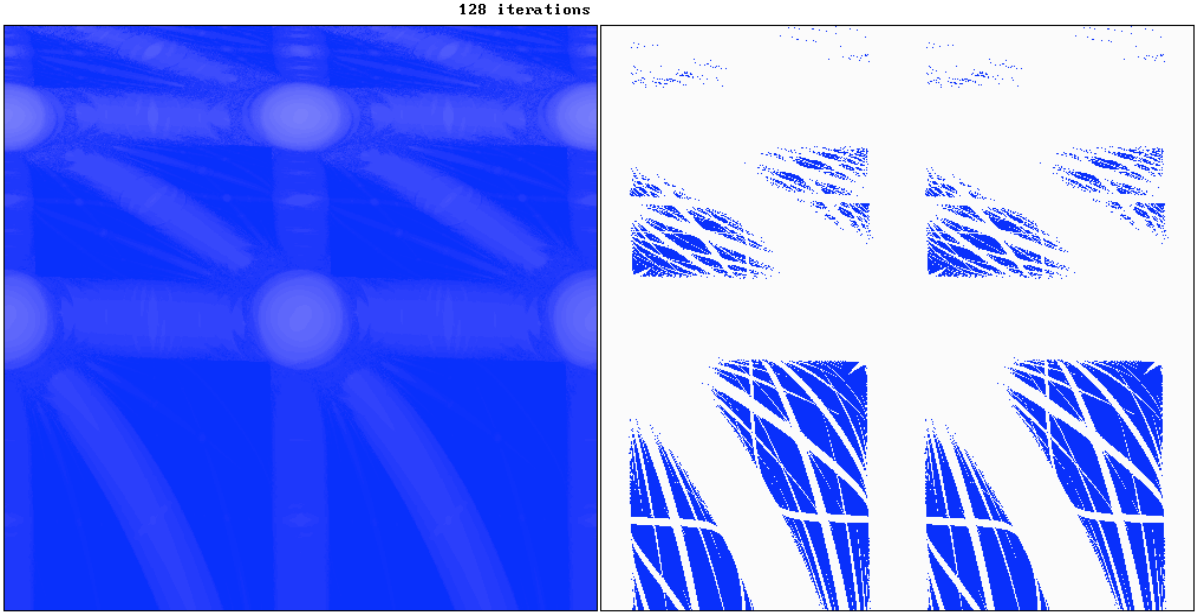
$$y_2 \;\geq\; \log\left(\frac{2(2-(K_1+\lambda))}{2(K_2+\lambda)-K_1K_2-(K_1+K_2)\lambda}\right).$$

Next figure confirms our expectations about that the number of invariant tori increase when we decrease the value of $y_2$.

STANDARD x EXPONENTIAL FROESCHLE MAP

K1= 0.100000, K2= 0.200000
L= 0.100000

128 iterations



vertical section: x1= 0.000000 x2= 0.000000
square -> y1: [-1.000000,1.000000]
y2: [-1.000000,1.000000]

3) **The exponential×exponential Froeschlé map.**

We change $W$ by

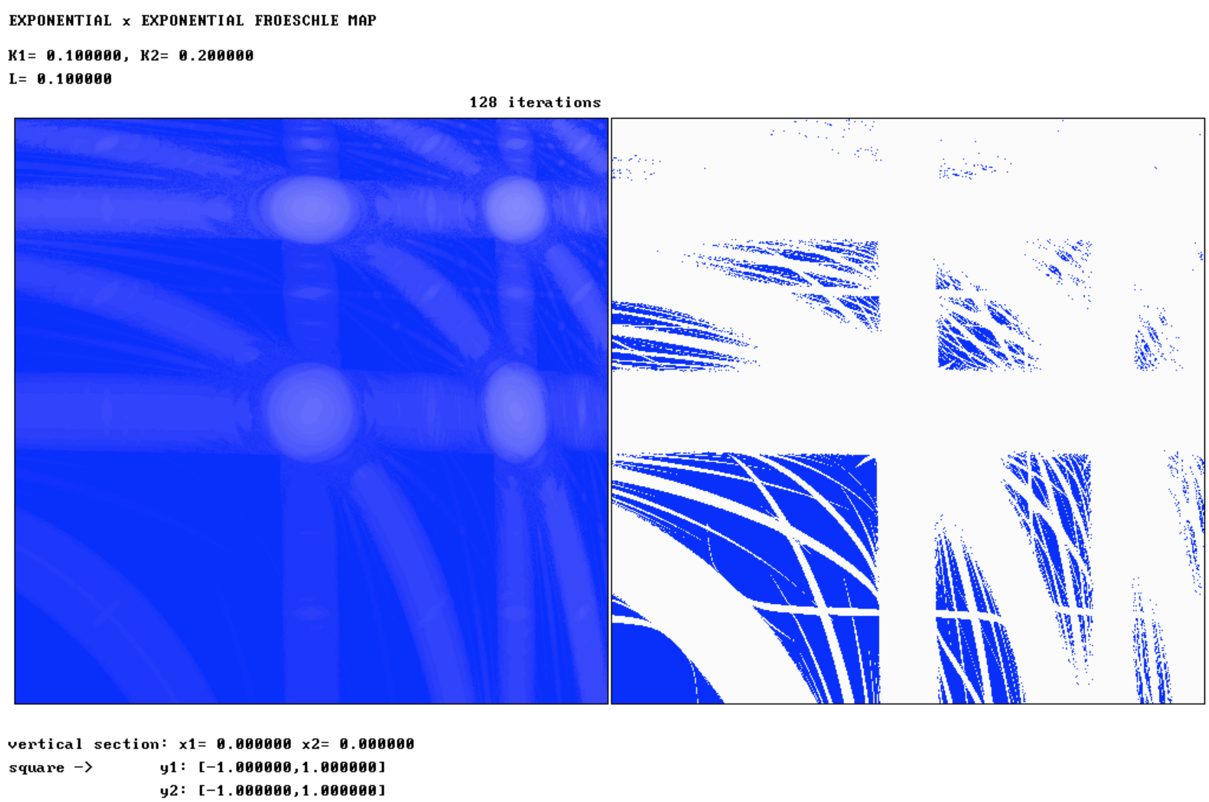
$$W(y_1, y_2) = e^{y_1} + e^{y_2}.$$

Hence,

$$\hat{A}(0,0,y_1,y_2) \quad = \quad \begin{pmatrix} 2\ e^{-y_1} - (K_1+\lambda) & -\lambda \\ -\lambda & 2\ e^{-y_2} - (K_2+\lambda) \end{pmatrix}.$$

Then, there is not invariant torus through any point $(0, 0, y_1, y_2)$ with

- $e^{y_1} \geq \frac{2}{K_1+\lambda}$, or
- $e^{y_2} \geq \frac{2}{K_2+\lambda}$, or
- $e^{y_1} < \frac{2}{K_1+\lambda}$, $e^{y_2} < \frac{2}{K_2+\lambda}$, $(1 - \frac{K_1+\lambda}{2} e^{y_1})(1 - \frac{K_2+\lambda}{2} e^{y_2}) \leq \frac{\lambda^2}{4} e^{y_1} e^{y_2}$.

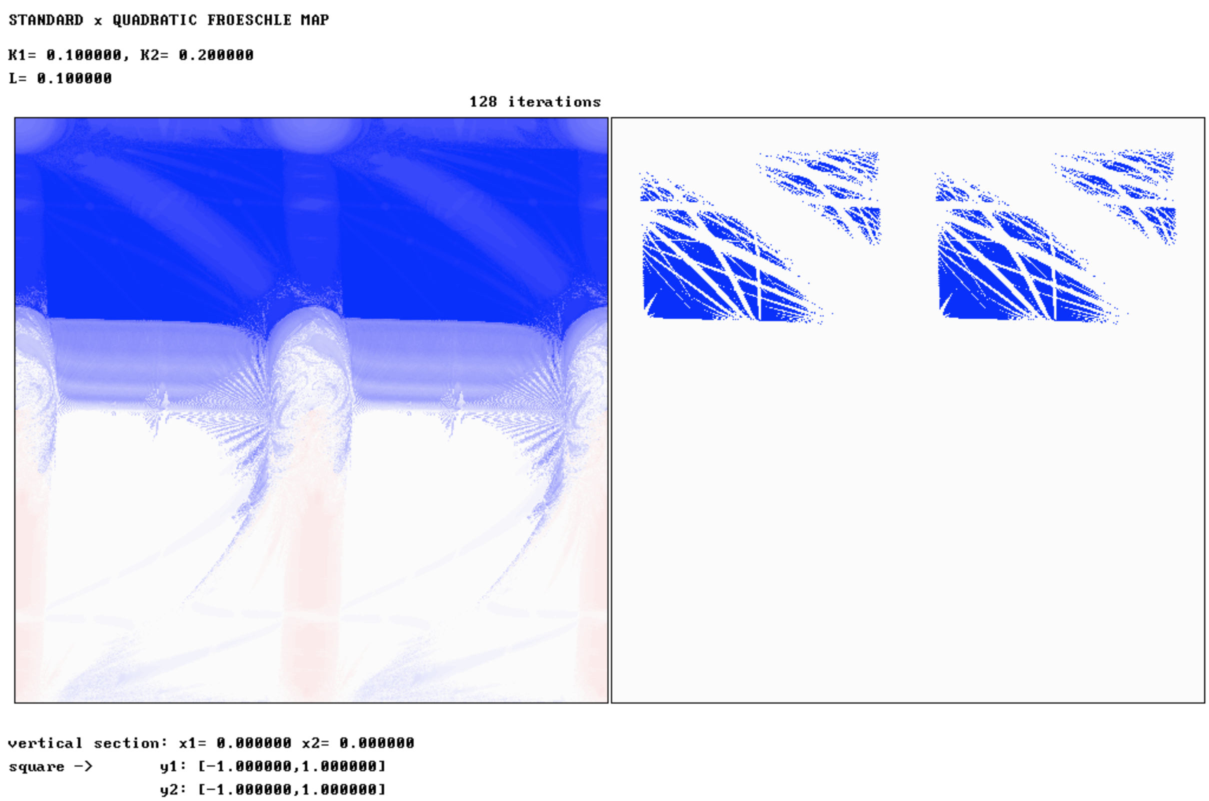


4) **The standard×quadratic Froeschlé map.**

In this case we have

$$W(y_1, y_2) \quad = \quad \frac{1}{2}y_1^2 + \frac{1}{3}y_2^3.$$

Then, our map is monotone ($+_d$) over the hypersurface $\{y_2 = \nabla_{x_2} V(x_1, x_2)\}$, and monotone undefinite below it. This is reflected in the next figure, because the intersection of the non-monotone set with the vertical plane $\{x_1 = x_2 = 0\}$ is the line $\{y_2 = 0\}$. There is a value $y_2$ bigger enough, say $\frac{1}{K_2+\lambda}$, such that there are not invariant tori over it.



5) **The quadratic×quadratic Froeschlé map.**

Finally, we have chosen

$$W(y_1, y_2) = \frac{1}{3}y_1^3 + \frac{1}{3}y_2^3$$

The phase space if divided in four regions: one is monotone positive

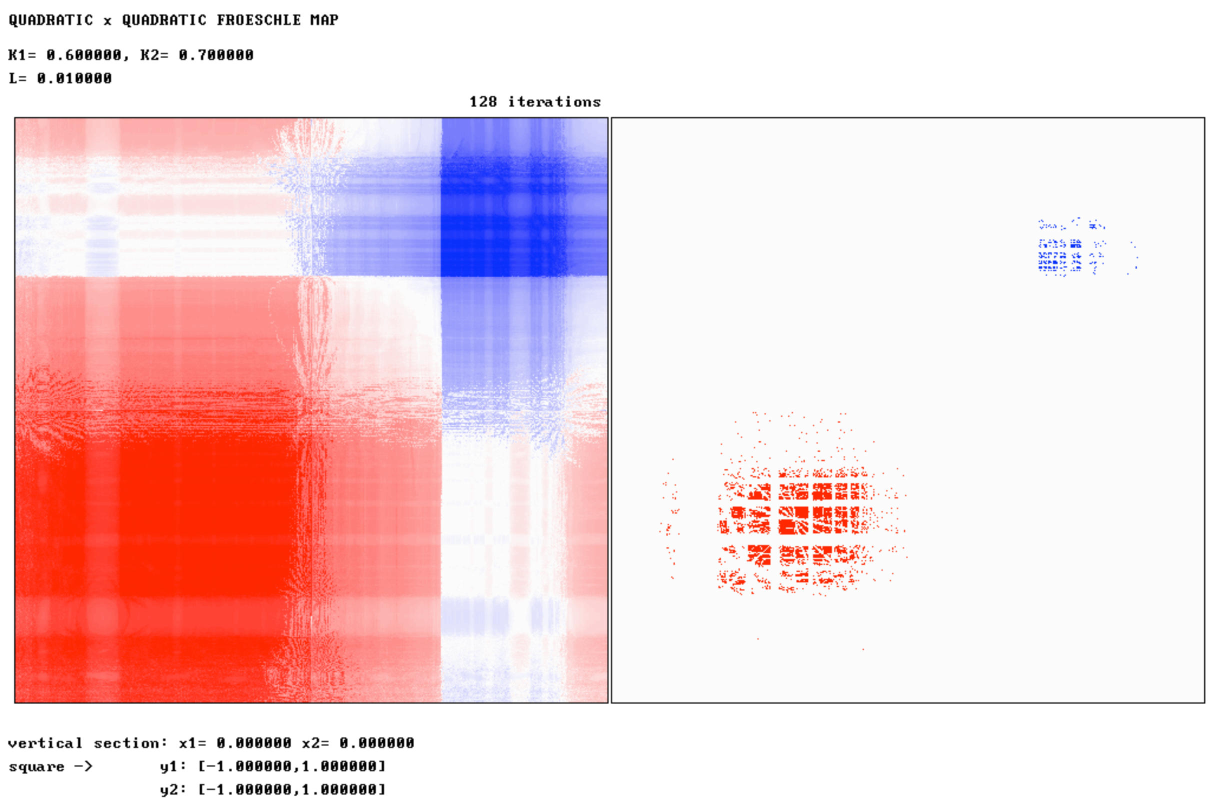
$$M_+ = \{y_1 > \nabla_{x_1} V(x_1, x_2), y_2 > \nabla_{x_2} V(x_1, x_2)\},$$

another is monotone negative

$$M_- = \{y_1 < \nabla_{x_1} V(x_1, x_2), y_2 < \nabla_{x_2} V(x_1, x_2)\},$$

and the other two are monotone undefinite. They are separated by the non monotone sets $\{y_1 = \nabla_{x_1} V(x_1, x_2)\}$ and $\{y_2 = \nabla_{x_2} V(x_1, x_2)\}$.

This is also reflected in the next figure, where the non-monotone sets are represented by the axis $\{y_1 = 0\}$ and $\{y_2 = 0\}$. We have chosen rather big parameters $K_1$ and $K_2$ and we see two Cantor×Cantor dusts: one is monotone positive and the other is monotone negative.




◁

# Appendix C

# The breakdown of invariant tori

The study of the breakdown of invariant tori is interesting in order to understand the transition to chaos in conservative dynamical systems.

The persistence of an invariant torus for small perturbations from the integrable case depends on the fact that the corresponding frequencies are 'far' from rationals. This is translated to a certain *Diophantine condition*. So, there is a nice connection between Dynamics and Arithmetic. The KAM tori are labelled by their frequencies, the more badly approachable by rationals the rotation vector is, the more difficult is to broke the corresponding torus.

In order to obtain good estimates on the domain of existence of such tori, it is better to look at a concrete frequency vector. For the standard map (and similar maps), we can ask about the *critical value* of the perturbative parameter, $K_\omega$, needed in order that the curve corresponding to such frequency, $\omega$, breaks. Greene [36] proposed a criterion based on the study of the stability of periodic orbits with nearby rotation number. He applied his method to show that, for the standard map, the 'last' invariant circle has frequency $\omega = \gamma$, where $\gamma$ is the golden mean

$$\gamma = \frac{1+\sqrt{5}}{2}.$$

The critical value when that torus is destroyed is

$$K_\gamma \simeq 0.97163540631.$$

This value was obtained by MacKay [63], and he had numerical evidence that it was, in fact, slightly high.

The Greene's method has been partially proven by MacKay [65] and Falcolini and de la Llave [32]. Tompaidis [94, 95] performed a Greene method in higher dimensions, and applied it to a three dimensional example (see also Section D.2.2).

We shall perform a Greene-like method, but instead of using the residues of periodic orbits (their dynamical character), we shall use their actions (their extremal character), but in a different way that Mather's $\Delta W$ [72]. Our symplectomorphisms must be monotone positive (or negative), or at least in the region where our torus exists. First, we shall check the method with the standard map and the golden curve, and we shall also notice scaling behaviour [63, 82]. Secondly, we shall apply the method to a 4D dimensional symplectic map: the Froeschlé map. Moreover, we shall do a numerical study of the kind of breakdown, that is to say, we ask for the kind of Aubry-Mather set our invariant torus transforms.

Although the approximation of quasi periodic orbits by periodic orbits is well understood in 2D twist maps, this is not the case in higher dimensions. Anyway, we shall use it in a heuristic way.

Finally, we must say that we only concern about KAM tori, that is to say, tori whose dynamics is given by ergodic translations. We recall that the dynamics on an invariant Lagrangian torus can be that of any diffeomorphism conjugated to any diffeomorphism of $\mathbb{T}^d$, as Herman proven in [42].

## C.1 Periodic orbits

First of all, we shall recall some definitions. We shall work on the $d$-cylinder $\mathbb{A}^d = \mathbb{T}^d \times \mathbb{R}^d$. Given a diffeomorphism $F : \mathbb{A}^d \to \mathbb{A}^d$, and its lift $\tilde{F} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, we shall say that a periodic point of period $n$, $(x, y) \in Per_n(F)$, has rotation vector $\frac{p}{n} \in \mathbb{Q}^d$ iff

$$\tilde{F}^n(x, y) = (x, y) + (p, 0).$$

In order to study an orbit with irrational rotation vector $\omega$, one consider periodic orbits with nearby rational rotation vectors.

### C.1.1 Approximation of invariant sets

The approximation of invariant sets by periodic orbits can rely on the two next propositions. Before stating them we need to recall a few concepts (see, for instance, [17]):

- Given a metric space $(X, d)$ one defines

$$\mathcal{H}(X) = \{K \subset X \mid K \neq \emptyset, K \text{ compact}\}.$$

- In the previous space of compact sets one defines the *Hausdorff distance*, as

$$h(A, B) = \max(\rho(A, B), \rho(B, A)),$$

  where

$$\rho(A, B) = \max_{a \in A} \min_{b \in B} d(a, b).$$

Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ a continuous map. We can extend this map to the bigger metric space as $F : \mathcal{H}(X) \to \mathcal{H}(Y)$, defined by $F(K) = f(K)$.

**Proposition C.1** *:*

*$F$ is a continuous extension of $f$.*

Let $f : X \to X$ be a continuous map. We shall say that $K \subset X$ is *(strictly) $f$-invariant* iff $f(K) = K$. Next result follows from the previous proposition.

**Corollary C.1** :

*Let $(K_n)_n$ be a sequence of $f$-invariant nonempty compact sets, convergent to $K$ (in the Hausdorff metric). Then:*

*$K$ is $f$-invariant.*

In particular, the limit of a sequence of periodic orbits, if it exists, is a compact invariant. The question is to know which kind of object it is.

### C.1.2 Reversible maps and symmetric periodic orbits

When our diffeomorphism has some symmetries, then we can simplify the computation of periodic orbits. We summarize here the main definitions about reversible maps.

**Reversibility.-** Given a set $X$, let $T : X \to X$ be a bijective transformation and $I : X \to X$ be an involution ($I^2 = id$). We shall say that $T$ is *$I$-reversible* if and only if $T^{-1} = ITI$. In such a case, we shall say that $I$ is the *reversor* of $T$.

For all $j \in \mathbf{Z}$, we define $I_j = T^j I$ and the $j$-th *symmetry axis* as

$$\Gamma_j \quad = \quad \{x \in X | I_j x = x\}.$$

Then we say that an orbit is *j-symmetric* if and only if it is invariant by $I_j$. The following holds:

- $x \in \Gamma_j,\ T^q x \in \Gamma_k,\ 2q + j - k \neq 0 \Rightarrow$
  $x$ is $\mid 2q + j - k \mid$-periodic and $\theta(x)$ is symmetric with respect $I_j$ and $I_k$.

In particular, if we look for $n$-periodic symmetric orbits, we can look for $x \in X$ such that:

- If $n = 2q$:
  $x \in \Gamma_0, T^q x \in \Gamma_0$ (symmetric with respect to $I_0$), or
  $x \in \Gamma_1, T^q x \in \Gamma_1$ (symmetric with respect to $I_1$).

- If $n = 2q + 1$ :
  $x \in \Gamma_1, T^q x \in \Gamma_0$, or
  $x \in \Gamma_0, T^{q+1} x \in \Gamma_1$
  (in both cases symmetric with respect to $I_0$ and $I_1$).

**Reversibility in generalized standard-like maps.-** Suppose that our diffeomorphism on $F : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ is given by

$$\begin{cases} y' = y - f(x) \\ x' = x + g(y') \pmod 1 \end{cases},$$

where $f, g : \mathbb{R}^d \to \mathbb{R}^d$ are two maps, being $f$ 1-periodic in all its variables.

Then $F = I_1 I_0$, where

$$I_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y - f(x) \end{pmatrix} , \ I_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + g(y) \\ y \end{pmatrix} .$$

$I_1$ is an involution, and $I_0$ is an involution iff $f$ is odd.

If this is our case, the (principal) symmetry axes of our reversible map are:

$$\Gamma_0 = \bigcup_{b \in \{0,1\}^d, \ f(\frac{1}{2}b)=0} S_{0,b}, \ \Gamma_1 = \bigcup_{b \in \{0,1\}^d} S_{1,b},$$

where

$$S_{a,b} \ = \ \{(\frac{1}{2}(ag(y) + b), y) \mid y \in \mathbb{R}^d\}.$$

For the sake of simplicity, we shall assume that all the symmetry axes exist, i.e. $\forall b \in \{0,1\}^d$

$$f\left(\frac{1}{2}b\right) \ = \ 0.$$

For instance, the standard map and the Froeschlé map (and their extension to other generalized standard-like maps) are reversible in this sense, and satisfy the previous condition.

**Remarks**

i) The standard map and the Froeschlé map are, in fact, *doubly reversible* [63], because they can factorize in other compositions of involutions. In general, if $g$ is even then $F = \bar{I}_1 \bar{I}_0$, where $\bar{I}_0$ and $\bar{I}_1$ are the involutions

$$I_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y + f(x) \end{pmatrix} , \ I_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - g(y) \\ -y \end{pmatrix} .$$

ii) In general, when we work with reversible symplectomorphisms, we ask for the involutions be antisymplectic. That is, $F^* \boldsymbol{\omega} = -\boldsymbol{\omega}$. In the case of $d = 1$, they are also called *orientation reserving* area preserving maps.

$\triangleleft$

The search for symmetric periodic orbits of rotation vector $\frac{p}{n}$ is summarized by the following diagram:

$$y \ \rightarrow \ (x = \tfrac{1}{2}(a_0 g(y) + b_0), y) \ \rightarrow (x_q, y_q) \ \rightarrow \ x_q - \tfrac{1}{2}(a_1 g(y_q) - b_1) = 0,$$

where:

- $a_0 \in \{0,1\}, b_0 \in \{0,1\}^d$
- if $n$ is even: $a_1 = a_0$
  if $n$ is odd: $a_1 = 1 - a_0$

- $b_1 = b_0 + p$
- if $n$ is even: $q = \frac{n}{2}$
  if $n$ is odd and $a_0 = 0$: $q = \frac{n+1}{2}$
  if $n$ is odd and $a_0 = 1$: $q = \frac{n-1}{2}$

Using this formulation the dimension of the problem is halved.

**A parallel shooting technique to look for periodic orbits.-** To solve the equation $F^n(z) = z$ is equivalent to solve the system:

$$\begin{cases} z_1 = F(z_0), \\ z_2 = F(z_1), \\ \vdots \\ z_0 = F(z_{n-1}). \end{cases}$$

The advantages can be summarized in two items:

- in our case, the differential of $F$ has spectral radius $\geq 1$, and so, the differential of $F^n$ can be extremely large if $n$ is large, giving rise to accuracy problems;
- the continuation respect to parameters is more efficient.

This is the background of the methods that we have used, but it can have the following variants:

- the symmetries of the problem can be used, and
- in some cases it is enough to consider the angular variables to determine the orbit.

Hence, it is possible to reduce the dimension of the problem to one fourth of the initial one.

## C.2 A variational Greene method

Suppose we have a monotone positive symplectomorphism on the cylinder, and it is a perturbation of a monotone positive integrable one. Hence, if the perturbation vanish, then all the orbits are minimizing and they live on Lagrangian graphs. We fix a certain invariant Lagrangian graph, whose dynamics is given by an ergodic translation. We want to know when it breaks.

Our method is heuristically based in next three points:

- as an exact Lagrangian graph is minimizing, then the orbits on it are minimizing, and segment of orbits close enough to it are also minimizing;
- if the dynamics on the torus is given by an ergodic translation, that is by a shift by an irrational vector of frequencies $\omega \in \mathbb{R}^d$, is reasonable to consider periodic orbits with rotation vectors close to $\omega$ as segments close to the initial object;
- although elliptic periodic orbits are not minimizing, small enough segments of them are minimizing.

### C.2.1 Area preserving maps

Given a 1-parametric family of monotone positive area preserving maps, $F_K$, being $F_0$ integrable, we wonder when a certain invariant curve of rotation number $\omega$ breaks down. In order to detect that critical value of the parameter, $K_\omega$, we propose next method:

1. Construction of a sequence of rationals $r_i = \frac{p_i}{n_i}$ tending to $\omega$. We shall choose the sequence of convergents of the continued fraction.

2. For any rational $r_i$, consider a periodic orbit with such a rotation vector, but its corresponding segment must be minimizing. We recall that all the $F_0$-orbits are minimizing.

3. Then, we must detect when this segment stop being minimizing. We shall call this critical value $K_{r_i}$.

4. This sequence of critical parameters seems to converge to $K_\omega$.

We notice that given a periodic orbit, its extremal character as a finite segment of points depends on the first point that we choose, that is, the order in which we apply the MMS iteration. In the examples we shall use the symmetries of our maps. The idea is to choose a symmetry axis and the segments must be symmetric with respect to it. If our map is not reversible, we had to consider all the possible orders, but we think that the segments must distribute around a certain axis.

**Continued fractions.-** One can classify the real numbers from their continued-fraction expansions [50]. The continued fraction of a real number $\omega$ is the sequence $[a_0; a_1, a_2, \dots]$ of integers generated by

$$\begin{cases} \omega_0 = \omega, \\ a_i = [\omega_i], \ \omega_{i+1} = \frac{1}{\omega_i - a_i} \ (i \geq 0). \end{cases}$$

Note that $a_1, a_2, \dots$ are positive. We can also write

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}.$$

- The continued-fraction expansion of an irrational is infinite, while that for rationals always ends. *Convergents* of a continued fraction are the rationals obtained by truncating the expansion:

  $$\frac{p_i}{n_i} = [a_0; a_1, \dots, a_i],$$

  and the fraction is irreducible.

- The continued-fraction expansion is *strongly convergent*:

  $$\lim_{i \to \infty} |p_i - n_i \omega| = 0.$$

  In fact, the convergents are the *best approximants*.

- Irrationals are more difficult to approximate if their continued-fraction elements are small, because a large element $a_{i+1}$ leads to a small correction to $\frac{p_i}{n_i}$. Examples are given by the numbers of *constant type*, whose elements of the continued fraction are bounded by a certain constant. They satisfy a *Diophantine condition*

$$\exists C > 0, \tau \geq 1 \mid \forall \frac{p}{n} \in \mathbb{Q}\ |n\omega - p| > \frac{C}{n^\tau}$$

for $\tau = 1$. The set of numbers of constant type has measure zero.

For instance, the *quadratic irrationals* have eventually periodic continued fraction, as Lagrange showed. A more special subset is given by the *noble numbers*: these have $a_i = 1$ from a certain element $i_0$. Noble numbers are dense in the reals. The noblest of numbers is the *golden mean*

$$\gamma = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \ldots],$$

which satisfies $\gamma^2 - \gamma - 1 = 0$. Sometimes

$$\sigma = 1 + \sqrt{2} = [2; 2, 2, 2, 2, \ldots]$$

is referred to as the *silver mean*. We shall call *bronze mean* the number

$$\beta = 1 + \sqrt{3} = [2; 1, 2, 1, 2, \ldots].$$

## Examples

1) **The standard map.**

We shall use $S_{0,0}$-symmetric periodic orbits, with that symmetry axis 'in the middle'. This is important to check the extremal character of the segments. Although these orbits are not minimizing (they are elliptic if $K$ is , small enough and inversion hyperbolic if $K$ is bigger), segments small enough of them are minimizing. This is related with the original Greene method, when one looks for period doubling bifurcations.

For the golden mean $\omega = \gamma$ (in fact, $K_\gamma = K_{\gamma-1}$) we have three different scalings:

- the convergence of the critical values $K_{r_i}$ to $K_\gamma$ is linear, and the asymptotic constant, C, is near to $\gamma - 1$;
- the initial points of the periodic orbits, $(0, y_{r_i})$, also converge lineally;
- finally, the residues $R_{r_i}$ seem to converge linearly to 1, although this convergence is given in period three, since there are three kinds of periodic orbits (depending on the other symmetry line).

These scalings have been also observed when one apply the original Greene method [63] or when one apply more geometrical methods [82]. They let us to improve the critical values by means of Aitken's method. Next three tables show the results.

| $i$ | $r_i$ | $K$ | $K_A$ | $C_K$ | $K - K_\omega$ | $K_A - K_\omega$ |
|---|---|---|---|---|---|---|
| 1 | 1/1 | 2.000000000000 | | | 1.02836e+00 | |
| 2 | 1/2 | 1.732050807569 | | | 7.60415e-01 | |
| 3 | 2/3 | 1.663903016141 | 1.640659327892 | | 6.92268e-01 | 6.69024e-01 |
| 4 | 3/5 | 1.278480954794 | 1.746688368559 | 5.655679 | 3.06846e-01 | 7.75053e-01 |
| 5 | 5/8 | 1.127250509332 | 1.029592668131 | 0.392376 | 1.55615e-01 | 5.79573e-02 |
| 6 | 8/13 | 1.075780907462 | 1.049226201013 | 0.340339 | 1.04146e-01 | 7.75908e-02 |
| 7 | 13/21 | 1.032334115527 | 0.797051995603 | 0.844125 | 6.06987e-02 | -1.74583e-01 |
| 8 | 21/34 | 1.009090548804 | 0.982349105874 | 0.534989 | 3.74551e-02 | 1.07137e-02 |
| 9 | 34/55 | 0.994835283933 | 0.972226722281 | 0.613299 | 2.31999e-02 | 5.91319e-04 |
| 10 | 55/89 | 0.985871590140 | 0.970687478426 | 0.628799 | 1.42362e-02 | -9.47925e-04 |
| 11 | 89/144 | 0.980379745567 | 0.971692629592 | 0.612677 | 8.74434e-03 | 5.72263e-05 |
| 12 | 144/233 | 0.977038964758 | 0.971850454494 | 0.608317 | 5.40356e-03 | 2.15051e-04 |
| 13 | 233/377 | 0.974948899669 | 0.971456201847 | 0.625622 | 3.31350e-03 | -1.79201e-04 |
| 14 | 377/610 | 0.973674374704 | 0.971682548867 | 0.609802 | 2.03897e-03 | 4.71456e-05 |
| 15 | 610/987 | 0.972889152047 | 0.971629042277 | 0.616090 | 1.25375e-03 | -6.36097e-06 |
| 16 | 987/1597 | 0.972405600684 | 0.971630512280 | 0.615814 | 7.70197e-04 | -4.89096e-06 |
| 17 | 1597/2584 | 0.972108645999 | 0.971636064785 | 0.614112 | 4.73243e-04 | 6.61542e-07 |
| 18 | 2584/4181 | 0.971926245661 | 0.971635816847 | 0.614236 | 2.90842e-04 | 4.13604e-07 |
| 19 | 4181/6765 | 0.971814047318 | 0.971634729505 | 0.615121 | 1.78644e-04 | -6.73738e-07 |
| 20 | 6765/10946 | 0.971745157297 | 0.971635574777 | 0.614002 | 1.09754e-04 | 1.71534e-07 |
| 21 | 10946/17711 | 0.971702830019 | 0.971635382226 | 0.614418 | 6.74268e-05 | -2.10170e-08 |
| 22 | 17711/28657 | 0.971676822865 | 0.971635378812 | 0.614430 | 4.14196e-05 | -2.44313e-08 |

| $i$ | $r_i$ | $y$ | $y_A$ | $C_y$ | $y - y_\omega$ | $y_A - y_\omega$ |
|---|---|---|---|---|---|---|
| 1 | 1/1 | 1.000000000000 | | | 4.05079e-01 | |
| 2 | 1/2 | 0.500000000000 | | | -9.49209e-02 | |
| 3 | 2/3 | 0.610279927619 | 0.590351921002 | | 1.53590e-02 | -4.56899e-03 |
| 4 | 3/5 | 0.581063269311 | 0.587182509556 | -0.264932 | -1.38576e-02 | -7.73840e-03 |
| 5 | 5/8 | 0.594845134663 | 0.590427778056 | -0.471713 | -7.57729e-05 | -4.49313e-03 |
| 6 | 8/13 | 0.592092361130 | 0.592550657327 | -0.199739 | -2.82855e-03 | -2.37025e-03 |
| 7 | 13/21 | 0.594054673449 | 0.593238003349 | -0.712849 | -8.66234e-04 | -1.68290e-03 |
| 8 | 21/34 | 0.594115173004 | 0.594117097587 | 0.030831 | -8.05735e-04 | -8.03810e-04 |
| 9 | 34/55 | 0.594507294292 | 0.594043636187 | 6.481391 | -4.13613e-04 | -8.77271e-04 |
| 10 | 55/89 | 0.594638724351 | 0.594704986114 | 0.335177 | -2.82183e-04 | -2.15921e-04 |
| 11 | 89/144 | 0.594756486232 | 0.595771095607 | 0.896004 | -1.64421e-04 | 8.50188e-04 |
| 12 | 144/233 | 0.594816273367 | 0.594877929546 | 0.507695 | -1.04634e-04 | -4.29780e-05 |
| 13 | 233/377 | 0.594857679620 | 0.594950954677 | 0.692561 | -6.32279e-05 | 3.00471e-05 |
| 14 | 377/610 | 0.594881677207 | 0.594914757525 | 0.579564 | -3.92304e-05 | -6.15004e-06 |
| 15 | 610/987 | 0.594896883564 | 0.594923186289 | 0.633662 | -2.40240e-05 | 2.27872e-06 |
| 16 | 987/1597 | 0.594906114656 | 0.594920375623 | 0.607055 | -1.47929e-05 | -5.31943e-07 |
| 17 | 1597/2584 | 0.594911828476 | 0.594921110592 | 0.618975 | -9.07909e-06 | 2.03025e-07 |
| 18 | 2584/4181 | 0.594915324063 | 0.594920832565 | 0.611778 | -5.58350e-06 | -7.50010e-08 |
| 19 | 4181/6765 | 0.594917479081 | 0.594920943351 | 0.616496 | -3.42849e-06 | 3.57846e-08 |
| 20 | 6765/10946 | 0.594918800785 | 0.594920897118 | 0.613315 | -2.10678e-06 | -1.04488e-08 |
| 21 | 10946/17711 | 0.594919613378 | 0.594920910364 | 0.614808 | -1.29419e-06 | 2.79767e-09 |
| 22 | 17711/28657 | 0.594920112505 | 0.594920907250 | 0.614238 | -7.95062e-07 | -3.16119e-10 |

| $i$ | $r_i$ | $R$ | $R_A$ | $C_R$ | $R - R_\omega$ | $R_A - R_\omega$ |
|---|---|---|---|---|---|---|
| 1 | 1/1 | 0.500000000000 | | | -5.00000e-01 | |
| 2 | 1/2 | 0.750000000000 | | | -2.50000e-01 | |
| 3 | 2/3 | 1.320047970837 | | | 3.20048e-01 | |
| 4 | 3/5 | 0.972228738684 | | | -2.77714e-02 | |
| 5 | 5/8 | 0.858699778873 | | | -1.41300e-01 | |
| 6 | 8/13 | 0.960742536492 | | | -3.92576e-02 | |
| 7 | 13/21 | 0.940600884808 | 0.942586213879 | | -5.93993e-02 | -5.74139e-02 |
| 8 | 21/34 | 0.948323756052 | 1.369404736786 | | -5.16764e-02 | 3.69405e-01 |
| 9 | 34/55 | 0.977116217402 | 0.976402583304 | | -2.28839e-02 | -2.35976e-02 |
| 10 | 55/89 | 0.976648111123 | 0.957447505504 | -1.139730 | -2.33520e-02 | -4.25527e-02 |
| 11 | 89/144 | 0.983736924027 | 1.006870547826 | 0.395131 | -1.62632e-02 | 6.87039e-03 |
| 12 | 144/233 | 0.992231389262 | 1.173770346049 | 0.923138 | -7.76877e-03 | 1.73770e-01 |
| 13 | 233/377 | 0.992624663555 | 1.005342234928 | 0.443212 | -7.37550e-03 | 5.34207e-03 |
| 14 | 377/610 | 0.995203709820 | 1.000694609257 | 0.323800 | -4.79645e-03 | 6.94449e-04 |
| 15 | 610/987 | 0.997659897828 | 1.000702091448 | 0.359143 | -2.34026e-03 | 7.01931e-04 |
| 16 | 987/1597 | 0.997823207165 | 1.000330614248 | 0.325386 | -2.17695e-03 | 3.30454e-04 |
| 17 | 1597/2584 | 0.998610585705 | 1.000050651837 | 0.297108 | -1.38957e-03 | 5.04915e-05 |
| 18 | 2584/4181 | 0.999318545862 | 1.000048311134 | 0.305544 | -6.81615e-04 | 4.81508e-05 |
| 19 | 4181/6765 | 0.999369475891 | 1.000024121845 | 0.297443 | -6.30684e-04 | 2.39615e-05 |
| 20 | 6765/10946 | 0.999599789674 | 1.000004528004 | 0.290355 | -4.00371e-04 | 4.36763e-06 |
| 21 | 10946/17711 | 0.999803394533 | 1.000003665756 | 0.292316 | -1.96766e-04 | 3.50538e-06 |
| 22 | 17711/28657 | 0.999818377881 | 1.000002011109 | 0.290313 | -1.81782e-04 | 1.85073e-06 |

For the errors, we have compared with the values obtained superconverging the results until an orbit of rotation number 75025/121393. They are:

- $K_\gamma \simeq 0.971635403243$,
- $y_\gamma \simeq 0.594920907566$,
- $R_\gamma \simeq 1.000000160378$.

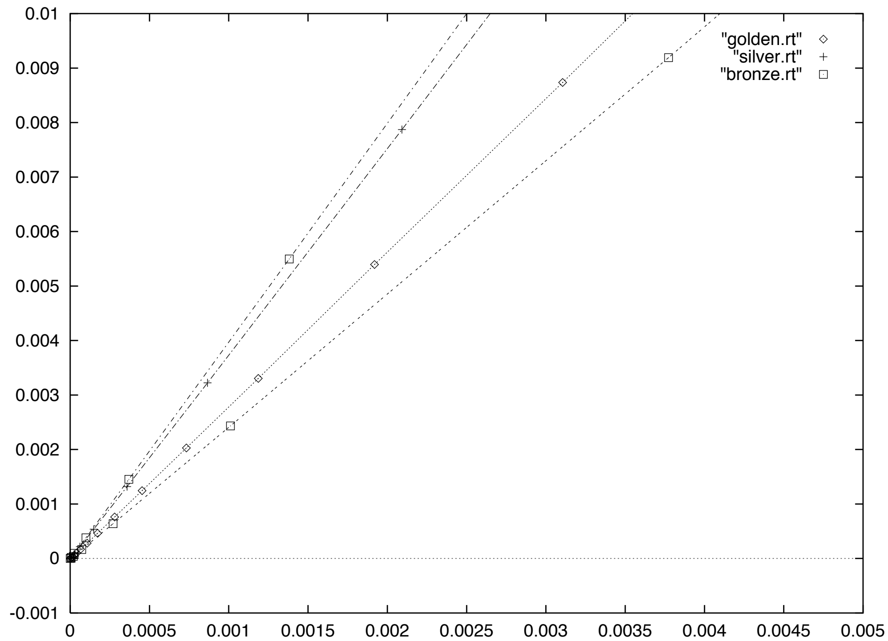
We have also computed the critical values for the silver mean and the bronze mean. As the bronze mean has continued-fraction expansion of period 2, this is also the period in the scaling behaviour (as in [82]). The three critical values are

$$\begin{aligned} K_\gamma &\simeq 0.971635403243, \\ K_\sigma &\simeq 0.957445407625, \\ K_\beta &\simeq 0.876067425540, \end{aligned}$$

and we note that they are in the correct positions!

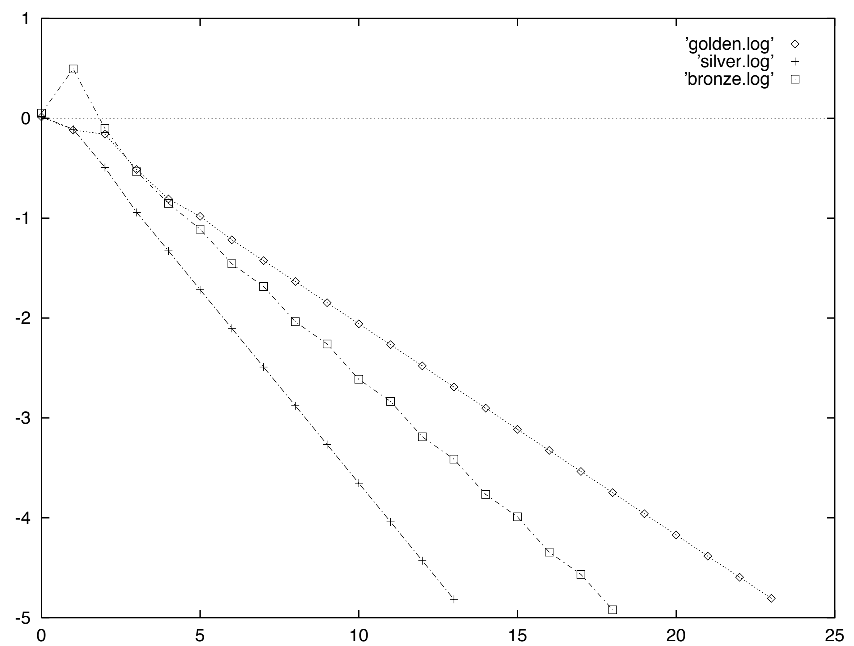
The results for these three numbers are summarized in the following two figures.

1. The first one shows the reduced error in the rational approximation, $|n_i\omega - p_i|$, versus the error in the estimate of the critical value, $K_{r_i} - K_\omega$, for the different convergents $r_i = \frac{p_i}{q_i}$ of the continued fraction.




$x = |n_i\omega - p_i|, y = K_\gamma - K_{r_i}$

2. Next figure shows the number of convergent, $i$, in front of the decimal logarithm of the error in the corresponding estimate of the critical value. For the three numbers the orders of the denominators of the last considered convergents are about $10^5$, and the corresponding errors in the estimates are also of the same order.




$x = i, y = \log_{10}(K_\gamma - K_{r_i})$

We have also considered a 'lead' number

$$\begin{aligned} \lambda &= [0; 1, 1, 1, 3, 1, 2, 2, 1, 3, 3, 2, 1, 1, 3, 2, 1, 1, 1, 2, \ldots] \\ &\simeq 0.6412359518762 \end{aligned}$$

which does not show scaling behaviour. The critical values estimated is $K_\lambda \simeq 0.912277185668$, obtained from a periodic orbit of period 72786, corresponding to the fifteenth convergent. The difference with the previous one is about $2 \cdot 10^{-5}$. In this case we can not apply the scaling behaviour in order to improve the results. Anyway, it is better than the bronze mean!

2) **The exponential standard map.**

In this example we have taken $\omega = \gamma$. The convergence to the critical value is also linear. The values obtained superconverging the results until an orbit of rotation number 28657/17711 are:

- $K_\gamma \simeq 0.608047936956$,
- $y_\gamma \simeq 0.466309789723$,
- $R_\gamma \simeq 1.000003497428$.

Next table shows the sequence of critical values.

| $i$ | $r_i$ | $K$ | $K_A$ | $C_K$ | $K - K_\omega$ | $K_A - K_\omega$ |
|---|---|---|---|---|---|---|
| 1 | 1/1 | 2.000000000000 | | | 1.39195e+00 | |
| 2 | 2/1 | 1.000000000000 | | | 3.91952e-01 | |
| 3 | 3/2 | 1.154700538379 | 1.133974596216 | | 5.46653e-01 | 5.25927e-01 |
| 4 | 5/3 | 1.031827843765 | 1.086219612449 | -0.794262 | 4.23780e-01 | 4.78172e-01 |
| 5 | 8/5 | 0.803328379967 | 1.297634940319 | 1.859644 | 1.95280e-01 | 6.89587e-01 |
| 6 | 13/8 | 0.702772533761 | 0.623741798218 | 0.440070 | 9.47246e-02 | 1.56939e-02 |
| 7 | 21/13 | 0.674184301968 | 0.662827985606 | 0.284302 | 6.61364e-02 | 5.47800e-02 |
| 8 | 34/21 | 0.645609371674 | -60.740449985072 | 0.999535 | 3.75614e-02 | -6.13485e+01 |
| 9 | 55/34 | 0.631557757817 | 0.617962526120 | 0.491746 | 2.35098e-02 | 9.91459e-03 |
| 10 | 89/55 | 0.622540884086 | 0.606392282597 | 0.641697 | 1.44929e-02 | -1.65565e-03 |
| 11 | 144/89 | 0.616959861175 | 0.607894328833 | 0.618953 | 8.91192e-03 | -1.53608e-04 |
| 12 | 233/144 | 0.613511686119 | 0.607937021622 | 0.617839 | 5.46375e-03 | -1.10915e-04 |
| 13 | 377/233 | 0.611431541138 | 0.608268596679 | 0.603260 | 3.38360e-03 | 2.20660e-04 |
| 14 | 610/377 | 0.610119695476 | 0.607879762423 | 0.630651 | 2.07176e-03 | -1.68175e-04 |
| 15 | 987/610 | 0.609323729415 | 0.608095609584 | 0.606753 | 1.27579e-03 | 4.76726e-05 |
| 16 | 1597/987 | 0.608832310575 | 0.608039354799 | 0.617387 | 7.84374e-04 | -8.58216e-06 |
| 17 | 2584/1597 | 0.608529815657 | 0.608045476912 | 0.615554 | 4.81879e-04 | -2.46004e-06 |
| 18 | 4181/2584 | 0.608344011410 | 0.608048158985 | 0.614239 | 2.96074e-04 | 2.22028e-07 |
| 19 | 6765/4181 | 0.608229930244 | 0.608048475233 | 0.613986 | 1.81993e-04 | 5.38276e-07 |
| 20 | 10946/6765 | 0.608159721789 | 0.608047368869 | 0.615425 | 1.11785e-04 | -5.68087e-07 |
| 21 | 17711/10946 | 0.608116627244 | 0.608048133256 | 0.613808 | 6.86903e-05 | 1.96300e-07 |
| 22 | 28657/17711 | 0.608090146221 | 0.608047936956 | 0.614487 | 4.22093e-05 | 0.00000e+00 |

3) **A $C^1$ example.**

We consider now a standard-like map with potential in the interval $[-\frac{1}{2}, \frac{1}{2}]$ given by

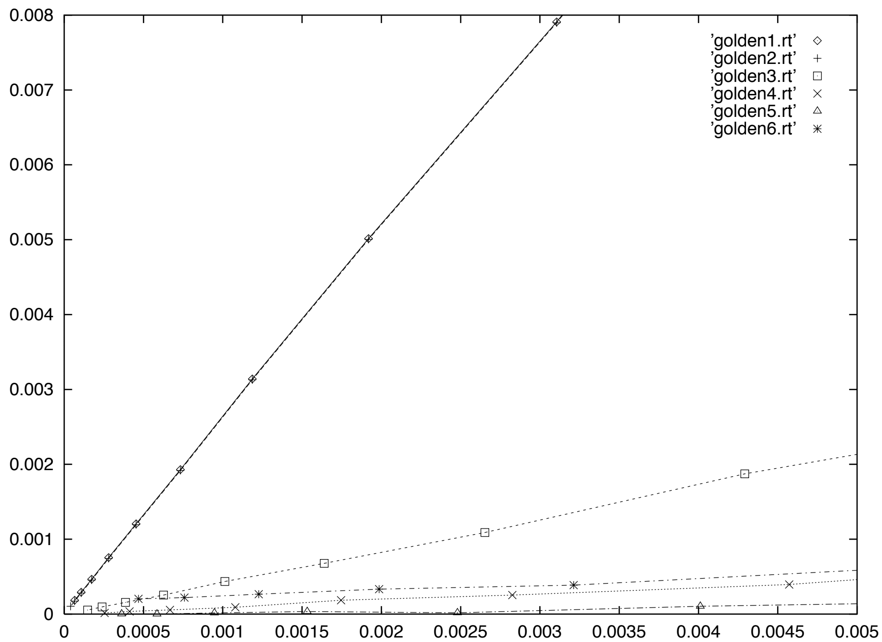
$$V(x) = -K(x^4 - \frac{1}{2}x^2 + \frac{1}{32}),$$

and extending by periodicity to the rest of $\mathbb{R}$. This potential is $C^2$ and our standard-like map is $C^1$.

We have performed the next experiment. We consider the golden mean $\omega = \gamma - 1$. For each value $k \in \mathbb{N}^*$, we approximate $\gamma$ by the sequence of rationals given by

$$[0; k], [0; 1, k], [0; 1, 1, k], [0; 1, 1, 1, k], \ldots .$$

The critical value is $K_\gamma \simeq 1.1254531070$. Next figure shows the reduced error of each rational approximation versus the error in the estimation of $K_\gamma$, for $k = 1, 2, 3, 4, 5, 6$. Several velocities are shown.




$x = |n_i\omega - p_i|, y = K_\gamma - K_{r_i}$ for different rational sequences approximating $\gamma - 1$

◁

### C.2.2 Higher dimensions

As we have recalled, approximation of irrational numbers by rationals has been very important in the study of breakdown of invariant curves in twist maps (and not only for twist maps) since the work of Greene [36]. MacKay [63] explained the phenomenon in terms of a renormalization group operator that changes the rotation number of an invariant curve by eliminating the first continued fraction coefficient. The question is if we can extend these ideas to higher dimensions.

First of all, we need a method to approximate an irrational vector by rational ones. There are some possible generalizations of the continued fraction algorithm in higher dimensions, as the Kim-Ostlund tree [51] (which generalizes the Farey-tree approximation squeme [38]) or the Jacobi-Perron algorithm [18, 87, 54, 57]. We have used the second one, and we have followed the description given by Tompaidis [95].

**The Jacobi-Perron algorithm.-** Given a point $\omega = (\omega_1, \omega_2) \in ]0,1[^2$, the Jacobi-Perron convergents $\frac{p_i}{n_i}$, with $p_i = (p_i^1, p_i^2) \in \mathbb{N}^2)$ and $n_i \in \mathbb{N}$, are recursively given by

$$\begin{pmatrix} p_{i+1} \\ n_{i+1} \end{pmatrix} = k_{i+1} \begin{pmatrix} p_i \\ n_i \end{pmatrix} + l_{i+1} \begin{pmatrix} p_{i-1} \\ n_{i-1} \end{pmatrix} + \begin{pmatrix} p_{i-2} \\ n_{i-2} \end{pmatrix}$$

where the integer coefficients $k_{i+1}, l_{i+1}$ are determined by [1]

$$(k_{i+1}, l_{i+1}) = ([\frac{1}{\omega_2^i}], [\frac{\omega_1^i}{\omega_2^i}])$$

and

$$(\omega_1^{i+1}, \omega_2^{i+1}) = (\{\frac{1}{\omega_2^i}\}, \{\frac{\omega_1^i}{\omega_2^i}\}).$$

The initial values are

$$\begin{aligned} &(\omega_1^0, \omega_2^0) = (\omega_1, \omega_2), \\ &p_{-2} = (0,1), \; p_{-1} = (1,0), \; p_0 = (0,0), \\ &n_{-2} = n_{-1} = 0, \; n_0 = 1. \end{aligned}$$

For all points in the unit square, $\forall i \geq 0, k_i \geq 1, k_i \geq l_i \geq 0$.

From a geometrical point of view, given a triangle determined by three successive approximants, all approximants of higher order will lie inside that triangle [57] [2]. Moreover, such a triangle does not contain rational points with denominator smaller than the largest denominator of the vertices.

We can measure the goodness of a rational approximation by means of the *reduced error* and the *Roth exponent*. Given a point in the unit square $\omega$ and a rational point $r = (\frac{p_1}{n}, \frac{p_2}{n})$,

- its reduced error is

$$\epsilon(r, \omega) = \|n\omega - p\|_2,$$

- and its Roth exponent is

$$\eta(r, \omega) = 1 - \frac{\log \epsilon(r, \omega)}{\log n}.$$

Measure-theoretical properties of the Jacobi-Perron algorithm were studied by Lagarias [57]. Let $r_i$ be the $i^{\text{th}}$ Jacobi-Perron approximant to our point $\omega$. Then, the *best approximation exponent* for $\omega$ (using the Jacobi-Perron scheme) is

$$\eta_b(\omega) = \limsup_{i \to \infty} \eta(r_i, \omega)$$

and the *uniform approximation exponent* is defined as

$$\eta_u(\omega) = \liminf_{i \to \infty} \min(\eta(r_i, \omega), \eta(r_{i+1}, \omega), \eta(r_{i+2}, \omega)).$$

[1] while $[\cdot]$ is the integral part, $\{\cdot\}$ is the mantissa.

[2] This property is also satisfied by another commonly used algorithm, the Farey-tree approximation scheme [51].

Lagarias showed that these exponents are constant in a set of measure one in the unit square. He also conjectured that these constants are in fact the same. Estimates of the best approximation exponent were computed numerically by Baldwin [15] and Kosygin [54], founding $\eta_b = 1.374 \pm .002$. If the Lagarias's conjecture were true, the successive triangles in the Jacobi-Perron scheme become, in the limit, needle-shaped.

Consider, for instance, a quadratic pair $\omega = (\sqrt{2}-1, \sqrt{3}-1)$. This is a good pair (or bad pair, depending on the point of view), because the two numbers are algebraic and lineally independent. Then, the rational approximation is difficult. The first convergents of the Jacobi-Perron algorithm are written in the next table:

| $i$ | $r_i$ | $\epsilon(r_i,\omega)$ | $\eta(r_i,\omega)$ |
|---|---|---|---|
| 0 | $(0,0)/1$ | $8.41113e-01$ | - |
| 1 | $(0,1)/1$ | $4.93325e-01$ | - |
| 2 | $(1,1)/1$ | $6.44160e-01$ | - |
| 3 | $(1,2)/3$ | $3.12010e-01$ | 2.06017 |
| 4 | $(7,13)/18$ | $4.88971e-01$ | 1.24753 |
| 5 | $(8,14)/19$ | $1.58658e-01$ | 1.62525 |
| 6 | $(17,30)/41$ | $2.22640e-02$ | 2.02456 |
| 7 | $(108,191)/261$ | $1.27678e-01$ | 1.36989 |
| 8 | $(116,205)/280$ | $3.27481e-02$ | 1.60675 |
| 9 | $(133,235)/321$ | $3.92290e-02$ | 1.56110 |
| 10 | $(490,866)/1183$ | $2.17678e-02$ | 1.54090 |
| 11 | $(606,1071)/1463$ | $1.11523e-02$ | 1.61690 |
| 12 | $(7431,13133)/17940$ | $1.21652e-02$ | 1.45016 |
| 13 | $(7921,13999)/19123$ | $9.64868e-03$ | 1.47075 |
| 14 | $(23879,42202)/57649$ | $3.80201e-03$ | 1.50832 |
| 15 | $(10898028,19260379)/26310167$ | $4.81910e-04$ | 1.44703 |
| 16 | $(32749763,57879540)/79064922$ | $5.98948e-04$ | 1.40803 |

Finally, we can also introduce golden-means of the Jacobi-Perron algorithm. They are those points $\omega$ such that the integer coefficients associated are constant: $k_n = k, l_n = l$. We shall write $\omega = (k,l)^\infty$. Then, we define the characteristic polynomial of $\omega$ as $P_\omega(t) = t^3 - kt^2 - lt - 1$ and then:

- $k < \tau < k+1,\ 0 < |\tau_1|, |\tau_2| < 1$, where $\tau$ is the maximal absolute root of $P_\omega$ and $\tau_1, \tau_2$ the remaining roots;
- $(\omega_1, \omega_2) = (\tau - k, \frac{1}{\tau})$;
- $\|n_i\omega - (p_i^1, p_i^2)\|_2 \le C(\omega)\kappa^i$, where $\kappa = \max(|\tau_1|, |\tau_2|) < 1$.

For instance, $(1,1)^\infty = (\tau - 1, \frac{1}{\tau})$, where

$$\begin{aligned}\tau &= \sqrt[3]{\frac{19}{27} + \sqrt{\frac{11}{27}}} + \sqrt[3]{\frac{19}{27} - \sqrt{\frac{11}{27}}} + \frac{1}{3} \\ &\simeq 1.839286755.\end{aligned}$$

We shall refer to it as the *golden vector*. Its first convergents are

| $i$ | $r_i$ | $\epsilon(r_i,\omega)$ | $\eta(r_i,\omega)$ |
|---|---|---|---|
| 0 | (0,0)/1 | 1.00000e+00 | - |
| 1 | (1,1)/1 | 4.83786e-01 | - |
| 2 | (2,1)/2 | 3.33091e-01 | 2.58601 |
| 3 | (3,2)/4 | 3.97610e-01 | 1.66529 |
| 4 | (6,4)/7 | 2.30928e-01 | 1.75319 |
| 5 | (11,7)/13 | 1.12195e-01 | 1.85285 |
| 6 | (20,13)/24 | 1.50901e-01 | 1.59506 |
| 7 | (37,24)/44 | 1.05500e-01 | 1.59433 |
| 8 | (68,44)/81 | 4.26860e-02 | 1.71770 |
| 9 | (125,81)/149 | 5.45886e-02 | 1.58113 |
| 10 | (230,149)/274 | 4.59181e-02 | 1.54887 |
| 11 | (423,274)/504 | 1.92695e-02 | 1.63466 |
| 12 | (778,504)/927 | 1.88242e-02 | 1.58147 |
| 13 | (1431,927)/1705 | 1.90621e-02 | 1.53217 |
| 14 | (2632,1705)/3136 | 9.33328e-03 | 1.58059 |
| 15 | (4841,3136)/5768 | 6.26089e-03 | 1.58584 |
| 16 | (8904,5768)/10609 | 7.55551e-03 | 1.52705 |
| 17 | (16377,10609)/19513 | 4.44399e-03 | 1.54826 |
| 18 | (30122,19513)/35890 | 2.11794e-03 | 1.58707 |
| 19 | (55403,35890)/66012 | 2.85827e-03 | 1.52782 |
| 20 | (101902,66012)/121415 | 2.02360e-03 | 1.52984 |
| 21 | (187427,121415)/223317 | 8.15577e-04 | 1.57741 |
| 22 | (344732,223317)/410744 | 1.03057e-03 | 1.53209 |

**A 'false' 4D example.-** Consider the Froeschlé map with parameters $K_1 = K_2 = 0$. Then it depends on the parameter $\lambda$ and it is given by

$$\begin{cases} y'_1 = y_1 - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)), & x'_1 = x_1 + y'_1 \pmod 1, \\ y'_2 = y_2 - \dfrac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)), & x'_2 = x_2 + y'_2 \pmod 1. \end{cases}$$

We shall work on the universal covering space $\mathbb{R}^2 \times \mathbb{R}^2$. On it, we perform the change of variables

$$\begin{cases} u_1 = x_1 - x_2, & v_1 = y_1 - y_2, \\ u_2 = x_1 + x_2, & v_2 = y_1 + y_2, \end{cases}$$

and our symplectomorphism is decomposed in the product of an integrable one and the standard map, with parameter $2\lambda$.

$$\begin{cases} v'_1 = v_1, & u'_1 = u_1 + v'_1, \\ v'_2 = v_2 - \dfrac{2\lambda}{2\pi}\sin(2\pi u_2)), & u'_2 = u_2 + v'_2. \end{cases}$$

We can project now into the cylinder, making $u_1 \pmod 1, u_2 \pmod 1$. Then, an orbit with rotation vector $\omega = (\omega_1, \omega_2)$ becomes another one with rotation vector $\bar\omega = (\omega_1 - \omega_2, \omega_1 + \omega_2)$. We must point out that this is seen on $\mathbb{R}^2 \times \mathbb{R}^2$ and not on $\mathbb{T}^2 \times \mathbb{R}^2$, because such a change does not define a diffeomorphism on the last manifold. Then, as the rotational part does not influence, then we obtain that

$$\lambda_{(\omega_1,\omega_2)} = \frac{1}{2} K_{\omega_1+\omega_2}.$$

For instance, if we take our quadratic pair $\omega = (\sqrt{2}-1, \sqrt{3}-1)$ we obtain that

$$\lambda_{(\sqrt{2}-1,\sqrt{3}-1)} \simeq 0.308287753633196.$$

We have arrived to this result applying our variational criterion to an orbit of rotation number 154850/135091, the tenth convergent of $\sqrt{2}+\sqrt{3}-2$. The previous one, 3989/3480, agrees with that in the first three figures.

**Remark**

If we had chosen the change of variables

$$\begin{cases} u_1 = \frac{x_1-x_2}{2}, & v_1 = \frac{y_1-y_2}{2}, \\ u_2 = \frac{x_1+x_2}{2}, & v_2 = \frac{y_1+y_2}{2}, \end{cases}$$

we would have arrived to the product of an integrable map and a standard-like map with potential

$$V(u_2) = -\frac{\lambda}{(2\pi)^2}\cos(4\pi u_2),$$

and we would have had to compute the critical value corresponding to $\frac{1}{2}(\omega_1+\omega_2)$.

For instance, if we use the previous example, then the critical value corresponding to the twelfth convergent of $\frac{1}{2}(\sqrt{2}+\sqrt{3}-2)$, (77425, 135091), agrees with that in 14 figures. ◁

**A numerical study of the breakdown of invariant tori.-** We shall consider the Froeschlé map (but we can choose any of its family). We want to study the breakdown of an invariant torus with frequencies $(\omega_1, \omega_2)$, which exists in the uncoupled case (we shall select the values of the $K$ parameters less than the corresponding critical values). We shall follow the next steps.

1. Construction of a sequence of rational rotation vectors tending to the selected one. We shall use the Jacobi-Perron algorithm, but one could also use other 'good' rational vectors (with 'big' Roth exponent and 'small' reduced error). We have chosen the Jacobi-Perron algorithm because it has good scaling properties, given by the convergence of the Roth exponents of the approximates.

2. For each rational rotation vectors:

   - we compute a periodic orbit for the product of standard maps ($\lambda = 0$),
   - and it is continued with respect to the coupling parameter, $\lambda$.

   In both cases we use the symmetries of the maps (they are reversible). We have used periodic orbits which are symmetric with respect to $\{x_1 = 0, x_2 = 0\}$ (the minimum of the potential). Those periodic orbits are elliptic.

   Recall that Olvera and Vargas [83] observed certain bifurcation phenomena when they continued periodic orbits of rotation vectors with at least one reducible component.

3. For $\lambda = 0$ the orbit must be minimizing. We detect the critical parameter $\lambda_\omega$ when the orbit stop being minimizing. We shall perform a table with such critical values, showing also the residues $(R_1, R_2)$ of the critical periodic orbit. Note that in all the examples seem to have a period doubling bifurcation, because one of the residues tend to 1. It is curious that this has been the case in all the examples we have studied. Another possibility would be a Krein crunch, in which two pair of elliptic eigenvalues collapse and transform in a complex hyperbolic quadruplet.

4. In order to 'see' the breakdown of the invariant torus, we shall draw several periodic orbits, when $\lambda$ increases. We shall obtain different kinds of metamorphoses, depending on how far of the breakdown we are for $\lambda = 0$. In the pictures, we show the angular components of the orbits in the square (in fact, the torus) $[0,1]\times[0,1]$, that is, we shall see the projections of such orbits on the zero-section. When we increase the parameter $\lambda$ the residues of the orbits increase very fast, being the continuation more difficult.

5. We must compare the drawings corresponding to the same value of the parameter $\lambda$ (and different rational rotation vectors), in order to discover to which kind of object the sequence of periodic orbits tends. We would like to know how are the higher dimensional *Aubry-Mather sets*, named *cantorus* by Percival. We distinguish two kinds of breakdown: a $I \times \mathcal{C}$ type and a $\mathcal{C} \times \mathcal{C}$ type. Here, $I$ means an interval (a 1 dimensional set) and $\mathcal{C}$ means a Cantor set (possibly with dimension 0). Sometimes we shall see certain resonances, associated to periodic orbits with rotation vectors near to our irrational rotation vector.

### Examples

1) **A quadratic pair**

First, we have applied the previous scheme to the torus with frequencies $\omega = (\sqrt{2}-1, \sqrt{3}-1)$. We have considered the next three cases:

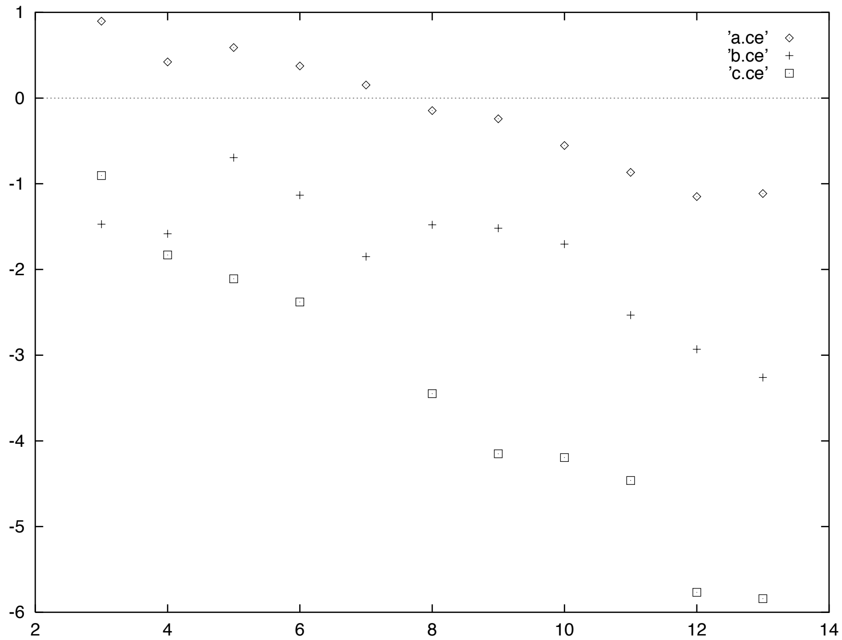
a) First, the values $K_1 = 0.1$ and $K_2 = 0.2$ have been taken. When we increase $\lambda$ a *resonance phenomenon* is observed, and the reason for that will be analyzed. The critical value has been estimated $\lambda_\omega \simeq 0.030$.

b) Second, we have taken small values for $K_1$ and $K_2$, for instance $K_1 = K_2 = 10^{-5}$. It is known that for $K_1 = K_2 = 0$ the torus breaks for $\lambda \approx 0.308$, and it must be a $I \times \mathcal{C}$ breakdown, since the map is the product of a rotation and a standard map. For small values of $K_1$ and $K_2$ the breakdown must be of the same type, and this agrees with the experimental results displayed in the figures. For the same reason, some strips of approximate slope $-1$ show up. The critical value is $\lambda_\omega \simeq 0.303$, quite far from 0.308. Then, the coupling influences strongly on our map.

c) Finally, we have taken $K_1 = 0.94$ and $K_2 = 0.86$, that is, close to the corresponding critical values. For larger values the torus would break in the form $\mathcal{C} \times \mathcal{C}$ (well, this is not true). A similar breakdown must be expected. The experimental results displayed in the figures confirm this expectation.

Moreover, the critical value is very small, $\lambda_\omega \simeq 0.000017$. Possibly there is a very short transition between a $I \times \mathcal{C}$ cantorus and a $\mathcal{C} \times \mathcal{C}$ cantorus.

As a summary, the next figure shows the decimal logarithm of the relative error in the estimate of $\lambda_\omega$, $e$, versus the number of convergent. For the error, we have compared with the value obtained in the last convergent considered, with has denominator equal to 57649.




$x = i, y = \log_{10} e_{r_i}$

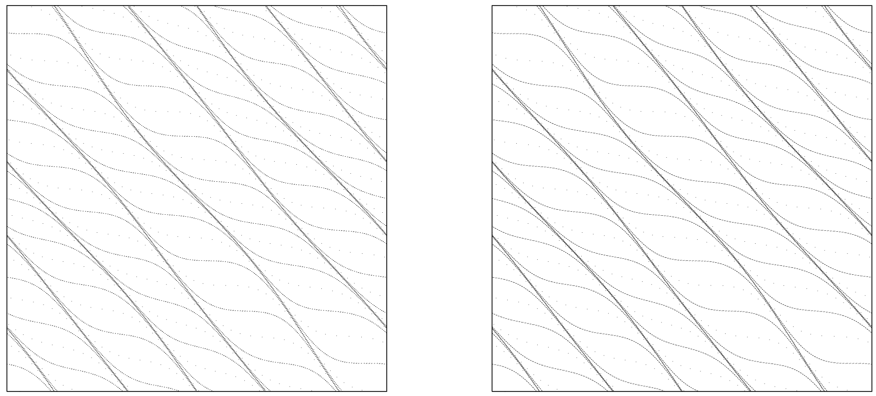
a) $K_1 = 0.1, K_2 = 0.2$

The table of results is:

| $i$ | $r_i$ | $\lambda$ | $R_1$ | $R_2$ |
|---|---|---|---|---|
| 3 | (1,2)/3 | 0.272976557421 | 0.825718606803 | 0.001363062671 |
| 4 | (7,13)/18 | 0.111785385270 | 0.960372322656 | 0.010533041999 |
| 5 | (8,14)/19 | 0.150247552464 | 0.882801627681 | 0.031086109893 |
| 6 | (17,30)/41 | 0.103536552368 | 0.959031282283 | 0.013623857907 |
| 7 | (108,191)/261 | 0.074380323309 | 0.997121454610 | -0.003454697886 |
| 8 | (116,205)/280 | 0.052630030576 | 0.999824785516 | -0.000000000000 |
| 9 | (133,235)/321 | 0.048332068032 | 0.999950580298 | -0.000000000000 |
| 10 | (490,866)/1183 | 0.039307855107 | 0.995169228849 | 0.000000000214 |
| 11 | (606,1071)/1463 | 0.034883517948 | 0.997998791594 | 0.000000000000 |
| 12 | (7431,13133)/17940 | 0.032887285817 | 0.999738839558 | 0.000000000000 |
| 13 | (7921,13999)/19123 | 0.033070341918 | 0.999991850955 | 0.000000000001 |
| 14 | (23879,42202)/57649 | 0.030712699108 | 0.999999999996 | -0.000000000001 |

First, we have drawn the two orbits of periods 17940 and 19123 for a value $\lambda = 0.05$, rather far of the critical value.

$K_1 = 0.1$, $K_2 = 0.2$ | $\frac{(7431,13133)}{17940}$ $(S_{0,(0,0)})$ | $\frac{(7921,13999)}{19123}$ $(S_{0,(0,0)})$

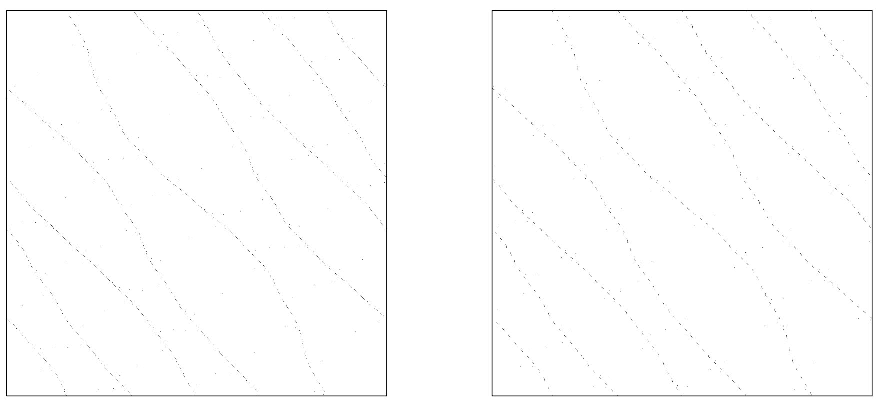


$\lambda = 0.05$

Where are the 1183 points that we have added to the first figure to obtain the second one?

Secondly, we show a pair of periodic orbits of period 1463, when $\lambda$ is bigger.

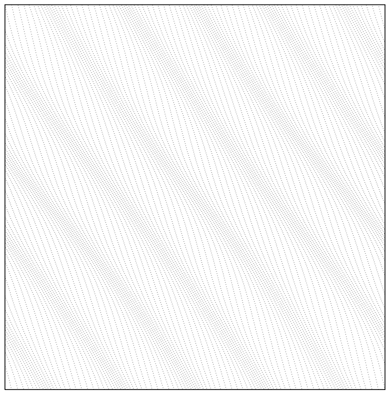
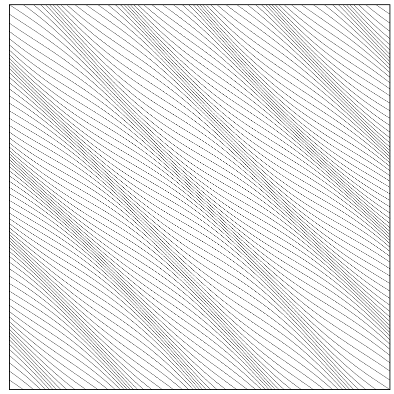
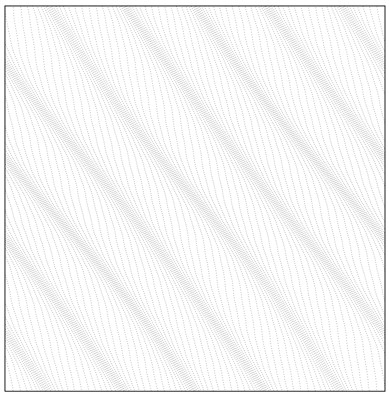
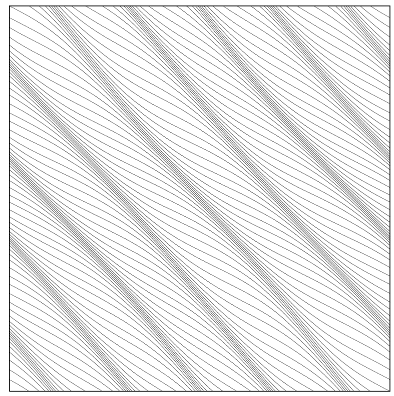
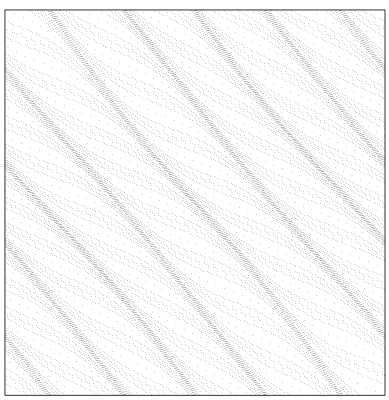
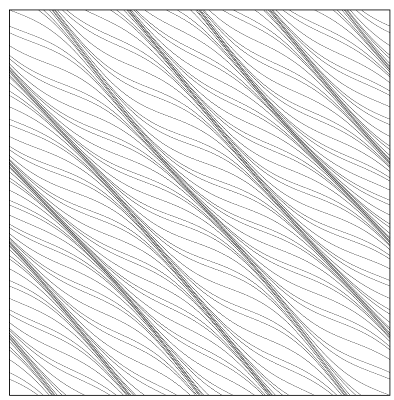
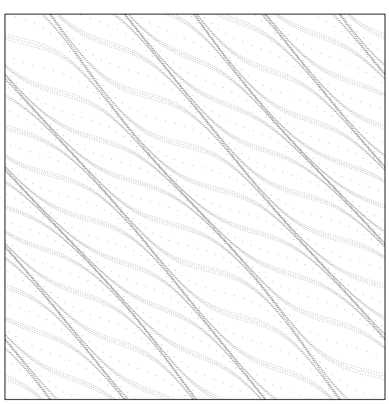
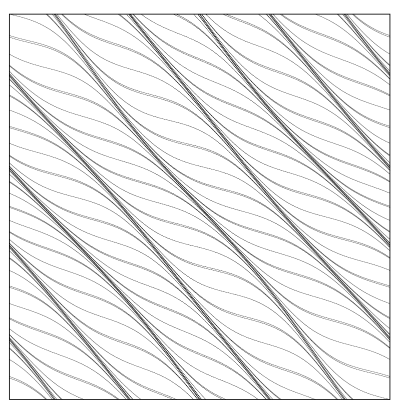
$K_1 = 0.1$, $K_2 = 0.2$ | $\frac{(606,1071)}{1463}$ $(S_{0,(0,0)})$ | $\frac{(606,1071)}{1463}$ $(S_{0,(1,1)})$



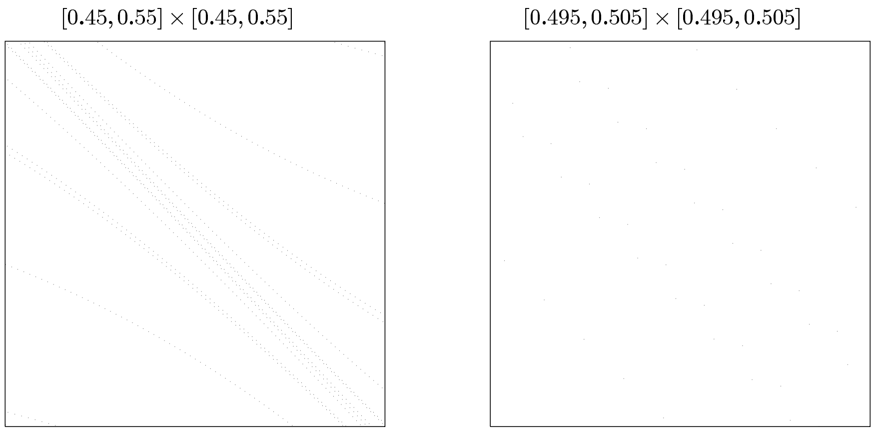
$\lambda = 0.1$

The torus is completely destroyed

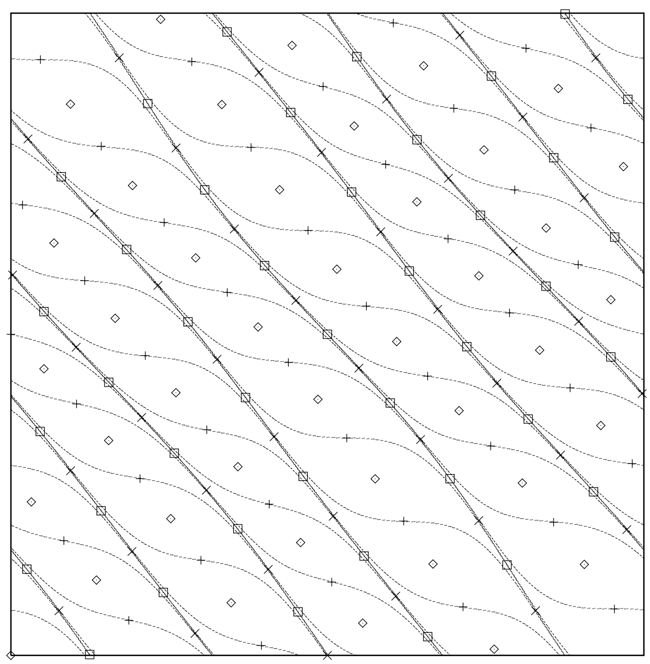
The transition in the breakdown of the torus is shown in the next page with orbits of period 19123 and 57649.

| $K_1 = 0.1$, $K_2 = 0.2$ | $\frac{(7921,13999)}{19123}$ $(S_{0,(0,0)})$ | $\frac{(23879,42202)}{57649}$ $(S_{0,(0,0)})$ |
|---|---|---|
| $\lambda = 0.025$ |  |  |
| $\lambda = 0.030$ |  |  |
| $\lambda = 0.035$ |  |  |
| $\lambda = 0.040$ |  |  |

Next figures show two details of the $S_{0,(0,0)}$-periodic orbit of period 57649, for the value $\lambda = 0.04$. We have zoomed in twice the center of that figure.




Finally, if we count the number of 'holes' in the figures, we obtain 41. This number is the denominator of one of the convergents of our rotation vector. These holes must correspond to the resonances associated to the elliptic orbit of period 41 ($\diamond$).The boundaries of these resonances must be given by the center-stable and center-unstable manifolds of the two elliptic-hyperbolic orbits of period 41 ($+$ and $\times$). We also have a hyperbolic orbit ($\square$). Next figure is taken for $\lambda = 0.05$.



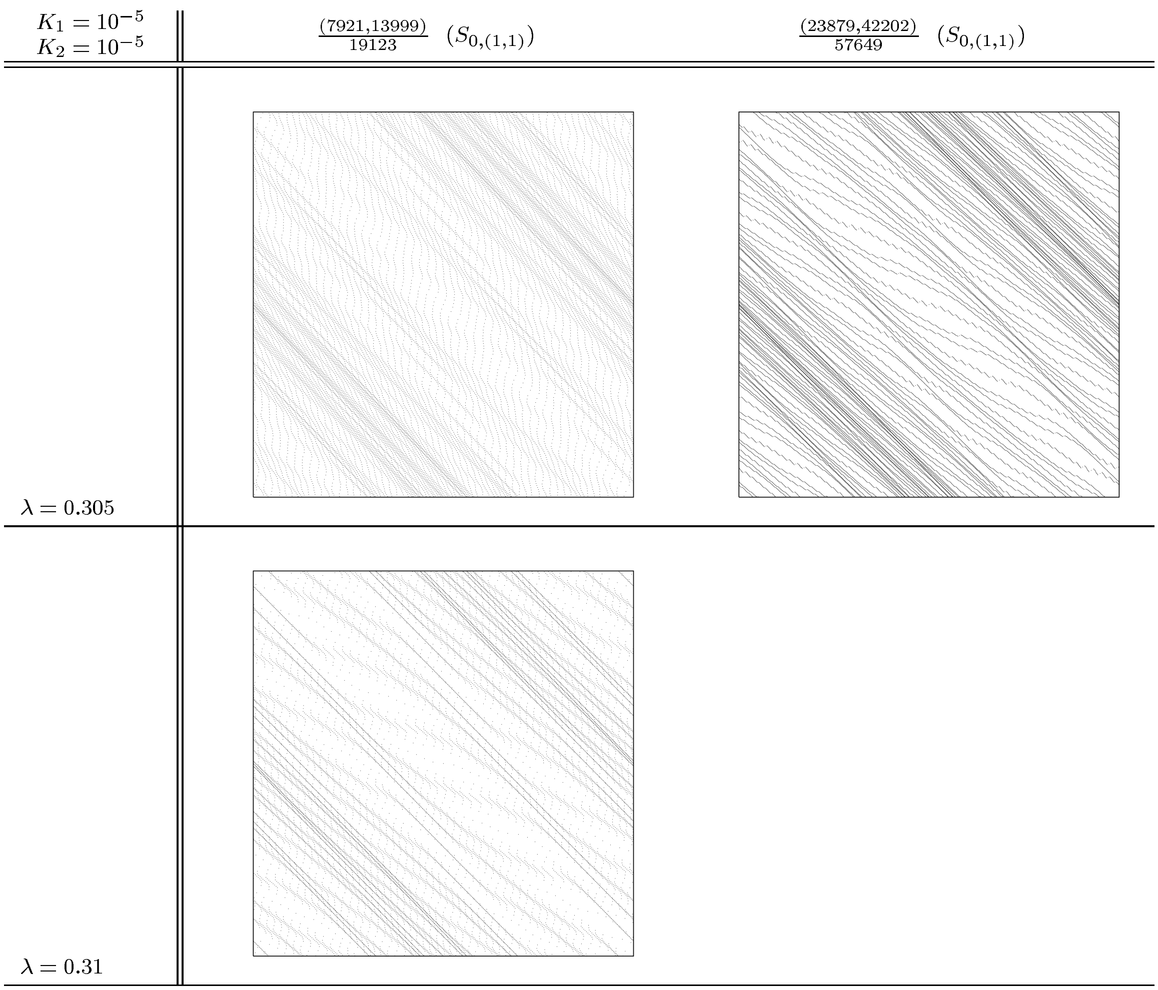
The $S_{0,(1,1)}$-periodic orbit of period 19123 and his 4 companions of period 41 ($\lambda = 0.05$)

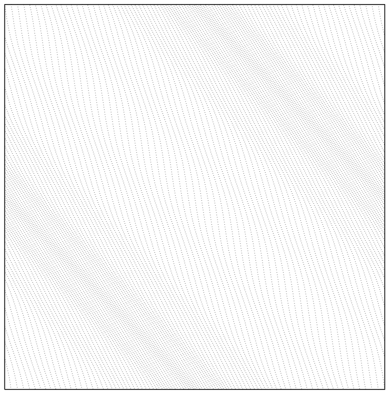
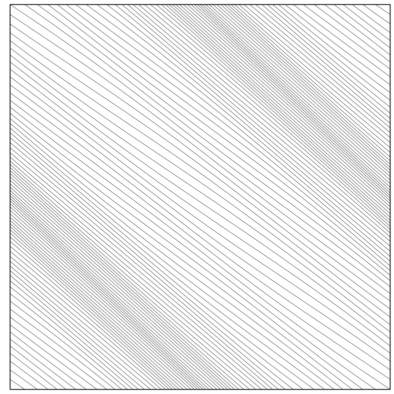
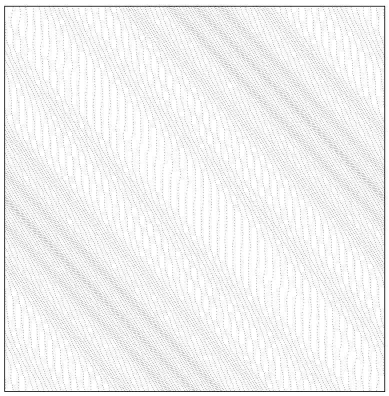
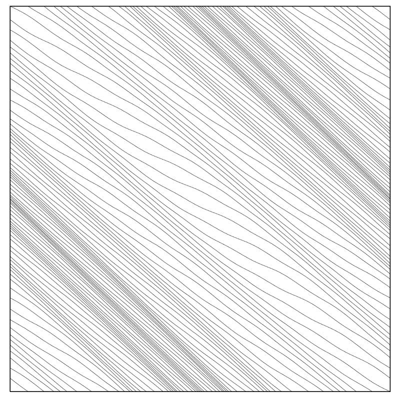
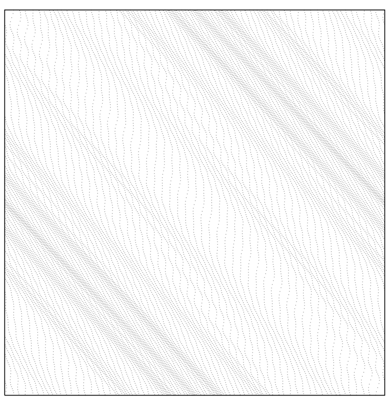
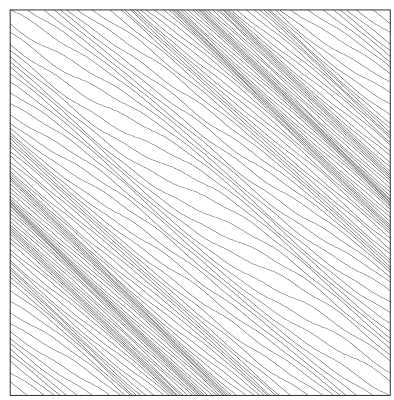
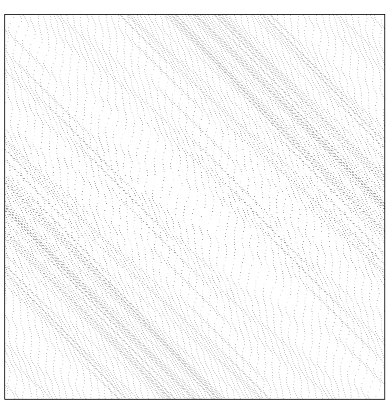
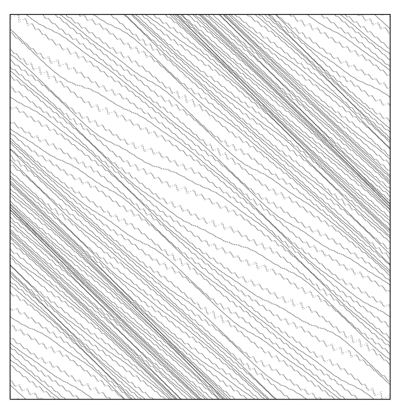
This periodic orbit is more concentrated that its previous $S_{0,(0,0)}$ companion, since it is minimizing, and it is nearer the cantorus.

b) $K_1 = K_2 = 10^{-5}$

| $i$ | $r_i$ | $\lambda$ | $R_1$ | $R_2$ |
|---|---|---|---|---|
| 3 | (1,2)/3 | 0.292891968810 | 0.853551515595 | 0.000000000000 |
| 4 | (7,13)/18 | 0.295191864175 | 0.999978886269 | 0.000000008677 |
| 5 | (8,14)/19 | 0.364380487804 | 0.843365834932 | 0.000001710212 |
| 6 | (17,30)/41 | 0.325382849433 | 0.987358946905 | -0.000007996227 |
| 7 | (108,191)/261 | 0.307395589261 | 0.997248343099 | 0.000000176114 |
| 8 | (116,205)/280 | 0.313165554710 | 0.952751102828 | -0.000315002378 |
| 9 | (133,235)/321 | 0.312299224162 | 0.993633039655 | -0.000452394769 |
| 10 | (490,866)/1183 | 0.309103110006 | 0.999978399073 | 0.000287486704 |
| 11 | (606,1071)/1463 | 0.304002649577 | 0.999918395275 | 0.000020112440 |
| 12 | (7431,13133)/17940 | 0.303466845552 | 0.999999848362 | -0.002338850071 |
| 13 | (7921,13999)/19123 | 0.303277941143 | 0.999259232489 | -0.000007686291 |
| 14 | (23879,42202)/57649 | 0.303111559878 | 0.999994042680 | -0.000433986339 |

Next two periodic orbits are $S_{0,(1,1)}$-symmetric. They are far of the breakdown of the tori and approximate the cantorus. The transition of two $S_{0,(0,0)}$-periodic orbits is shown in the next page. Compare the results.



| $K_1 = 10^{-5}$ $K_2 = 10^{-5}$ | $\frac{(7921,13999)}{19123}$ $(S_{0,(0,0)})$ | $\frac{(23879,42202)}{57649}$ $(S_{0,(0,0)})$ |
|---|---|---|
| $\lambda = 0.25$ |  |  |
| $\lambda = 0.30$ |  |  |
| $\lambda = 0.3025$ |  |  |
| $\lambda = 0.305$ |  |  |

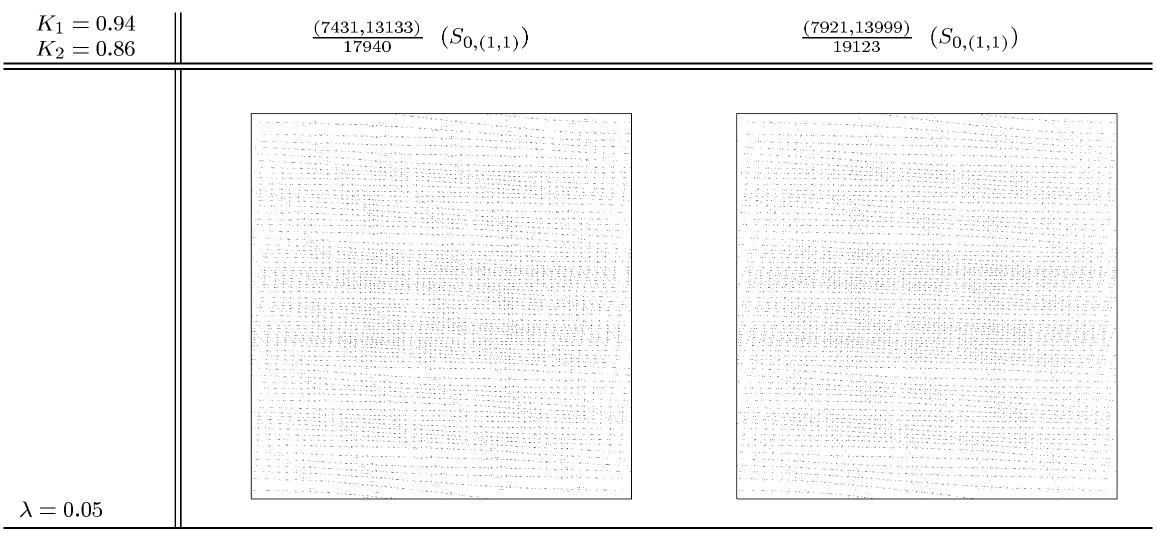
c) $K_1 = 0.94, K_2 = 0.86$

We have obtained the next results. Note that the behaviour of the second residue is different to the previous two examples.

| $i$ | $r_i$ | $\lambda$ | $R_1$ | $R_2$ |
|---|---|---|---|---|
| 3 | (1,2)/3 | 0.124921658131 | 0.711509182606 | 0.206761394118 |
| 4 | (7,13)/18 | 0.014768948858 | 0.942919846053 | 0.295565747971 |
| 5 | (8,14)/19 | 0.007814850007 | 0.896167498887 | 0.320005353044 |
| 6 | (17,30)/41 | 0.004191587946 | 0.937863129428 | 0.181525757409 |
| 7 | (108,191)/261 | - | - | - |
| 8 | (116,205)/280 | 0.000373119336 | 0.993048446490 | 0.153376290067 |
| 9 | (133,235)/321 | 0.000087867986 | 0.996415524230 | -0.410568359983 |
| 10 | (490,866)/1183 | 0.000081067026 | 0.994841117815 | -0.037850561360 |
| 11 | (606,1071)/1463 | 0.000051633508 | 0.998707133037 | -0.335583474019 |
| 12 | (7431,13133)/17940 | 0.000015468654 | 0.999969606977 | 0.113443090278 |
| 13 | (7921,13999)/19123 | 0.000015734182 | 0.999765776280 | -0.041139390566 |
| 14 | (23879,42202)/57649 | 0.000017174735 | 0.999973116450 | -1.58212087845 $10^5$ |

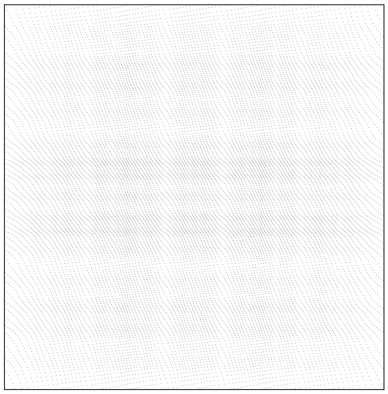
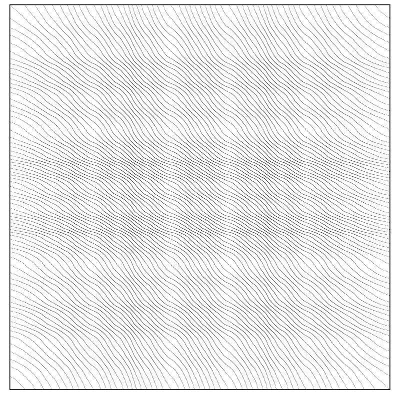
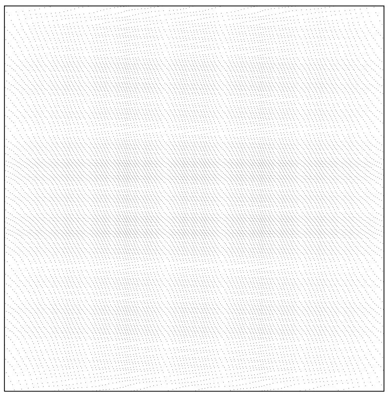
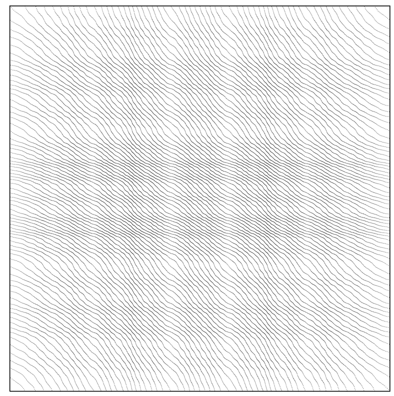
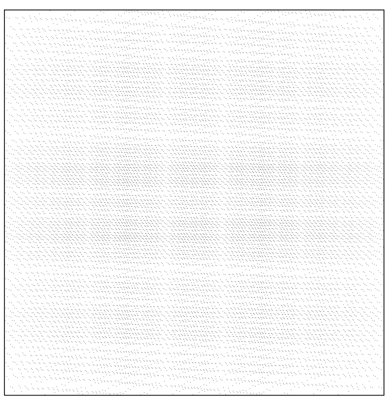
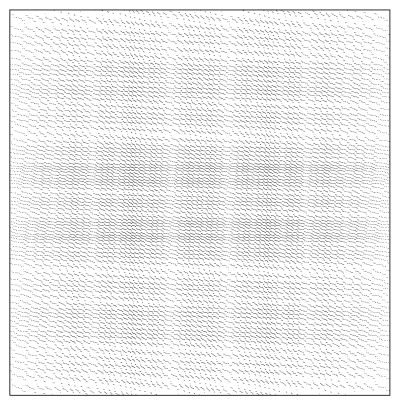
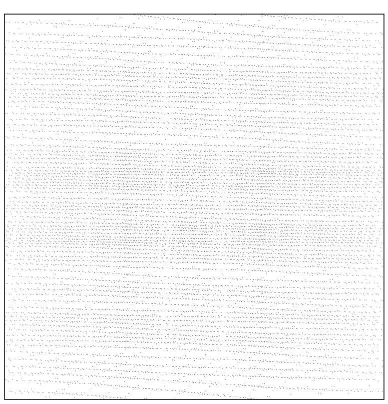
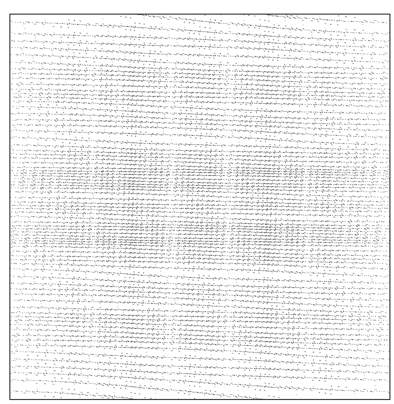
Notice that the value corresponding to period 261 has not been found. The problem is that the corresponding orbit is not minimizing for $\lambda = 0$. Notice also that this number is the worst of the convergents of the list.

The $S_{0,(1,1)}$-periodic orbits of periods 17940 and 19123 for a value $\lambda = 0.0001$ are shown in the next figure.

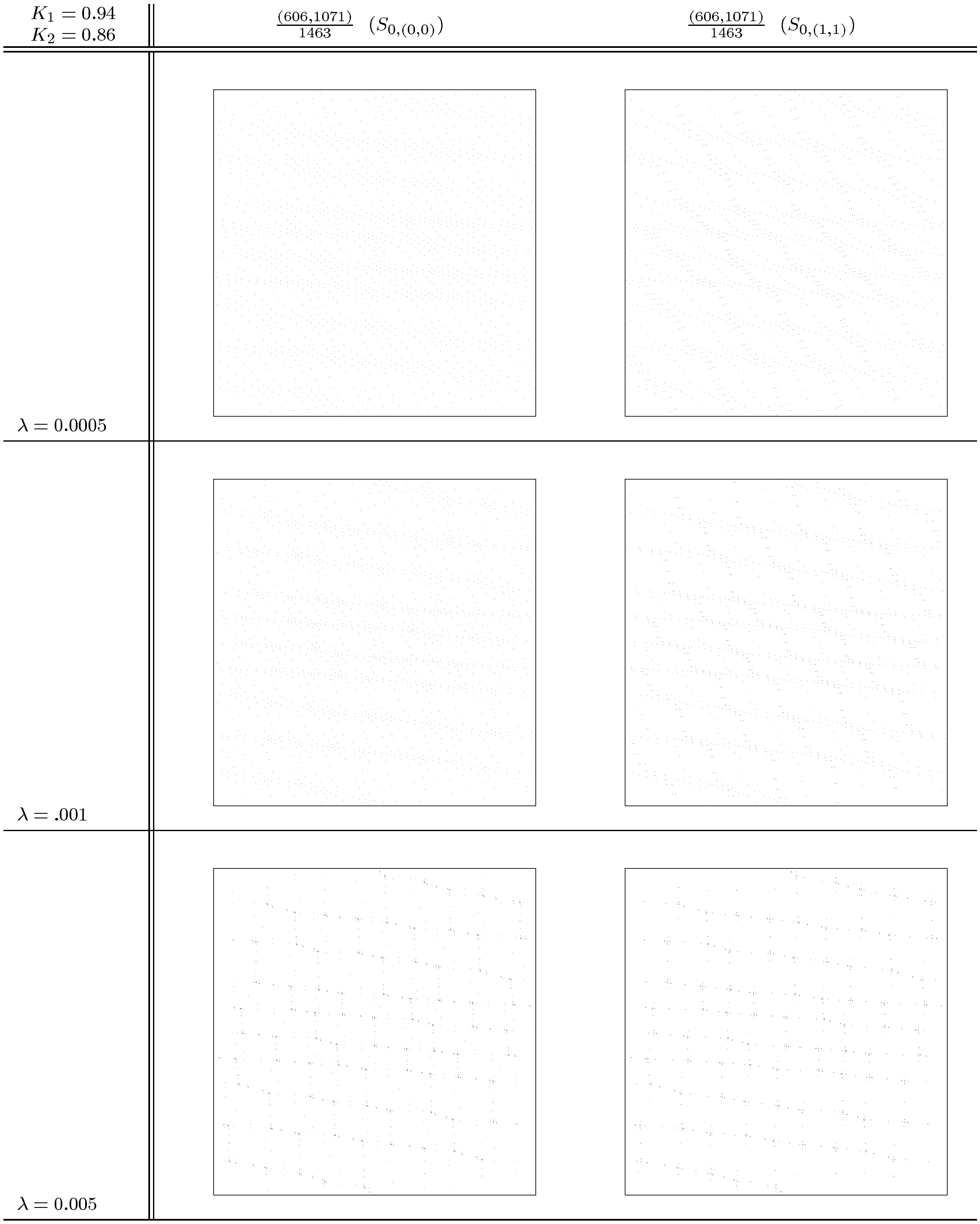



Where are the 1183 points that we have added to the first figure to obtain the second one?

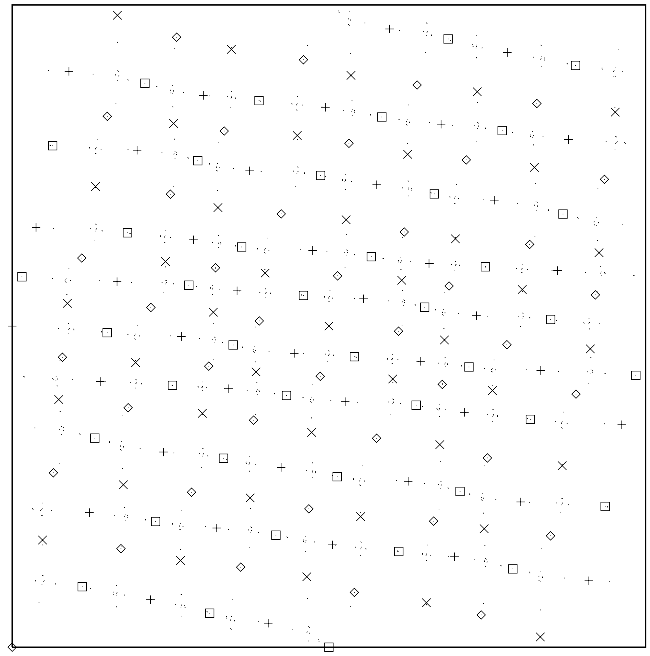
The transition is given in the next page using $S_{0,(0,0)}$-periodic orbits.

| $K_1 = 0.94$ <br> $K_2 = 0.86$ | $\frac{(7921,13999)}{19123}$ $(S_{0,(1,1)})$ | $\frac{(23879,42202)}{57649}$ $(S_{0,(1,1)})$ |
|---|---|---|
| $\lambda = 0$ |  |  |
| $\lambda = 10^{-5}$ |  |  |
| $\lambda = 2.5 \cdot 10^{-5}$ |  |  |
| $\lambda = 5 \cdot 10^{-5}$ |  |  |

Next orbits have period 1463 and $\lambda$ is very far of the breakdown.




We also glimpse 41 holes. Next figure shows the $S_{0,(1,1)}$-periodic orbit of period 1463, and the four periodic orbits of period 41, for $\lambda = 0.005$. In this case, these orbits of period 41 are elliptic-hyperbolic.



The $S_{0,(1,1)}$-periodic orbit of period 1463 and his 4 companions of period 41 ($\lambda = 0.005$)

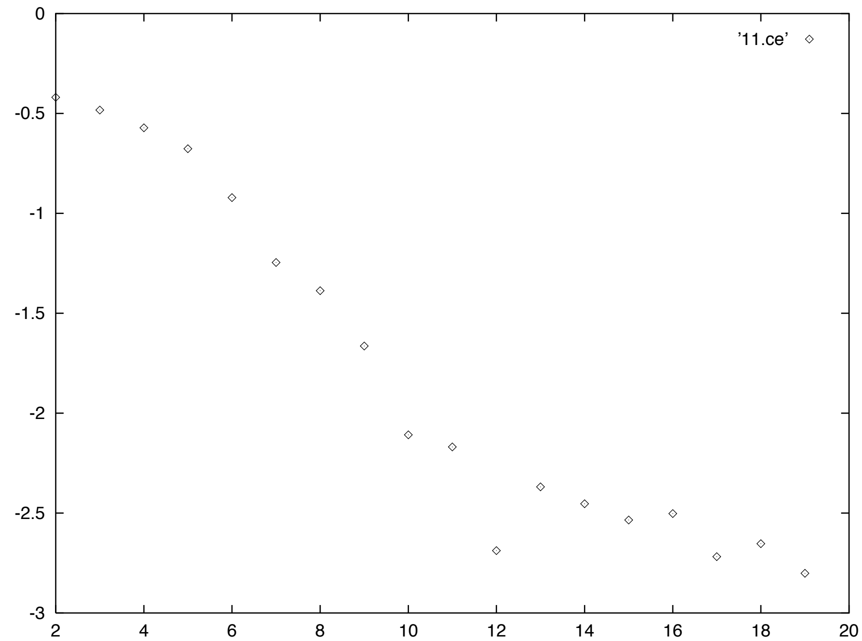
## 2) The golden vector

We have consider the golden vector $\omega = (1,1)^\infty$, and the first case considered for our quadratic pair.

- $K_1 = 0.1, K_2 = 0.2$

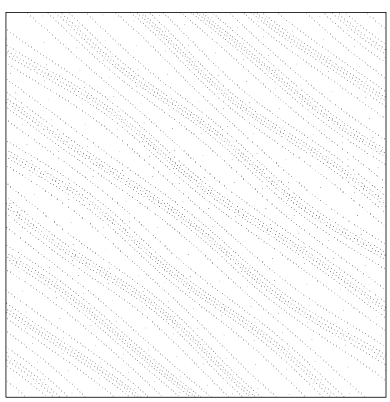
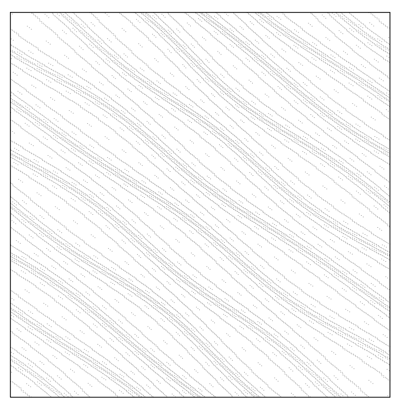
| $i$ | $r_i$ | $\lambda$ | $R_1$ | $R_2$ |
|---|---|---|---|---|
| 2 | (2,1)/2 | 0.786227600212 | 0.750000000000 | 0.054276599357 |
| 3 | (3,2)/4 | 0.481430637556 | 0.790957312802 | 0.043314778374 |
| 4 | (6,4)/7 | 0.429570033995 | 0.823454787306 | 0.042317240583 |
| 5 | (11,7)/13 | 0.368616829728 | 0.897819757731 | 0.043241227669 |
| 6 | (20,13)/24 | 0.311328212367 | 0.951851420005 | 0.072736980399 |
| 7 | (37,24)/44 | 0.220740693678 | 0.985991614228 | 0.006120647432 |
| 8 | (68,44)/81 | 0.157621577267 | 0.978690403484 | 0.000005049403 |
| 9 | (125,81)/149 | 0.141810443381 | 0.999964581010 | 0.000000001302 |
| 10 | (230,149)/274 | 0.122484691284 | 0.994117007449 | 0.000037318042 |
| 11 | (423,274)/504 | 0.108609469112 | 0.993252159255 | 0.000046591875 |
| 12 | (778,504)/927 | 0.107588222922 | 0.999446646095 | -0.000057864885 |
| 13 | (1431,927)/1705 | 0.102866041606 | 0.999999168218 | 0.000000000017 |
| 14 | (2632,1705)/3136 | 0.105091037580 | 0.999330828476 | 0.000082011541 |
| 15 | (4841,3136)/5768 | 0.104336702842 | 0.998734403769 | 0.000006384543 |
| 16 | (8904,5768)/10609 | 0.103735028331 | 0.999831870987 | 0.000174165462 |
| 17 | (16377,10609)/19513 | 0.103959442764 | 0.999999692232 | 0.000121240453 |
| 18 | (30122,19513)/35890 | 0.102726762387 | 0.999999702857 | -0.000000000001 |
| 19 | (55403,35890)/66012 | 0.103035786604 | 0.999745115697 | -0.000004215211 |
| 20 | (101902,66012)/121415 | 0.102393165561 | 0.999978181375 | -0.000000006441 |
| 21 | (187427,121415)/223317 | 0.100814420276 | 0.999999999999 | -0.000000000001 |

Next figure shows the relation between the error in the estimation of the critical value and the corresponding convergent.

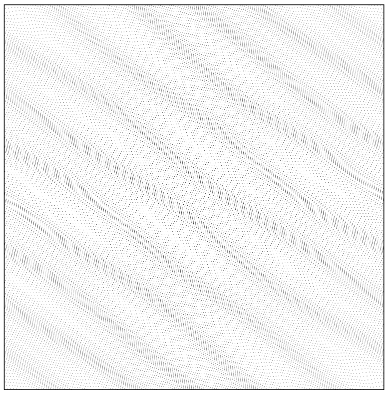
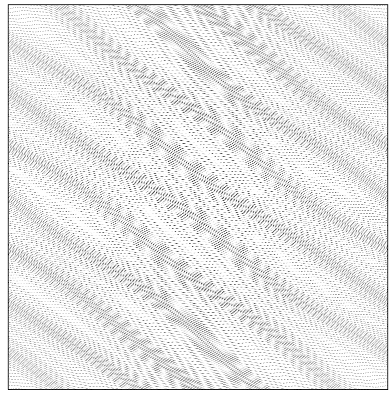
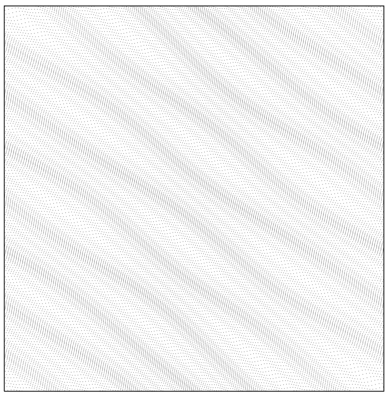
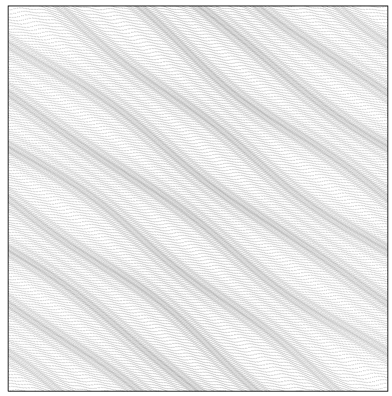
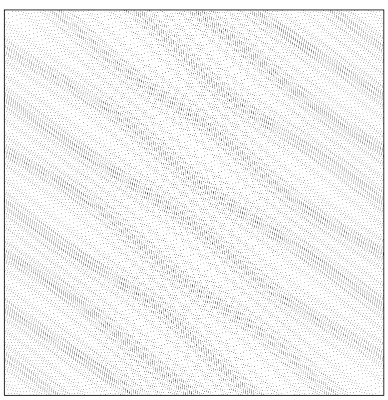
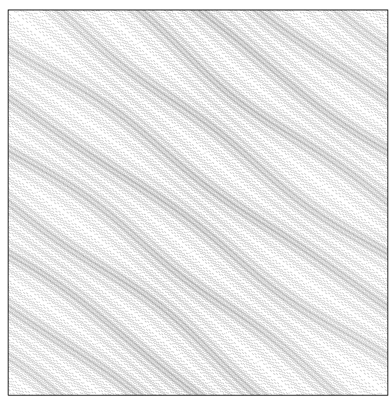
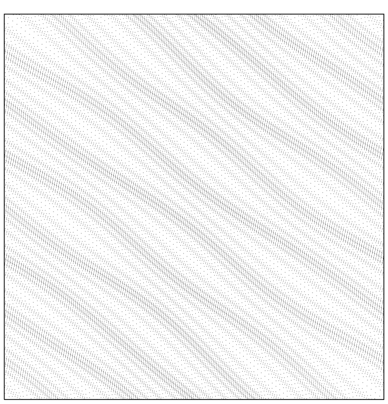
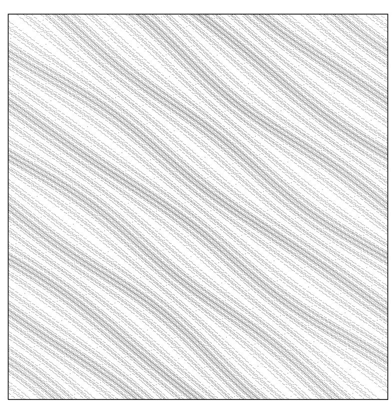



$x = i, y = \log_{10} e_{r_i}$

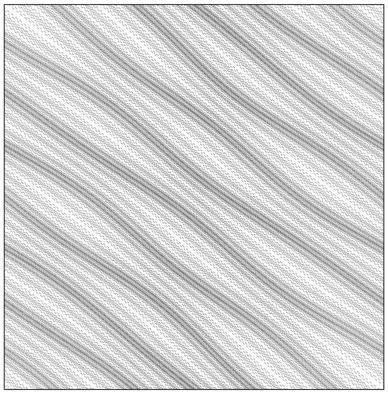
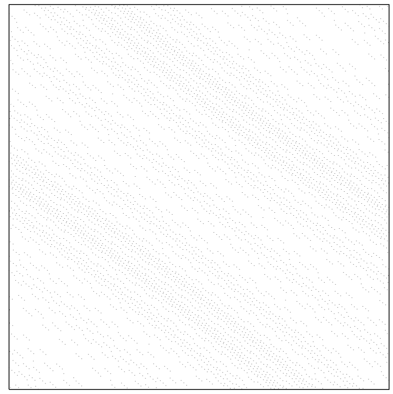
Two orbits of periods 10609 and 19513 for a value $\lambda = 0.12$, greater than the critical value, have the next appearance:

| $K_1 = 0.1$ <br> $K_2 = 0.2$ | $\frac{(8904,5768)}{10609}$ $(S_{0,(0,0)})$ | $\frac{(16377,10609)}{19513}$ $(S_{0,(0,0)})$ |
|---|---|---|
| $\lambda = 0.12$ |  |  |

The transition in the breakdown of the torus is shown in the next page with orbits of period 35890 and 66012.

| $K_1 = 0.1$ $K_2 = 0.2$ | $\frac{(30122,19513)}{35890}$ $(S_{0,(0,0)})$ | $\frac{(55403,35890)}{66012}$ $(S_{0,(0,0)})$ |
|---|---|---|
| $\lambda = 0.1$ |  |  |
| $\lambda = 0.1025$ |  |  |
| $\lambda = 0.105$ |  |  |
| $\lambda = 0.1075$ |  |  |

Next figure shows an orbit of period 121415 and a magnification of its center. We have chosen $\lambda = 0.105$, one of the values of the previous table.

| $K_1 = 0.1$ <br> $K_2 = 0.2$ | $\frac{(101902,66012)}{121415}$ $(S_{0,(0,0)})$ in $[0,1]\times[0,1]$ | $\frac{(101902,66012)}{121415}$ $(S_{0,(0,0)})$ in $[0.4,0.6]\times[0.4,0.6]$ |
|---|---|---|
| $\lambda = 0.105$ |  |  |

**Remark**

Notice that this torus breaks later that the torus corresponding to the quadratic pair! The simple idea that good cubic vectors are more robust than quadratic ones seems to be quite natural. However, Bollt and Meiss [21] observed, for the 4D semi-standard map (which is complex), that a torus with a quadratic rotation vector $(\gamma, \zeta)$ is more robust than a torus with the spiral mean vector $(\sigma^{-2}, \sigma^{-1})$. On one hand, $(\gamma, \zeta)$ is given by the golden mean $\gamma$ and $\zeta = (1+\sqrt{2})/(5+4\sqrt{2})$ is one real root of $7\zeta^2 - 6\zeta + 1 = 0$. On the other hand, the *spiral mean* is the golden vector for the Kim-Ostlund tree, and $\sigma$ is the real root of $\sigma^3 - \sigma - 1 = 0$. ◁

◁

# Appendix D

# Applications to symplectic skew-products

As we know, the time-1 flow of a 1-periodic Hamiltonian is a model of symplectomorphism. An example is given by the Newton's equation

$$\ddot{x} \;=\; f(x,t),$$

where the force is 1-periodic and the variable $x$ is $d$-dimensional. Suppose that we perturb quasi-periodically our system, and the equation of the motion is

$$\ddot{x} \;=\; f(x,t,\omega t),$$

where $f = f(x,t,\theta)$ is 1-periodic in $t$ and $\theta$, and $\omega$ is an irrational number. We can unfold the equation on the extended phase space $\mathbb{R}^{2d} \times \mathbb{R}$ as

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x,t,\theta) \\ \dot{\theta} = \omega \end{cases} .$$

Then, the time-1 flow is like

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \;\longrightarrow\; \begin{pmatrix} \bar{x}(x,y,\theta) \\ \bar{y}(x,y,\theta) \\ \theta + \omega \end{pmatrix},$$

and the first two variables behave symplectically. It is a model of *symplectic skew product.*

We are going to extend to these kind of systems the results already obtained for exact symplectomorphisms. We shall give the results without proofs, and, for the sake of simplicity, we shall work on the standard symplectic manifold (or in the annulus).

## D.1 Symplectic skew-products

### D.1.1 Definitions

We shall consider the standard symplectic structure in $\mathbb{R}^d \times \mathbb{R}^d$, endowed with the space-momentum coordinates $(x, y)$. We extend our phase space by adding new variables

$\theta \in \mathbb{R}^p$. Then, our *extended phase space* will be $\mathbb{R}^{2d+p}$. The $\theta$ variables will behave as temporal coordinates.

A *symplectic skew-product* on $\mathbb{R}^{2d}$ over $\mathbb{R}^p$ is a diffeomorphism $F : \mathbb{R}^{2d+p} \to \mathbb{R}^{2d+p}$ given by

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} f(x,y,\theta) \\ g(x,y,\theta) \\ \omega(\theta) \end{pmatrix},$$

where each diffeomorphism $F_\theta = (f_\theta, g_\theta)$, with $f_\theta = f(\cdot,\cdot,\theta)$ and $g_\theta = g(\cdot,\cdot,\theta))$, is symplectomorphic. It is a coupled family of $\theta$-parametric symplectic maps. It is a model of non-autonomous discrete mechanical system, and in order to known the trajectory of a point $(x,y)$ we also must know the initial time $\theta$.

In such a case, there exists a function $S : \mathbb{R}^{2d+p} \to \mathbb{R}$ satisfying the exactness equations

$$\begin{cases} \dfrac{\partial S}{\partial x}(x,y,\theta) = g(x,y,\theta)^\top \dfrac{\partial f}{\partial x}(x,y,\theta) - y^\top \\ \\ \dfrac{\partial S}{\partial y}(x,y,\theta) = g(x,y,\theta)^\top \dfrac{\partial f}{\partial y}(x,y,\theta) \end{cases}.$$

It is the primitive function of our skew-product and it is defined up to additive $\theta$-functions. Of course, $S_\theta = S(\cdot,\cdot,\theta)$ is the primitive function of $F_\theta$.

We shall say that $F$ is monotone iff each $F_\theta$ is monotone. Analogously, we can define monotone positiveness.

## D.1.2 Variational principles

Let $F : \mathbb{R}^{2d+p} \to \mathbb{R}^{2d+p}$ be a symplectic skew-product, being $S$ its primitive function. In order to look for the fixed points of $F$ we can look for the critical points of $S$ restricted to the vertically transformed set

$$K = \{(x,y,\theta) \in \mathbb{R}^{2d} \times \mathbb{R}^p \mid f(x,y,\theta) = x, \omega(\theta) = \theta\}$$

We can also define variational principles for the orbits. Then, if $F$ is monotone, we get that the extremal character of a segment of orbit of length $n$ beginning at $(x_0, y_0, \theta_0) = (x,y,\theta)$ is given by the Hessian matrix

$$H_{0,n} = \begin{pmatrix} \hat{A}_1 & \hat{B}_1 & & & \\ \hat{B}_1^\top & \hat{A}_2 & \hat{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{B}_{n-3}^\top & \hat{A}_{n-2} & \hat{B}_{n-2} \\ & & & \hat{B}_{n-2}^\top & \hat{A}_{n-1} \end{pmatrix},$$

where the matrices $\hat{A}$s and $\hat{B}$s are given by

$$\hat{A}_i = D_{i-1} B_{i-1}^{-1} + B_i^{-1} A_i$$

and

$$\hat{B}_i = -B_i^{-1}.$$

In this case, the coefficients of the matrix depend not only on the space-momentum coordinates, but also on the temporal coordinates, that is, on the iteration number itself. We can use the MMS iteration in order to know if our segment is minimizing, that is, if the matrix is definite positive.

**Remark**

> This method is similar to that used by Mather in [75], where he extended several properties of twist maps of the annulus to finite compositions of twists maps. ◁

### D.1.3 Extended Lagrangian graphs

The extended Lagrangian graph on $\mathbb{R}^{2d+p}$ generated by the function $l : \mathbb{R}^{2d+p} \to \mathbb{R}$ is given by

$$y = \nabla_x l(x, \theta).$$

It is family of Lagrangian graphs on $\mathbb{R}^{2d}$, parametrized by $\theta$.

Our graph is invariant for a certain symplectic skew-product $F$ iff $\forall x \in \mathbb{R}^d, \theta \in \mathbb{R}^p$

$$g(x, \nabla_x l(x, \theta), \theta) = \nabla_x (f(x, \nabla_x l(x, \theta), \theta), \omega(\theta)).$$

Then, we define a function $\hat{\Phi}$ on our extended phase space by

$$\hat{\Phi}(x, y, \theta) = S(x, y, \theta) - (l(f(x, y, \theta), \omega(\theta)) - l(x, \theta)).$$

We obtain that this function restricted to our graph depends only on $\theta$, because the partial derivatives of $\hat{\Phi}$ respect to $x, y$ vanish on the graph.

In particular, fixed an *extended fiber* by $(x, \theta)$, then $\hat{\Phi}(x, \cdot, \theta)$ has a critical point in $y = \nabla_x l(x, \theta)$. If it is a minimum, and this condition is satisfied for all the extended fibers, we shall say that our graph is minimizing. We also obtain that the orbits on a minimizing graph are minimizing.

**Remark**

> We can extend symplectically the extended phase space by adding conjugate variables to $\theta$, $I$. If we have a symplectic skew-product $F$, we can extend it to this big extended phase space, by taking $\bar{F} : \mathbb{R}^{2d} \times \mathbb{R}^{2p} \to \mathbb{R}^{2d} \times \mathbb{R}^{2p}$ defined as
>
> $$\begin{pmatrix} x \\ y \\ \theta \\ I \end{pmatrix} \longrightarrow \begin{pmatrix} f(x, y, \theta) \\ g(x, y, \theta) \\ \omega(\theta) \\ \mathrm{D}\omega(\theta)^{-\top} I \end{pmatrix}.$$

$\bar{F}$ is not symplectic, but it is volume-preserving. It is a skew-product of two symplectomorphisms.

Moreover, an extended Lagrangian graph $y = \nabla_x l(x,\theta)$ can be also extended by adding $I = \nabla_\theta l(x,\theta)$, but this kind of extension does not necessarily preserve the invariance condition. If we extend our graph adding $I = 0$, then it will be invariant for the extension, but it is not necessarily a Lagrangian graph. $\triangleleft$

## D.2 Converse KAM theory

In this section our extended phase space will be $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{T}^p$, endowed with the coordinates $(x, y, \theta)$. Let $F : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{T}^p \to \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{T}^p$ be a symplectic skew-product, where the dynamics on the time-periodic components $\theta$ is given by an ergodic translation by a vector $\omega \in \mathbb{T}^p$. We shall consider its lift $\tilde{F} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^p$, whose primitive function is $S : \mathbb{R}^{2d+p} \to \mathbb{R}$.

Let $\psi : \mathbb{T}^d \times \mathbb{T}^p \to \mathbb{R}^d$ be a differentiable map, whose graph $\mathcal{L}_\psi$ is an $F$-invariant extended Lagrangian torus. Thus, we can write

$$\psi(x,\theta) = a(\theta) + \nabla_x l(x,\theta),$$

where $a : \mathbb{T}^d \in \mathbb{R}^d$ is the average function and $l : \mathbb{T}^d \times \mathbb{T}^p \to \mathbb{R}$. So, the generating function (on $\mathbb{R}^{2d} \times \mathbb{T}^p$) of the graph is

$$L(x,\theta) = a(\theta)x + l(x,\theta).$$

The dynamics on the $(d+p)$-torus is like $(x,\theta) \to (\bar{f}(x,\theta), \theta+\omega)$, that is to say, $\bar{f}(x,\theta) = f(x, \psi(x,\theta), \theta)$. We shall suppose that our torus is monotone.

### D.2.1 A non-existence criterion of invariant tori

Similarly to the results in Appendix B, we can prove that a monotone positive invariant extended Lagrangian torus is minimizing and, then, its orbits are minimizing. This also provides a non-existence criterion:

> if the orbit by $(x,y,\theta) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{T}^p$ yields on a monotone positive region, and has a segment which does not have non-degenerate minimal action then it does not lie on any invariant extended Lagrangian graph included into such a region.

On one side, we can study the points on the extended phase space in order to check if it is possible that a Lagrangian invariant torus pass through them, following Appendix B. On the other side, we can wonder if a certain torus can exist, as in Appendix C. That is, we can ask ourselves if a torus whose dynamics is given by a diophantine rotation $\hat{\omega} = (\omega_0, \omega)$ can exists.

### D.2.2 An example: the rotating standard map

A *generalized rotating standard-like map* on $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{T}^p$ of potentials $V : \mathbb{T}^d \times \mathbb{T}^p \to \mathbb{R}$ and $W : \mathbb{R}^d \times \mathbb{T}^p \to \mathbb{R}$ and rotation vector $\omega \in \mathbb{R}^p$ is given by

$$\begin{cases} y' = y - \nabla V(x,\theta) \\ x' = x + \nabla W(y',\theta) \pmod 1 \\ \theta' = \theta + \omega \pmod 1 \end{cases} .$$

It is not only an exact symplectic skew-product, but also it is an exact volume preserving map (the volume is given by the product of the two standard volumes: $\Omega = dy \wedge dx \wedge d\theta = \mathrm{d}(y\ dx \wedge d\theta)$). If $W(y,\theta) = \frac{1}{2}y^2$ we obtain a *rotating standard-like map*, and it is monotone ($+_d$). For the rest of the section, we shall consider $d = p = 1$.

The potential of the *rotating standard map* [10, 95] is the 2-parameter function

$$V(x,\theta) = \frac{-1}{(2\pi)^2}\cos(2\pi x)(K + \lambda\cos(2\pi\theta)).$$

For $\lambda = 0$, it is decomposed in the product of a standard map and a rotation.

The extended Lagrangian graphs are surfaces that divide the extended phase space in two connected components. We are interested in the existence of such invariant tori. First of all, we shall apply the first step in variational criterion. Like in Appendix B, we shall choose the slice $\{x = 0,\ \theta = 0\}$. Since

$$\begin{aligned} \hat{A}(x,y,\theta) &= 2 - V''(x,\theta) \\ &= 2 - \cos(2\pi x)(K + \lambda\cos(2\pi\theta)) \end{aligned}$$

then

$$\hat{A}(0,y,0) = 2 - (K + \lambda).$$

Hence, we obtain that

> there are no invariant extended Lagrangian graphs (tori) if $K + \lambda \geq 2$.

Notice that this bound does not depend on $\omega$. We can also apply the MMS iteration to the points of a 2-dimensional slice $\theta = \theta_0$.

Fix now an irrational rotation vector $\hat{\omega} = (\omega_0, \omega)$ ($\omega$ already fixed) and a small enough $K$, such that the corresponding invariant torus exist for $\lambda = 0$. We shall apply the variational Greene method in order to detect which is the critical value $\lambda_{\hat{\omega}}$ when the torus breaks down. As our map has not periodic orbits, we shall use *almost periodic orbits* with nearby rotation vectors. That is, we shall construct a sequence of rationals converging to $\omega_0$, say $\left(\frac{p_i}{n_i}\right)_i$, and for each $i$ we shall continue with respect to $\lambda$ a point $(x, y, \theta_0)$ satisfying

$$F_\lambda^{n_i}(x,y,\theta_0) = (x + p_i, y, \theta_0 + n_i\omega).$$

For $\lambda = \lambda_{r_i}$ the segment stop being minimizing. In the tables of results we shall also show the 'residues' of the critical orbits, and they seem to tend to 1.

In order to avoid unpleasant accumulations in our rational approximations, we must obtain good sequences of rational vectors approximating the pair $(\omega_0, \omega)$, and then choose the corresponding components. We shall use the Jacobi-Perron algorithm.

We have also used the symmetries of our rotating standard map. For each $a, b \in \{0,1\}$ and $\theta_0 \in \mathbb{T}^1$ we define the axis

$$S_{a,b}(\theta_0) = \{(x = \frac{1}{2}(ay+b), \theta_0) \mid y \in \mathbb{R}\}.$$

We have used the symmetry axis $S_{(0,0)}(0)$, and we have distributed the points of the almost periodic orbits symmetrically respect to that axis, in order to detect the critical value of the breakdown. Before this, we have continued some periodic orbits in order to know how is the breakdown. Their projections onto the zero-section are also displayed.

**Remarks**

i) As far as I know, the rotating standard map was introduced by Artuso, Casati and Shepelyansky in [10], where properties of the map and existence of tori were also investigated. This example was also studied by Tompaidis [95], by considering the residues of periodic orbits, that is, with a method nearer to the Greene's method. He should approximate also the rotation $\omega$ by rationals. Although in the next examples we have used almost periodic points, we have also used a 'Tompaidis variational method'. The results are similar.

ii) The existence of codimension-1 invariant tori for small values of the parameters $K$ and $\lambda$ is given by some Herman's theorems about translated tori [102]. As a particular case, he considered the cylinder $B^d = \mathbb{T}^d \times \mathbb{R}$, endowed with coordinates $(x_1, \ldots, x_d, y)$ and volume form $\Omega = \mathrm{d}(y dx_1 \wedge \ldots \wedge dx_d)$. He assured that an exact volume preserving diffeomorphism $F$ close enough to another one $F_0$ satisfying $F_0(x, 0) = (x + \alpha, 0)$, being $\alpha$ a Diophantine vector, has many rotational invariant tori which, moreover, are graphs.

◁

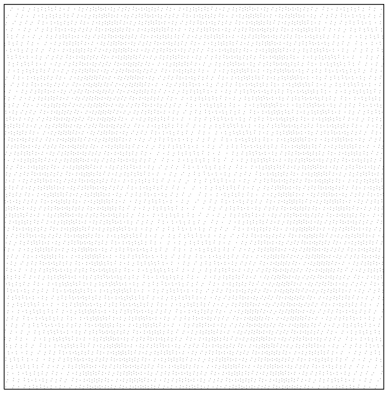
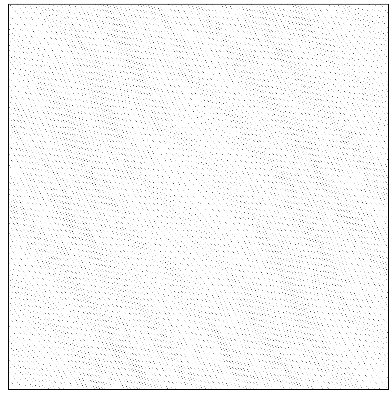
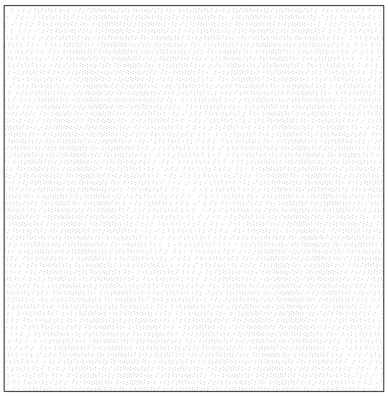
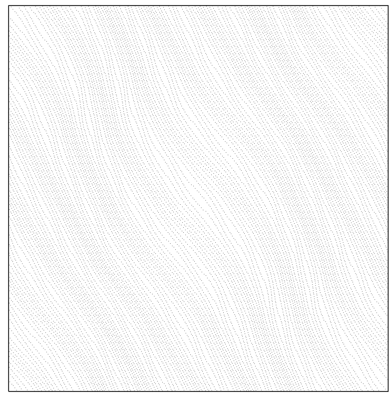
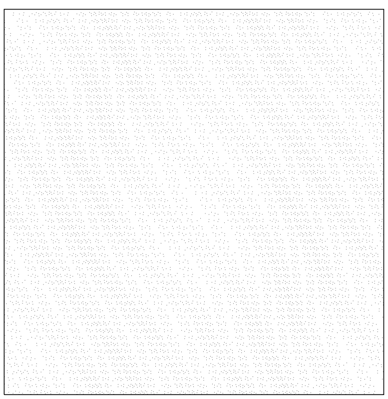
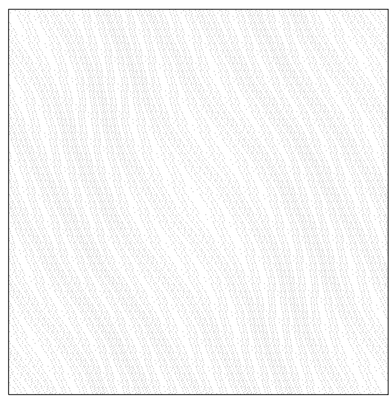
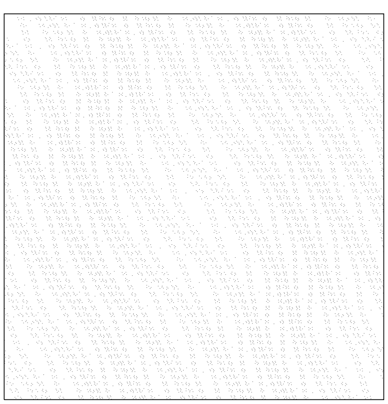
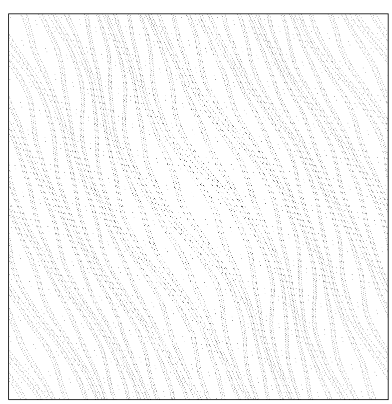
**Examples**

1) $K = 0$, $\hat{\omega} = (\omega_0, \omega) = (1, 1)^\infty$.

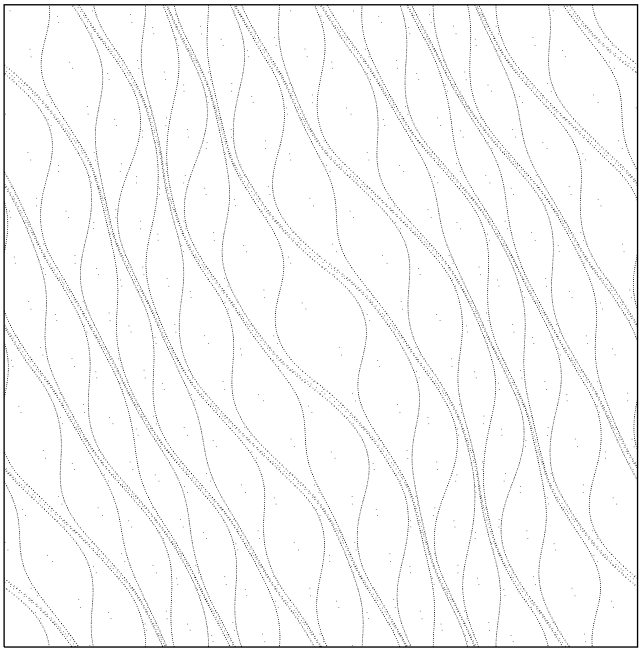
| $i$ | $r_i$ | $\lambda$ | $R$ |
|---|---|---|---|
| 13 | 927/1705 | 0.574507055719 | 0.996556455390 |
| 14 | 1705/3136 | 0.572804188415 | 0.998111118230 |
| 15 | 3136/5768 | 0.571201029424 | 0.997442128220 |
| 16 | 5768/10609 | 0.570310482777 | 0.998642500860 |
| 17 | 10609/19513 | 0.569776631252 | 1.000258991261 |
| 18 | 19513/35890 | 0.569800602434 | 0.999374706969 |
| 19 | 35890/66012 | 0.569272536214 | 1.001051155865 |
| 20 | 66012/121415 | 0.568578818316 | 1.000160489209 |

The value of $\lambda_{\hat{\omega}}$ estimated by Tompaidis was $\lambda_{\hat{\omega}} \simeq 3.55/(2\pi) \simeq 0.565$, by means a periodic orbit of period 10609. It seems too small.

The transition in the breakdown is shown in the next page with almost periodic orbits of 'periods' 10609 and 19513.

| $K = 0$ | $\frac{5768}{10609}$ $(S_{0,0}(0))$ | $\frac{10609}{19513}$ $(S_{0,0}(0))$ |
|---|---|---|
| $\lambda = 0.565$ |  |  |
| $\lambda = 0.57$ |  |  |
| $\lambda = 0.58$ |  |  |
| $\lambda = 0.60$ |  |  |

If we continue the almost periodic orbit of period 19513 until the value $\lambda = 0.7$ we obtain the next figure.



A $S_{0,0}(0)$ almost periodic orbit of period 19513 ($\lambda = 0.7$)

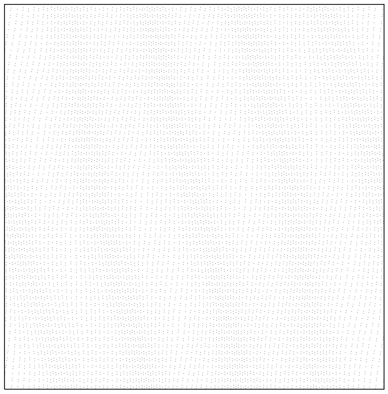
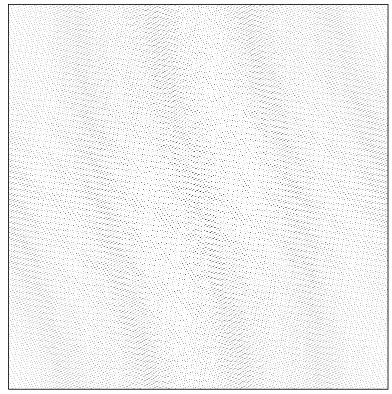
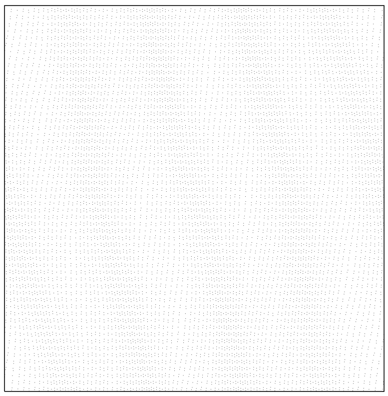
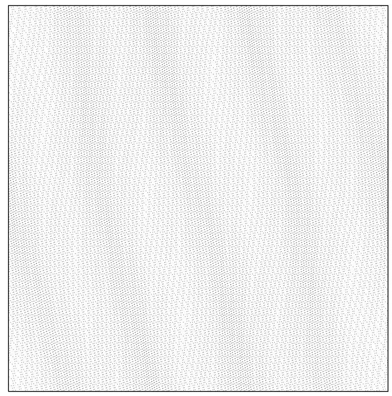
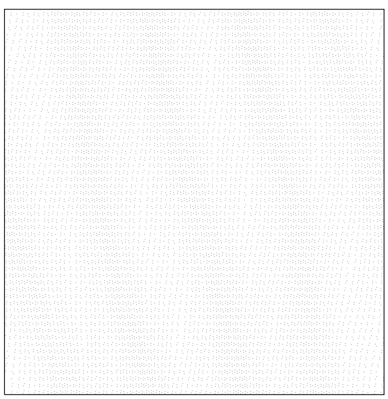
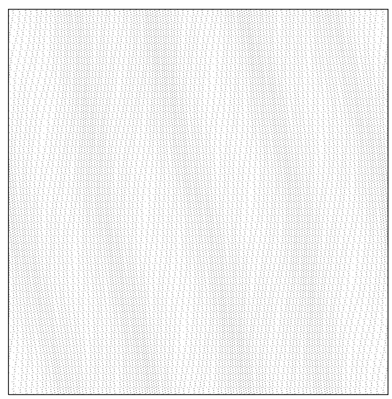
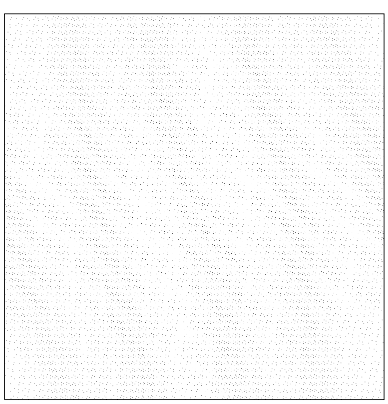
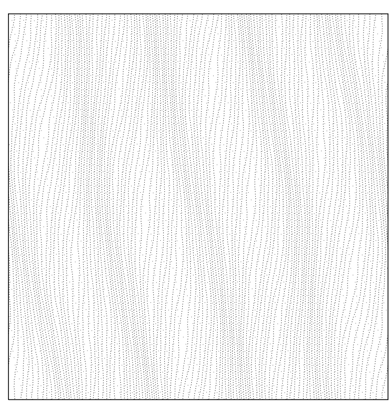
The number of 'holes' in the picture is 68. But 68 does not appear in the list of convergents! Do not worry, 68 is a good denominator, in the sense that the corresponding best rational vector, $r = (57, 37)/68$, has good reduced error and Roth exponent: $\epsilon(r, \hat{\omega}) \simeq 7.72121\ 10^{-2}$ and $\eta(r, \hat{\omega}) \simeq 1.60699$.

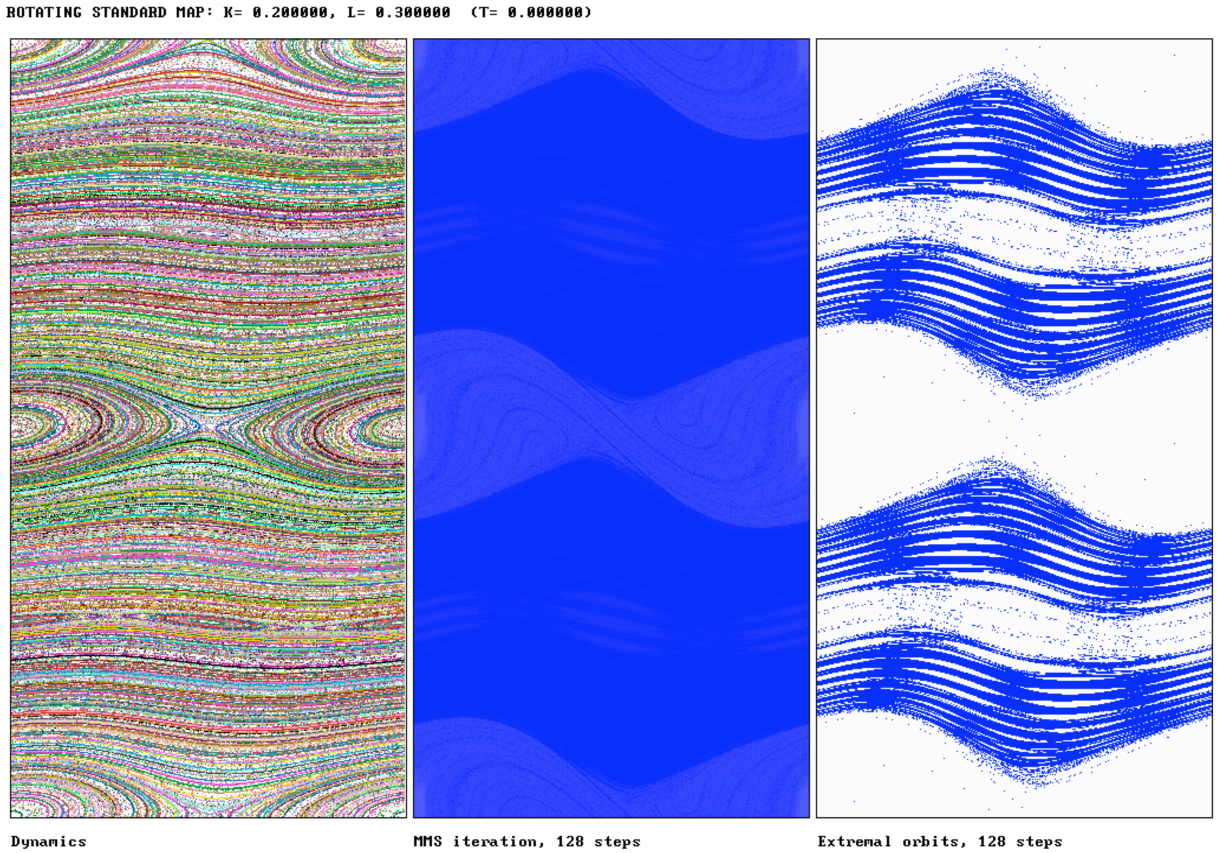
2) $K = 0.2$, $\hat{\omega} = (\omega_0, \omega) = (1, 1)^\infty$.

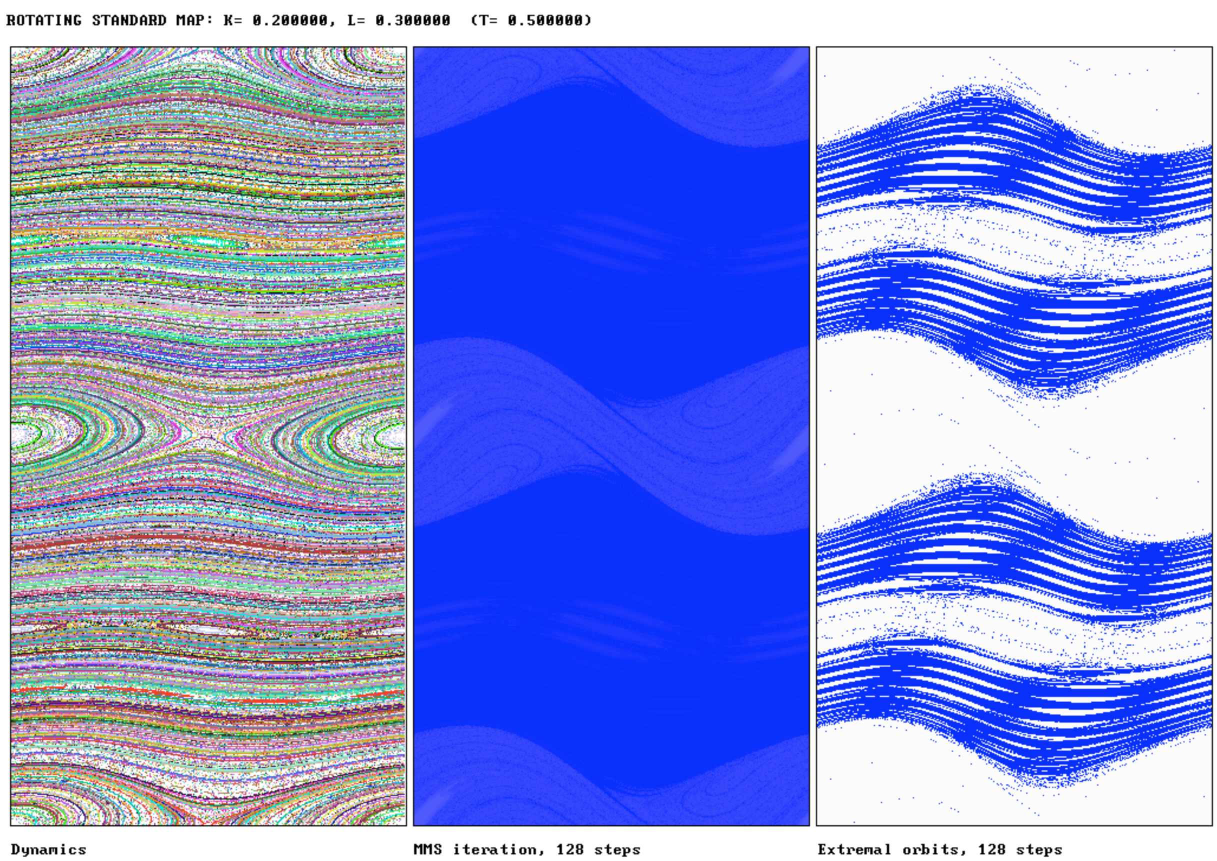
| $i$ | $r_i$ | $\lambda$ | $R$ |
|---|---|---|---|
| 13 | 927/1705 | 0.285273715732 | 0.996946926969 |
| 14 | 1705/3136 | 0.282087268528 | 0.999047942222 |
| 15 | 3136/5768 | 0.273526134343 | 0.998988310660 |
| 16 | 5768/10609 | 0.269290262474 | 1.000603738593 |
| 17 | 10609/19513 | 0.267458379552 | 1.000294359525 |
| 18 | 19513/35890 | 0.264701735661 | 0.999655512304 |
| 19 | 35890/66012 | 0.264355564769 | 1.000339035331 |

The value of $\lambda_{\hat{\omega}}$ estimated by Tompaidis was $\lambda_{\hat{\omega}} \simeq 1.75/(2\pi) \simeq 0.279$, by means a periodic orbit of period 10609. It seems too big.

The breakdown is shown in the next page with almost periodic orbits of 'periods' 10609 and 35890.

| $K = 0$ | $\frac{5768}{10609}$ $(S_{0,0}(0))$ | $\frac{19513}{35890}$ $(S_{0,0}(0))$ |
|---|---|---|
| $\lambda = 0.26$ |  |  |
| $\lambda = 0.265$ |  |  |
| $\lambda = 0.27$ |  |  |
| $\lambda = 0.28$ |  |  |


ROTATING STANDARD MAP: K= 0.200000, L= 0.300000 (T= 0.000000)
Dynamics
MMS iteration, 128 steps
Extremal orbits, 128 steps


ROTATING STANDARD MAP: K= 0.200000, L= 0.300000 (T= 0.500000)
Dynamics
MMS iteration, 128 steps
Extremal orbits, 128 steps

In the previous pictures we have taken $\lambda = 0.3$, quite far of the breakdown of the golden torus. They show, for $\theta_0 = 0, \frac{1}{2}$:

- the dynamics, taking points on $\theta = \theta_0$ and projecting their orbits on such a slice;
- the extremal character of the orbits of the fiber $\theta = \theta_0$;
- the corresponding minimizing orbits.

It seems that the golden torus is not so robust, because there are many minimizing orbits. Compare also the number of minimizing bands in the two pictures. Inside these bands can be the sections of invariant tori with the slices. Recall that these tori bound the motion of the points.

◁

# Appendix E

# Towards a geometrical explanation of the breakdown

It is expected that the reasons for the breakdown of invariant tori are related to geometrical obstructions that can be seen as a generalization of the results given by Olvera and Simó [82]. In the case of area preserving maps, they performed a method to determine the critical value of breakdown of a certain invariant curve, based in the computation of heteroclinic connections of nearby hyperbolic periodic orbits. In higher dimensions, these geometrical obstructions must be given by codimension-1 invariant manifolds, as center-stable and center-unstable manifolds of nearby elliptic-hyperbolic periodic orbits (with only a hyperbolic plane).

Moreover, while heteroclinic connections of periodic hyperbolic orbits in area preserving maps have been useful in order to bound resonance zones which are useful in order to explain transport [76], in higher dimensions these 'bags' should be bounded by pieces of center-stable and center-unstable manifolds of elliptic-hyperbolic periodic orbits. Generally speaking, the codimension-1 invariant manifolds are the skeleton of the dynamical system.

In order to show the importance of these manifolds in higher dimensional symplectic maps, in this chapter we have performed an easier example. We shall consider several aspects related with the global and local behaviour of a 4D symplectic map, the Froeschlé map. In fact, we shall study a neighborhood of the $(0,0)$ resonance, that is, the resonance zone associated to the origin, which is an elliptic fixed point (for small enough values of the parameters). We shall see that this zone is bounded by the center-stable and center-unstable manifolds associated to the two $(0,0)$ elliptic-hyperbolic fixed points.

As all of this is hard to see in 4D, we shall restrict our attention on the points of the zero-section $\{y = 0\}$. In order to 'see' the resonance zone we shall consider two properties of the points of our phase space: their rate of escape and their extremal character. Before this, we shall intersect the center-stable and center-unstable manifolds of the two elliptic-hyperbolic fixed points (on $\{y = 0\}$) with such a slice.

## E.1 Escape-time and extremal character

### E.1.1 The escape-time algorithm

The *escape-time algorithm* is used to draw beautiful pictures like the Mandelbrot's set or Newtonians fractals [17]. Here, we apply this algorithm to know what points turn some of its angular coordinates (not turn on the totally elliptic point). We say that an orbit by a point $(x, y)$ is *non-rotating* iff

$$||\pi_x(F^n(x,y)) - x||_\infty < 1, \ \forall n \in \mathbb{N}$$

(i.e., any of the angular coordinates have gone around to the corresponding $S^1$).

In order to simplify we shall consider the section of such set with the torus $\{y = 0\}$. We have drawn pictures sized $500 \times 500$ pixels (the unit-square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ representing the torus), choosing $N = 10000$ and following the next steps (for each pixel):

1. we transform the pixel to a point of the unit-square;

2. we apply the lift of Froeschlé map to the point for $N$ times (maximum);

3. if for some iteration $n$, some of the angular coordinates have gone around once, we draw the corresponding pixel with a grey colour (the smaller $n$, the darker is the colour);

4. if at the end of iterations the point has not round, we draw the pixel with white colour.

The figure appears at the end of this chapter. Although all the points around the center of the box, which correspond to the elliptic fixed point, seem do not escape, in fact they escape in an exponentially long time, due to the phenomenon known as *Arnold diffusion* and the theory of Nekhoroshev.

### E.1.2 Extremal character in polar coordinates

The second picture that we have made shows the extremal character of the points belonging to the zero-section. This extremal character has been computed with respect to the symplectic polar coordinates with respect to the origin.

In such coordinates, our symplectomorphism is not monotone positive. In fact, there is a monotone positive region and a monotone undefinite region. This behaviour appear in the figure. The grid is also $500 \times 500$, and we have iterated the points 128 times.

## E.2 Calculus of center-stable and center-unstable manifolds of an EH fixed point

### E.2.1 Computation via power series

Given an elliptic-hyperbolic fixed point (that we suppose the origin), we can write our diffeomorphism as

$$\begin{pmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & \rho & 0 & 0 \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & \rho^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} + \dots,$$

where $a^2 + b^2 = 1$ and $|\rho| < 1, \rho \neq 0$. Moreover, we can do, if necessary, symplectic lineal change of variables (in order to preserve the symplectic structure), via *Williamson normal form* [101].

Then, the (local) center manifold $\mathcal{W}^c_{loc}$ is given by the graph of

$$\begin{cases} x_2 = \phi(x_1, y_1) = \sum_{k \geq 2} \phi_k(x_1, y_1) \\ \\ y_2 = \psi(x_1, y_1) = \sum_{k \geq 2} \psi_k(x_1, y_1) \end{cases},$$

where subscripts denote the degree of the homogeneous terms of the series. The equations that we have to solve $\forall k \geq 2$ are like

$$\begin{cases} \phi_k(ax + by, -bx + ay) - \rho\, \phi_k(x, y) = r_k(x, y) \\ \psi_k(ax + by, -bx + ay) - \rho^{-1}\psi_k(x, y) = s_k(x, y) \end{cases}$$

On the other side, the (local) center-stable manifold $\mathcal{W}^{cs}_{loc}$ (and, similarly, $\mathcal{W}^{cu}_{loc}$) is given by

$$y_2 = \Lambda(x_1, x_2, y_1) = \sum_{k \geq 2} \Lambda_k(x_1, x_2, y_1)$$

and the equations are, $\forall k \geq 2$,

$$\Lambda_k(ax + by, \rho v, -bx + ay) - \rho^{-1}\Lambda_k(x, v, y) \quad = \quad L_k(x, v, y)$$

If we write

$$\Lambda_k(x, v, y) = \sum_{m=0}^{k} \lambda_m(x, y) v^{k-m}, \;\; L_k(x, v, y) = \sum_{m=0}^{k} l_m(x, y) v^{k-m}$$

then, $\forall m = 0 \div k$

$$\lambda_m(ax + by, -bx + ay) - \rho^{k-m+1}\lambda_m(x, y) \quad = \quad \rho^{k-m} l_m(x, y)$$

All these *homological equations* are, then, of the same type.

### E.2.2 Calculus of the sections with the torus $\{y = 0\}$

Let be:

- $z_f$, the EH fixed point, and $M = \mathrm{D}F(z_f)$;
- $V$, the matrix of the change of base (to reduce $L$ to normal form);
- $\zeta$, the new variables: $\zeta = V^{-1}(z - z_f)$;
- $\zeta_4 = \phi(\zeta_1, \zeta_2, \zeta_3)$, the local parametrization of the center-stable manifold $\mathcal{W}^{cs}(z_f)$ (at the new coordinates), calculated until certain order.

Fixed a small enough $R$, we shall accept that if $||z||_1 \leq R$ and $\zeta = V^{-1}(z - z_f)$ verifies $\zeta_4 = \phi(\zeta_1, \zeta_2, \zeta_3)$, then $z \in \mathcal{W}^{cs}(z_f)$. It is very important to have calculated $\phi$ till high order (because the expansion of a center-stable manifold through the center directions is hard, because its points go around the fixed point).

So, the equation we have to solve is (for a certain $k \in \mathbb{N}$):

$$\pi_4(V^{-1}F^k(x_1, x_2, 0, 0)^T - z_f) - \phi(\pi_{1,2,3}(V^{-1}F^k(x_1, x_2, 0, 0)^T - z_f)) \quad = \quad 0$$

with the condition $||F^k(x_1, x_2, 0, 0)||_1 \leq R$. Obviously, the solution of this equation is a curve, and we can find different patches of it by continuation and varying $k$.

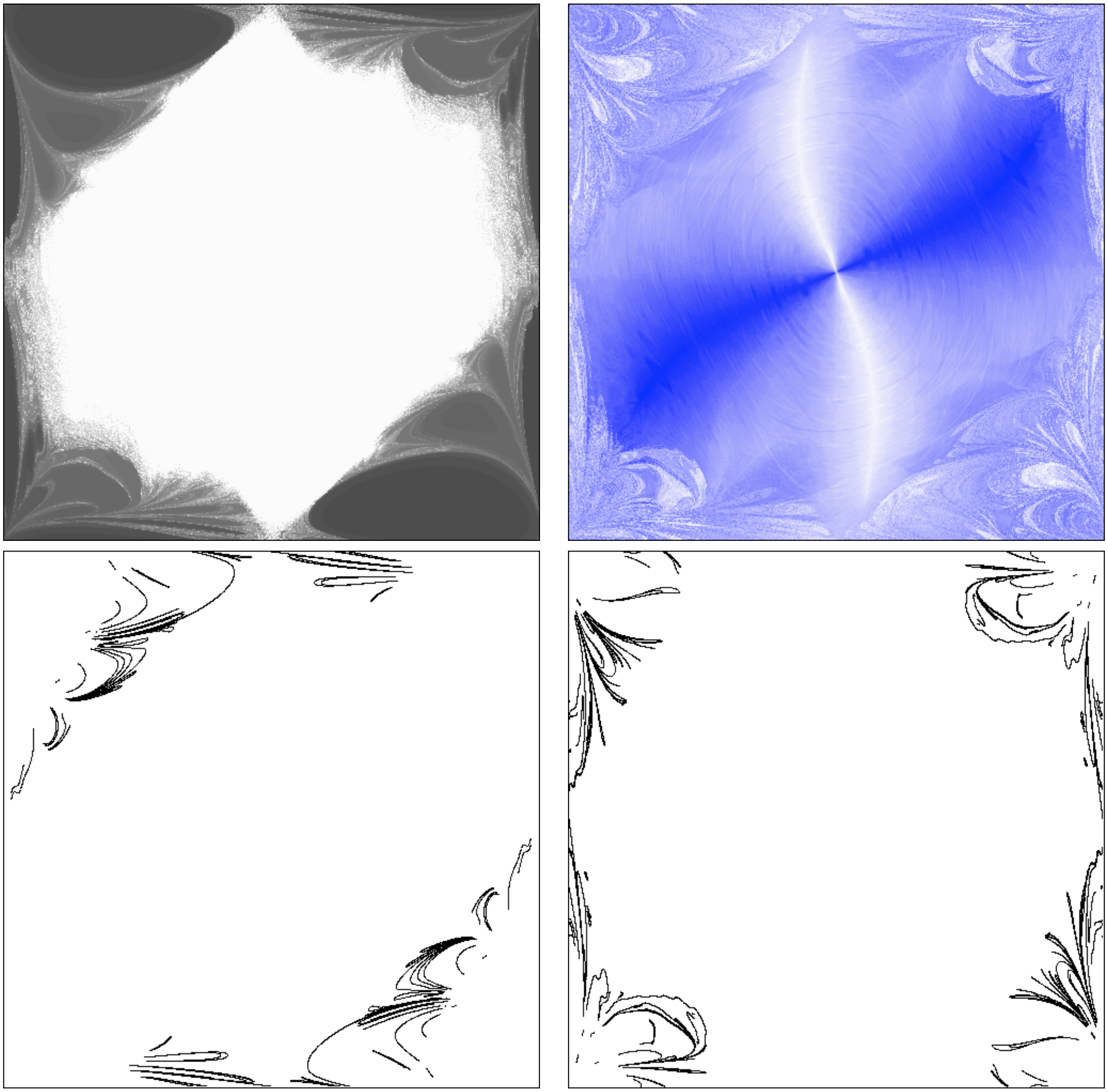
Of course, we can do the same with $\mathcal{W}^{cu}(z_f)$, but then $-k \in \mathbb{N}$. In our example, this is not necessary, due to the symmetries of the Froechlé map.

## E.3 Pictures at an exhibition

Let $z_{eh} = (0, \frac{1}{2}, 0, 0)$, $z_{eh} = (\frac{1}{2}, 0, 0, 0)$ be the two elliptic-hyperbolic fixed points of the Froeschlé map, and $\mathbb{T}_0^2$ the torus $\{y = 0\}$(represented by the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$). We have chosen the parameters $K_1 = 0.5$, $K_2 = 0.3$, $\lambda = 0.1$. The four pictures are:

- Levels of escape-time with $N = 10000$. The grey color is chosen using a logarithmic scale, that is $g = \frac{\log n}{\log N} \in [0, 1]$.
- Extremal character with respect to symplectic polar coordinates, after 128 iterations.
- Different sections of $\mathcal{W}^{cs}(z_{eh})$ with $\mathbb{T}_0^2$.
- Different sections of $\mathcal{W}^{cs}(z_{he})$ with $\mathbb{T}_0^2$.

Compare the pictures!



As we see, the resonance zone associated to the elliptic fixed point seems to be bounded by pieces of center-stable and center-unstable manifolds of its elliptic-hyperbolic companions. These manifolds have really many folds, and we think they arrive to the elliptic fixed point (after a very long time). A similar picture of the resonance zone using escape-times appear in [31], but they did not explain its shape in terms of these manifolds.

On the other side, the monotone positive invariant tori surrounding the origin, if they exist, seem to accumulate around a curve. We recall that the invariant tori intersect the zero-section in a point, generically.

# Appendix F

# Normal forms

Here, we show the necessary steps in order to simplify the dynamics around an exact Lagrangian invariant manifold of an exact symplectomorphism. The first step is to transport the invariant manifold to the zero-section of its cotangent bundle, but this is possible thanks to Weinstein's theorems [97, 98]. In the case that our symplectic manifold is a cotangent bundle and our invariant manifold is a graph this can be easily done by means of a fiberwise translation. The second one, if the dynamics on the invariant manifold is conjugated to an easier one, we can get it via the lift of the corresponding conjugation. The rest of steps try to kill the 'vertical' jet of the symplectomorphism, that is to say, the dependence on the $y$-variables. Generally, this is not possible.

We shall apply our method in order to obtain the already known normal forms for invariant tori and for hyperbolic points (cf. [20, 65]).

## F.1 Set up

### F.1.1 Step 1: Simplification of the dynamics on the zero-section

Suppose we have an exact symplectomorphism $F : T^*M \to T^*M$, with $pf(F) = S$ and $F$-invariant zero-section: $F \circ z = z \circ q \circ F \circ z$. We suppose that the dynamics on the base space, $q \circ F \circ z$ is conjugated (via $\delta$) to the diffeomorphism $\phi : M \to M$. We can perform an exact symplectic change of variables such that the dynamics on the base space be $\phi$.

Since

$$q \circ F \circ z \quad = \quad \delta \circ \phi \circ \delta^{-1},$$

we must just define

$$\bar{F} \quad = \quad \hat{\delta}^{-1} \circ F \circ \hat{\delta},$$

where $\hat{\delta}$ is the lift of $\delta$ to $T^*M$ (and $\delta \circ q = q \circ \hat{\delta}$). Then:

- $pf(\bar{F}) = pf(F) \circ \hat{\delta}$,

- the zero-section is $\bar{F}$-invariant, and its dynamics is given by $\phi$. As a matter of fact,

$$\begin{aligned}\bar{F}\circ z &= \hat{\delta}^{-1}\circ F\circ\hat{\delta}\circ z = \hat{\delta}^{-1}\circ F\circ z\circ\delta \\ &= \hat{\delta}^{-1}\circ z\circ q\circ F\circ z\circ\delta = z\circ\delta^{-1}\circ\delta\circ\phi\circ\delta^{-1}\circ\delta \\ &= z\circ\phi.\end{aligned}$$

### F.1.2 Step k: Elimination of the k-terms

In order to make easier the problem, we now assume that our manifold $M^d$ is parallelizable, i.e., $T^*M \simeq M\times\mathbb{R}^d$ (for instance, $M=\mathbb{R}^d$ or $M=\mathbb{T}^d$).

Suppose that we have done $k-1$ steps of normal form and that our exact symplectomorphism is $F_{k-1} : T^*M \to T^*M$. As the the zero-section is fixed, being the diffeomorphism $\phi : M\to M$ its dynamics, our primitive function is

$$\hat{S}_{k-1}(x,y) \;=\; N_{\leq k-1}(x,y) + R_{>k-1}(x,y),$$

where

$$N_{\leq k-1}(x,y) \;=\; \sum_{2\leq i\leq k-1} N_i(x,y)$$

and

$$R_{>k-1}(x,y) \;=\; \sum_{i\geq k} R_i(x,y).$$

Each function $N_i(x,y)$ is $y$-homogeneous of degree $i$. The same for the functions $R_i(x,y)$. Then, we also shall write

$$N_i(x,y) \;=\; \sum_{|n|=i} \nu_n(x)y^n$$

and

$$R_i(x,y) \;=\; \sum_{|n|=i} \rho_n(x)y^n.$$

$N_{\leq k-1}$ corresponds to the terms that we have not able to eliminate, and it is the normal form until degree $k-1$. $R_{>k-1}$ is the corresponding residue.

In order to eliminate all the terms of order $k$ in $\hat{S}_{k-1}$, we want to find an exact symplectomorphism $G_k$, with primitive function

$$\hat{T}_k(x,y) = \sum_{i\geq k} T_i(x,y),$$

and leaving all the points of the zero-section fixed. We keep the notation by $y$-homogeneous degrees: $T_i(x,y) = \sum_{|n|=i}\tau_n(x)y^n$. We shall look for this diffeomorphism as the time-1 flow of a Hamiltonian $\hat{H}_k = H_k(x,y)$, which is $y$-homogeneous of degree $k$. Hence, we can compute $G_k$ and $G_k^{-1}$ by the Lie series method. The relationship between $H_k$ and $\hat{T}_k$ is given by

$$\hat{T}_k = \sum_{m\geq 1} \frac{\mathbf{\Lambda}_m(H)}{m!},$$

and, in particular

$$\begin{aligned} T_k(x,y) &= \mathbf{\Lambda}(H_k)(x,y) \\ &= (k-1)H_k(x,y). \end{aligned}$$

The new symplectomorphism is $\hat{F}_k = G_k^{-1}\circ\hat{F}_{k-1}\circ G_k$, whose primitive function is $\hat{S}_k$.

$$\begin{aligned} \hat{S}_k(x,y) &= pf(F_k) = pf(G_k^{-1}\circ F_{k-1}\circ G_k) = \hat{S}_{k-1}\circ G_k + \hat{T}_k - \hat{T}_k\circ G_k^{-1}\circ F_{k-1}\circ G_k \\ &= N_{\leq k-1}(x,y) + R_k(x,y) + \ldots + \\ &\quad T_k(x,y) + \ldots - \\ &\quad T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) + \ldots \\ &= N_{\leq k-1}(x,y) + R_k(x,y) + T_k(x,y) - T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) + \ldots, \end{aligned}$$

where "…" means terms with $y$-degree greater that $k$. Then, the *homological equation* that we must solve is:

$$T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) - T_k(x,y) = R_k(x,y).$$

If we know how to solve these equation, we obtain the main terms of $\hat{T}_k$.

So, we have a linear operator on the space of $y$-homogeneous functions of degree $k$ $\mathcal{F}_k$, given by

$$L_\phi T_k(x,y) = T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) - T_k(x,y).$$

Of course, we can define this operator on the graded algebra $\bigoplus_{k=0}^{\infty}\mathcal{F}_k$. Formally speaking, this space is the space of all the functions, $\mathcal{F}$.

Then, we must solve equations as

$$L_\phi T_k = R_k,$$

but it is not always possible. For instance, if we have the splitting

$$\mathcal{F} = \ker L_\phi \oplus L_\phi(\mathcal{F})$$

(i.e., $L_\phi$ is an isomorphism on its image $L_\phi(\mathcal{F})$), then $R_k$ can be written as

$$R_k = N_k + L_k,$$

and we can only solve $L_\phi T_k = L_k$, being $N_k$ the remainder [1]. So, the normal form until degree $k$ is $N_{\leq k}$, which belongs to $\ker L_\phi$. The new primitive function is

$$\hat{S}_k(x,y) = N_{\leq k} + \hat{R}_{>k}.$$

[1] In fact, we do not need that the sum be direct, but in such a case we can obtain different normal forms.

## F.2 On a neighborhood of an invariant torus

If we are working on $\mathbb{T}^d \times \mathbb{R}^d$ (or in its covering space), then the functions are 1-periodic in their $x$-variables. If the dynamics on the zero-section is a shift by $\omega$, then the homological equations that we have to solve are like

$$T_k(x+\omega, y) - T_k(x,y) \quad = \quad R_k(x,y).$$

That is to say, the operator that we must consider is

$$\begin{aligned} L_\omega T_k(x,y) &= T_k(x+\omega,y) - T_k(x,y) \\ &= \sum_{|n|=k} (\tau_n(x+\omega) - \tau_n(x)) y^n. \end{aligned}$$

It is decomposed in 'smaller' ones: $l_\omega \tau(x) = \tau(x+\omega) - \tau(x)$. $l_\omega$ acts on the space of functions defined on the $d$-torus: $\mathcal{F}(\mathbb{T}^d)$. If $\omega$ is a Diophantine vector [2] , then we can eliminate all the terms except the constants in $x$ [3] because $\mathcal{F}(\mathbb{T}^d) = \ker l_\omega \oplus l_\omega(\mathcal{F}(\mathbb{T}^d))$, being $\ker l_\omega$ the space of constant functions and $l_\omega(\mathcal{F}(\mathbb{T}^d))$ the space of null average functions (see [6, 88]). So, the normal form until degree $k$ is

$$N_{\leq k}(y) \quad = \quad \sum_{2 \leq |n| \leq k} \nu_n y^n,$$

where the $\nu_n$ are constants.

Hence, if we apply the results of Section 4.3, we get that the normal form until degree $k$ is

$$\begin{cases} f(x,y) = \omega + x + \Omega_{k-1}(y) + O(y^k) \\ \\ g(x,y) = y(I + O(y^k)) \end{cases},$$

where

$$\begin{aligned} \Omega_{k-1}(y) &= \sum_{i=2}^{k} \frac{1}{i-1} \nabla_y N_i(y) \\ &= \nabla_y H_k(y). \end{aligned}$$

That is to say, $\Omega_{k-1}$ is the gradient of a certain polynomial of degree $k$ and order 2, $H_k$, and $\mathbf{\Lambda}(H_k) = N_{\leq k}$. This is the *Birkhoff's normal form.*

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[2] That is to say, there exists $C > 0$ and $\tau \geq 1$ such that

$$|q \cdot \omega - n| \geq C/\|q\|_1^\tau \quad \forall q \in \mathbb{Z}^d, \forall n \in \mathbb{Z}.$$

[3] Not only formally, but also analytically.

**Remarks**

i) Note that the normal form until order $k$ (eliminating the terms of the remainder) is interpolated by the time-1 flow of the time-independent Hamiltonian $\omega \cdot y + H_k(y)$. This Hamiltonian is integrable. This is equivalent to construct $d$ approximate integrals of $F$ in a small neighborhood of the invariant manifold [32, 94].

ii) This kind of normal form has been useful in order to obtain partial justifications of Greene's criterion (see [65, 32] for the case $d = 1$ and [94] in higher dimensions).

$\triangleleft$

## F.3 On a neighborhood of a hyperbolic point

Suppose that we have a symplectomorphism in a neighborhood of the origin of $\mathbb{R}^d \times \mathbb{R}^d$. As we know, the stable and the unstable manifolds are exact Lagrangian. Anyway, we shall work in a neighborhood of the origin.

- We can put one of them (or a piece of them), for instance, the unstable, on $\mathbb{R}^d \times \{0\}$.
- Suppose that the linear part can be diagonalized: $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ is the diagonal matrix of unstables eigenvalues ($|\lambda_i| > 1, \forall i = 1 \div d$). By a Poincaré's theorem, we can get that the dynamics on the unstable manifold be $\bar{x} = \Lambda x$, provided the next non resonance condition be satisfied [4]: $\forall |n| \geq 2$ $\lambda_i \neq \lambda^n$.

Then, the homological equation of order $k$ is

$$T_k(\Lambda x, \Lambda^{-1} y) - T_k(x, y) = R_k(x, y).$$

If we expand these functions in powers of $y$, the equations are

$$\sum_{|n|=k} (\lambda^{-n} \tau_n(\lambda x) - \tau_n(x)) y^n = \sum_{|n|=k} \rho_n(x) y^n,$$

and we get the set of operators

$$l_{\lambda,n} \tau_n(x) = \lambda^{-n} \tau_n(\lambda x) - \tau_n(x)$$

Expanding $\tau_n(x) = \sum_{m \in \mathbb{N}^d} \tau_{n,m} x^m$ and $\rho_n(x) = \sum_{m \in \mathbb{N}^d} \rho_{n,m} x^m$, we obtain that

$$\lambda^{-n} \lambda^m \tau_{n,m} - \tau_{n,m} = \rho_{n,m},$$

that is to say

$$\tau_{n,m} = \frac{\rho_{n,m}}{\lambda^{m-n} - 1}.$$

[4] We use multi-index notation, and $\lambda^n$ means $\lambda_1^{n_1} \ldots \lambda_d^{n_d}$

In fact, this is not possible if the denominator vanishes, for instance if $m = n$. If this is the only case when the denominator vanishes, we shall say that $\lambda$ satisfies a strong non resonance condition. In such a case, the *formal* normal form has a primitive function

$$\begin{aligned} N(x,y) &= \sum_{2\leq|n|} \nu_n (xy)^n \\ &= P(xy). \end{aligned}$$

($P(xy)$ means a function $P(x_1y_1, \ldots, x_dy_d)$, and we shall write $z_i = x_iy_i$).

Then, we can prove that our formal symplectomorphism is

$$\begin{cases} f(x,y) = \Lambda(x + \sum_{|n|\geq 1} f_n(x)(xy)^n) \\ \\ g(x,y) = \Lambda^{-1} \sum_{|n|\geq 1} g_n(x)(xy)^n \end{cases},$$

where the vector functions $f_n = (f_n^1, \ldots f_n^d)$ and $g_n = (g_n^1, \ldots, g_n^1)$ are given by

$$f_n^j(x) = \varphi_n^j x^{e_j}$$

and

$$g_n^j(x) = \psi_n^j x^{-e_j}.$$

(If a subscript has not sense, the corresponding coefficient will be zero). The constants are given by the next recurrence (for the notations, see Section 4.3):

- Step 1: $\forall i, j = 1 \div d$:

$$\begin{cases} \psi_{e_i}^j = \delta_{ij} \\ \\ \varphi_{e_i}^j = (1+\delta_{ij})\nu_{e_i+e_j} \end{cases}.$$

- Step $k$: $\forall |n| = k,\ \forall j = 1 \div d$:

$$\begin{cases} \psi_n^j = \Psi_n^j \\ \\ \varphi_n^j = (n_j+1)\sigma_n^j - \Phi_n^j \end{cases},$$

  where

$$\Psi_n^j = n_j\nu_n - \sum_i \sum_{u+v=n} (u_j + \delta_{ij})\phi_u^i \psi_{v-e_i}^i,$$

$$\begin{aligned} \Phi_n^j = {} & (n_j+1)\nu_{n+e_j} - \sum_i \phi_{e_j}^i \psi_n^i - \\ & \sum_i \sum_{\substack{u+v=n \\ |v|\neq 1}} (u_j+1)\phi_{u+e_j}^i \psi_v^i(x) \end{aligned}$$

and

$$\sigma_n^j = \frac{1}{k} \sum_i \Phi^i_{n+e_j-e_i}.$$

For instance, if $d = 1$, the primitive function of the formal normal form is $N(x, y) = P(xy)$ and the symplectomorphism is given by

$$\begin{cases} \bar{x} = x \cdot p(xy) \\ \bar{y} = y/p(xy) \end{cases},$$

where $p(0) = \lambda$, $P(0) = 0$ and $P'(0) = 0$. The relation between $p$ and $P$ is given by

$$P'(z) \quad = \quad z\, \frac{p'(z)}{p(z)}$$

(where $z = xy$ and $'$ means the derivative respect to $z$). As Moser proved [77] this normal form is not only formal, but also is analytic.

As we know, this normal form is not only formal, but also is analytic.

## F.4 On a neighborhood of a hyperbolic isotropic torus

Suppose we have on $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, with coordinates $(x_1, x_2, y_1, y_2)$, an exact symplectomorphism leaving the $d_1$-dimensional torus $\{x_2 = 0, y_1 = 0, y_2 = 0\}$ fixed. We suppose that its dynamics is a shift by $\omega = (\omega_1, \ldots, \omega_{d_1}) \in \mathbb{R}^{d_1}$, and that it is hyperbolic. We suppose that we can put the unstable manifold $\mathcal{W}^u(\mathbb{T}^{d_1})$ on the zero section $\{y_1 = 0, y_2 = 0\}$ and that the dynamics on it is decomposed in a shift by $\omega$ and an homothetic transformation by a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{d_2})$:

$$\begin{cases} \bar{x}_1 = x_1 + \omega \\ \bar{x}_2 = \Lambda x_2 \end{cases}.$$

This is a reducibility hypothesis. Of course, we suppose $\forall i = 1 \div d_2, |\lambda_i| > 1$.

Under some non resonance conditions, as $\omega$ be a Diophantine vector and $\lambda$ be strongly non resonant (see Section F.3), we can get a formal normal form with primitive function

$$N(x_1, x_2, y_1, y_2) \quad = \quad \sum_{|n_1| \geq 0} \nu_{n_1}(x_2 y_2) y_1^{n_1},$$

being $\nu_0(z)$ of order 2 and $\nu_{e_i}(z)$ of order 1, where $z = (x_2^1 y_2^1, \ldots, x_2^{d_2} y_2^{d_2})$. This is a mixed situation of the two previous sections.

# Appendix G

# Action forms, foliations and variational principles

The philosophy underlying in the constructions that we have made is that the geometry of the Hamiltonian mechanics is given by the Liouville form and the standard foliation of the phase space (the cotangent bundle of a manifold). This have been useful in order to study Lagrangian graphs, which are transversal to such a foliation.

If we are interesting in the study of other Lagrangian manifolds, other foliations and action forms must be considered. There are some results about this subject that we shall recall.

Finally, plans for future work about this subject are stated.

## G.1 Examples

For the sake of simplicity, we shall work on the standard symplectic manifold $\mathbb{R}^d \times \mathbb{R}^d$, and we shall use the standard notations.

Let $F$ be the symplectomorphism in $\mathbb{R}^{2d}$ given by

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases},$$

with $S$ as primitive function.

### G.1.1 Changing the beginning and the ending Lagrangian manifolds

Next example is essentially due to Tabacman [93], where he used this kind of construction in order to prove the existence of heteroclinic connections.

Suppose we have two Lagrangian graphs $\mathcal{L}_b$ and $\mathcal{L}_e$, given by the corresponding generating functions $l_b, l_e : \mathbb{R}^d \to \mathbb{R}$ (it can be also defined in a certain open neighborhood in $\mathbb{R}^d$).

We want to seek the orbits connecting these two Lagrangian graphs (in Section 5.4 we considered orbits connecting two 'horizontal' Lagrangian manifolds). That is, given

$n > m+1$, we want to look for the orbits connecting them after $n-m$ steps, the $(n-m)$-sequences of $\mathbb{R}^{2d}$

$$(x_m, y_m), (x_{m+1}, y_{m+1}), \dots (x_{n-1}, y_{n-1})$$

such that

- $y_m = \nabla l_b(x_m)$,
- $\forall i = m \div n-2,\ F(x_i, y_i) = (x_{i+1}, y_{i+1})$,
- $g(x_{n-1}, y_{n-1}) = \nabla l_e(f(x_{n-1}, y_{n-1}))$.

These segments of orbit are extremal of the action

$$\boldsymbol{S}_{m,n}(x_m, y_m, x_{m+1}, y_{m+1}, \dots, x_{n-1}, y_{n-1}) = \sum_{i=m}^{n-1} S(x_i, y_i) + l_b(x_0) - l_e(f(x_{n-1}, y_{n-1})),$$

restricted to the set of sequences satisfying

- $y_m = \nabla l_b(x_m)$,
- $\forall i = m \div n-2,\ f(x_i, y_i) = x_{i+1}$,
- $g(x_{n-1}, y_{n-1}) = \nabla l_e(f(x_{n-1}, y_{n-1}))$.

### G.1.2 Changing the Lagrangian foliation

We are going to change the 'orientation' in our phase space, and instead of to seek orbits connecting two 'vertical' fibers we shall connect two 'horizontal' ones. That is to say, given two $y$-points $\mathbf{y}_m, \mathbf{y}_n \in \mathbb{R}^d$, where $n > m+1$, we want to look for the orbits connecting them after $n-m$ steps, i.e., the $(n-m)$-sequences of $\mathbb{R}^{2d}$

$$(x_m, y_m), (x_{m+1}, y_{m+1}), \dots (x_{n-1}, y_{n-1})$$

such that

- $y_m = \mathbf{y}_m$,
- $\forall i = m \div n-2,\ F(x_i, y_i) = (x_{i+1}, y_{i+1})$,
- $g(x_{n-1}, y_{n-1}) = \mathbf{y}_n$.

Instead of considering 'horizontal' chains we shall consider 'vertical' ones:

- $y_m = \mathbf{y}_m$,
- $\forall i = m \div n-2,\ g(x_i, y_i) = y_{i+1}$,
- $g(x_{n-1}, y_{n-1}) = \mathbf{y}_n$.

Finally, the action on such a set will be

$$\begin{aligned}\hat{\boldsymbol{S}}_{m,n}(x_m, y_m, x_{m+1}, y_{m+1}, \ldots, x_{n-1}, y_{n-1}) &= \sum_{i=m}^{n-1}(S(x_i,y_i) + y_i^\top x_i - g(x_i,y_i)^\top f(x_i,y_i) \\ &= \sum_{i=m}^{n-1}(S(x_i,y_i) + y_i^\top (x_i - f(x_i,y_i)).\end{aligned}$$

The set of chains is a $d(n–m–1)$-submanifold of $\mathbb{R}^{2d(n-m)}$, provided the rank of the matrix

$$\begin{pmatrix} C_m & 0 & -I & & & & & \\ & C_{m+1} & D_{m+1} & 0 & -I & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & C_{n-2} & D_{n-2} & 0 & -I & \\ & & & & & C_{n-1} & D_{n-1} & \end{pmatrix}$$

is maximal ($= n-m$) in all the chains. For instance, this *transversality condition* is satisfied when $F$ is 'vertically' monotone, that is to say, if $C(z)$ is regular for all the points.

We obtain that:

- The connecting orbits are critical chains of $\hat{\boldsymbol{S}}_{m,n}$.
- If $F$ is 'vertically' monotone, the critical chains of $\hat{\boldsymbol{S}}_{m,n}$ are connecting orbits of $F$.

In particular, if we look for fixed points, we can consider the fixed action

$$\hat{\boldsymbol{s}}(x,y) = S(x,y) + y^\top (x - f(x,y))$$

restricted to the horizontally transformed set

$$\hat{K} = \{(x,y) \in \mathbb{R}^{2d} \mid g(x,y) = y\}.$$

From a geometrical point of view, the horizontal foliation is associated to the 1-form $\hat{\boldsymbol{\alpha}} = -x \; \mathrm{d}y$, which is also an action form for the symplectic form $\omega = \mathrm{d}y \wedge \mathrm{d}x$. The function $\hat{S}(x,y) = S(x,y) + y^\top x - g(x,y)^\top f(x,y)$ is the corresponding primitive function, that is, $F^*\hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}} = \mathrm{d}\hat{S}$. The exactness equations relative to this action form are:

$$\begin{cases} \dfrac{\partial \hat{S}}{\partial x}(x,y) = -f(x,y)^\top \dfrac{\partial g}{\partial x}(x,y) \\[2ex] \dfrac{\partial \hat{S}}{\partial y}(x,y) = -f(x,y)^\top \dfrac{\partial g}{\partial y}(x,y) + x^\top \end{cases}.$$

Any 1-form like $\boldsymbol{\alpha}_U = y \; \mathrm{d}x - \mathrm{d}U(x,y)$, being $U : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ a function, is an action form for our symplectic form $\omega = \mathrm{d}y \wedge \mathrm{d}x$. In fact, thanks to the topological properties of $\mathbb{R}^d \times \mathbb{R}^d$, all the action forms are constructed in this way. So, to each function $U = U(x, y)$ we can associate a primitive function $S_U$. The relation between $S_U$ and the original primitive function is

$$S_U(x, y) \quad = \quad S(x, y) - U \circ F(x, y) + U(x, y).$$

In the previous example we have taken $U(x, y) = x \cdot y$. If, for instance, $y = \nabla l(x)$ is an invariant graph, then we can take $U(x, y) = l(x)$, and the resulting action form is $\boldsymbol{\alpha}_l = (y - \nabla l(x))\, \mathrm{d}x$, which vanish on such a graph and on vertical vectors. Moreover, the corresponding primitive function is

$$S_l(x, y) \quad = \quad S(x, y) - l(f(x, y)) + l(x).$$

It is the function $\hat{\Phi}$ introduced in Section 6.1!

# G.2 Lagrangian foliations

## G.2.1 Whatever we need

As we have seen, the election of the action form determines the geometry of our phase space (given by the action form and the corresponding Lagrangian foliation) and the mechanics that we do on it (given by variational principles). We recall that dynamics is independent of such elections.

It seems that whatever we need in order to define variational principles for the orbits of a certain symplectomorphis is:

- a Lagrangian foliation, that is, a foliation whose leaves are Lagrangian manifolds;
- a Lagrangian manifold, transversal to the Lagrangian foliation, and which is the basis of such foliation;
- an associated 1-form, which vanish on the basis and acting on tangent vectors to the foliation.

There are several theorems which relate these ingredients, and generalize the canonical exact symplectic geometry of the cotangent bundle. In fact, they let us generalize the results that we have obtained. A survey of results is given in [61]. We have pick out a few ones.

## G.2.2 Some Darboux-Weinstein's theorems

The first ones are due to Weinstein [97], and extend Darboux's theorem.

**Theorem G.1** :

*Let $\mathcal{L}$ be a Lagrangian submanifold of a symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$ of dimension $2d$, $T^*\mathcal{L}$ be its cotangent bundle and $\boldsymbol{\alpha}_{\mathcal{L}}$ its Liouville form. Then:*

- *There exists a diffeomorphism $\psi$ from an open neighborhood $\mathcal{V}$ of $\mathcal{L}$ in $\mathcal{M}$ onto an open neighborhood $\psi(\mathcal{V})$ of the image of the zero-section in $T^*\mathcal{L}$ which satisfies the following properties:*
  1. *the restriction $\psi_{|\mathcal{L}}$ of $\psi$ to $\mathcal{L}$ is the zero-section $s_0$ of $T^*\mathcal{L}$;*
  2. *the diffeomorphism $\psi$ is a symplectomorphism, i.e., $\psi^* \mathrm{d}\boldsymbol{\alpha}_{\mathcal{L}} = \boldsymbol{\omega}$.*
- *If we are given a Lagrangian complement $E$ of $T\mathcal{L}$ in the symplectic vector bundle $T_{\mathcal{L}}\mathcal{N}$, which is the restriction of $T\mathcal{N}$ to $\mathcal{L}$, then:*

  *we may choose $\psi$ such that, for every point $x \in \mathcal{L}$, $T_x\psi(E_x) = VT_{\psi(x)}(T^*\mathcal{L})$.*
- *If we assume that our manifold $\mathcal{N}$ is equipped with a Lagrangian foliation, defined by a completely integrable Lagrangian subbundle $E$ of $T\mathcal{N}$ transverse to $\mathcal{L}$, then:*

  *we may choose $\psi$ such that it maps each leaf of the foliation $E_{|\mathcal{V}}$ into a fiber of $T^*\mathcal{L}$.*

We remark that if our symplectic manifold is exact and our Lagrangian manifold is also exact then the symplectomorphism $\psi$ is also exact. Using lifts of diffeomorphisms (see Section 7.2.2), the next result was easily proven in [61].

**Corollary G.1** *:*

*Let $(\mathcal{N}_1, \boldsymbol{\omega}_1)$ and $(\mathcal{N}_2, \boldsymbol{\omega}_2)$ be symplectic manifolds of the same dimension, $\mathcal{L}_1$ and $\mathcal{L}_2$ be Lagrangian submanifolds of $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, such that there exists a diffeomorphism $\phi$ from $\mathcal{L}_1$ onto $\mathcal{L}_2$. Then:*

*There exists a diffeomorphism $\psi$ from an open neighborhood $\mathcal{U}_1$ of $\mathcal{L}_1$ in $\mathcal{N}_1$ onto an open neighborhood $\mathcal{U}_2$ of $\mathcal{L}_2$ in $\mathcal{N}_2$ which satisfies:*

$$\psi_{|\mathcal{L}_1} = \phi, \;\; \psi^*\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1.$$

Next theorem by Guillemin and Sternberg [37] show us the relationship between Lagrangian foliations and action forms.

**Corollary G.2** *:*

*Let $\mathcal{L}$ be a Lagrangian submanifold of the symplectic manifold $(\mathcal{N}, \boldsymbol{\omega})$, and $E$ be a completely integrable Lagrangian subbundle of $T\mathcal{M}$ transverse to $\mathcal{L}$. Then, there exists an open neighborhood $\mathcal{V}$ of $\mathcal{L}$ in $\mathcal{M}$ upon which a unique* 1*-form $\boldsymbol{\alpha}$ is defined which satisfies the following properties:*

1. $\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{\omega}_{|\mathcal{V}}$,
2. $E_{\mathcal{V}} \subset \ker \boldsymbol{\alpha}$,
3. $\boldsymbol{\alpha}_{|\mathcal{L}} = 0$.

*The* 1*-form $\boldsymbol{\alpha}$ is said to be associated with $\mathcal{L}$ and $E$.*

Kostant, Guillemin and Sternberg [37] have proven a converse of the previous result. Independently, Cohen [25] has proven a similar result.

**Theorem G.2** *:*

*Let* $(\mathcal{N}, \boldsymbol{\omega})$ *be a symplectic manifold of dimension* $2d$*, and* $\boldsymbol{\alpha}$ *be an action form. We assume that the set of zeros of* $\boldsymbol{\alpha}$*,*

$$\mathcal{L} \;=\; \{z \in \mathcal{N} \mid \boldsymbol{\alpha}(z) = 0\},$$

*is a submanifold of dimension* $d$*. Then* $\mathcal{L}$ *is Lagrangian, and there exists an open neighborhood of* $\mathcal{L}$ *in* $\mathcal{N}$ *equipped with a Lagrangian foliation transverse to* $\mathcal{L}$ *such that* $\boldsymbol{\alpha}$ *is the* 1*-form associated to* $\mathcal{L}$ *and with this foliation.*

## G.3 Final discussion

As we have recalled in Section F.2, the Birkhoff normal form until degree $k$ of a diophantine torus $T$ for a symplectomorphism $F$ of a certain symplectic manifold $(N, \boldsymbol{\omega})$, is given by

$$\begin{cases} f(x,y) = \omega + x + \sum_{i=2}^{k} \nabla_y H_i(y) + O(y^k) \\ \\ g(x,y) = y(I + O(y^k)) \end{cases},$$

where each $H_i$ is a homogeneous polynomial of degree $i$. Then, the *torsion* of the torus $T$ is the quadratic form given by $H_2$ (or the symmetric matrix $\mathrm{D}^2 H_2$. Its inertia is independent of the way that we have moved our torus to its zero-section and the steps in the normal form. Then, we can say that our torus has degenerated, positive, negative, undefinite or null torsion. This is an intrinsic characteristic of the torus.

Hence, suppose our torus be positive. In a neighborhood of it we can write the dynamics as in the normal form until degree 2. In such coordinates, the torus is minimizing and the orbits on it are also minimizing. This is a local property. This extremal character depends on the vertical Lagrangian foliation and the Liouville form, which is the associated action form. All of this around our zero-section. Finally, if we go back to our initial torus, we obtain that it has a transversal Lagrangian foliation upon which the orbits of the torus are minimizing with respect to the variational principles induced by such a foliation. The problem is to choose this foliation.

For instance, in the examples given in Section B.2.1 relative to the quadratic standard map (similarly for the trigonometric standard map), the upper r.i.c. have positive torsion, and the lower ones have negative torsion. In the middle, although the r.i.c. are not graphs and they cross the non-monotone curve, may be their torsions have a type. May be only one has null torsion and it could be 'the last'. It should be interesting to adapt foliations to the folds of these curves, and check the extremal character of the orbits respect to these foliations. On the other side, we could take profit that the dynamical character of the orbits do not change by transformations of the phase space.

So then, we saw in the examples given in Appendix C, that the elliptic periodic orbits near the torus transformed to reflection hyperbolic-elliptic periodic orbit when the torus broke dowm. That is, two eigenvalues, which were on the unit circle, collide in the negative real axis and transformed into a reflection hyperbolic pair, giving a period doubling bifurcation. Another possibility of collision is the called Krein crunch, where two pairs of elliptic eigenvalues collide and transform into a hyperbolic quadruplet. Why this is not our case? We think that the reason is the following. As our torus have positive torsion, we can do the previous reduction. Periodic orbits are, of course, fixed points of a power of $F$ and the corresponding matrix $B$ must be positive definite if such orbits are near enough the torus, and complex hyperbolic quadruplets are not possible (Section 5.5.2). The relationship between extremal and dynamical character of the orbits must be more deeply studied. We think they are also related with the kind of breakdown, that is, with the kind of object which remains after such breakdown.

A more difficult problem correspond to undefinite or degenerated torsion. These tori do no let to obtain a priori inequalities as in Section B.1 and they produce more complicated dynamics, as Herman shown in [43]. It should be interesting to test the proportion of positive and negative eigenvalues that we obtain when we apply the MMS iteration. For instance, once we have done two steps in the normal form around our torus, we have

$$\begin{cases} f(x,y) = \omega + x + By + O(y^2) \\ \\ g(x,y) = y(I + O(y^2)) \end{cases},$$

where the symmetric matrix $B$ gives the torsion. Suppose it is non-degenerated. Then, the Hessian matrix associated to segments of orbit of length $n$ over our torus has constant entries are they are

$$H_{0,n+1} = \begin{pmatrix} 2B^{-1} & -B^{-1} & & & 0 \\ -B^{-1} & 2B^{-1} & -B^{-1} & & \\ & \ddots & \ddots & \ddots & \\ & & -B^{-1} & 2B^{-1} & -B^{-1} \\ 0 & & & -B^{-1} & 2B^{-1} \end{pmatrix}.$$

The eigenvalues of such a matrix are (see Section 5.5)

$$\sigma(H_{0,n+1}) = \bigcup_{j=1}^{n} (2 - c_j)\sigma\left(B^{-1}\right),$$

where the $c_j$ are cosinus. Then, the proportion of positive eigenvalues in the matrix $H_{0,n+1}$ is the same that in the matrix $B$ (we can also use the MMS iteration). Another question is which are the bifurcations of periodic orbits associated to the breakdown of these tori.

We think that it could be useful to experiment with different dynamics around an invariant Lagrangian torus. In Section 4.3 we show how to do this, and in Chapter 10 we proved that the algorithm produces convergent expansions. In fact, in this chapter also find another algorithm, getting a time-dependent Hamiltonian which produce such dynamics. In both of cases, an algebraic manipulator of Fourier-Taylor series is needed.

Finally, in order to study invariant Lagrangian manifolds with folds, that is, those that are not graphs, should be interesting to generate them with Morse families, so called phase functions. A phase function of a Lagrangian manifold defined into a cotangent bundle is similar to a generating function, but it contains additional parameters which let the folds (see [98] or [61] for details). Some results of this report can be extended to this more general context, but we have not found how to apply them.

# Notes and notations

# Notes on Differential Geometry

We shall use the next standard notations and results of Differential Geometry (see, for instance, [2]). For the sake of simplicity, all the objects (manifolds, vector fields, forms, etc) will be $C^\infty$.
Let $\mathcal{M}$ be a manifold of dimension $m$.

- $\mathcal{X}(\mathcal{M})$ is the set of *vector fields* on $\mathcal{M}$. A vector field $X \in \mathcal{X}(\mathcal{M})$ is a section of the *tangent bundle* $T\mathcal{M}$.
- $\mathcal{X}^*(\mathcal{M}) = \Omega^1(\mathcal{M})$ is the set of 1-*forms* or *Pfaffian forms* on $\mathcal{M}$. A 1-form is a section of the *cotangent bundle* $T^*\mathcal{M}$.
- $\Omega(\mathcal{M}) = \bigoplus_{k=0}^m \Omega^k(\mathcal{M})$ is the set of all *exterior differential forms* on $\mathcal{M}$. A *k-form* is an element of $\Omega^k(\mathcal{M})$, and it is a section of the vector bundle of *exterior k-forms on the tangent space* of $\mathcal{M}$, $\Lambda^k(\mathcal{M}) = \Lambda^k(T\mathcal{M})$.

1. **Vector fields and forms**
   - Vectors fields act on functions by derivation:
   $$X(f) = \mathrm{D}f(X).$$
   - (Pull-forward)
   $$(\phi\circ\psi)_*X = \phi_*\psi_*X.$$
   - (Pull-back)
   $$(\phi\circ\psi)^*\alpha = \psi^*\phi^*\alpha.$$
2. **Lie bracket**
   - The Lie bracket of two vector fields on $\mathcal{M}$ is defined by
   $$[X,Y]\,(f) = X(Y(f)) - Y(X(f)).$$
   Vector fields on $\mathcal{M}$ with the Lie bracket form a Lie algebra; that is, $[X,Y]$ is real bilinear, skew symmetric, and Jacobi's identity holds:
   $$[[X,Y]\,,Z] + [[Z,X]\,,Y] + [[Y,Z]\,,X] = 0.$$
   - For diffeomorphisms $\phi$, $\psi$:
   $$\phi_*[X,Y] = [\phi_*(X),\phi_*(Y)]\,.$$
3. **Exterior product**
   - The set of forms on $\mathcal{M}$, $\Omega(\mathcal{M})$, are a real associative algebra with $\wedge$ as multiplication. Furthermore,
   $$\alpha\wedge\beta = (-1)^{kl}\beta\wedge\alpha$$
   for $k$ and $l$-forms $\alpha$ and $\beta$, respectively.
   - For maps $\phi$, $\psi$:
   $$\phi^*(\alpha\wedge\beta) = \phi^*\alpha\wedge\phi^*\beta.$$

4. **Exterior derivative**

- For $\alpha$ a $k$-form we define a $(k+1)$-form $\mathrm{d}\alpha$ by

$$\begin{aligned}\mathrm{d}\alpha(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i X_i(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k)) + \\ &\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j]\,,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)\end{aligned}$$

- d is an antiderivation, that is, d is a real linear map on forms and

$$\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^k\alpha\wedge\mathrm{d}\beta$$

  for $\alpha$ a $k$-form. Moreover:

$$\mathrm{d}\mathrm{d}\alpha = 0.$$

- For a map $\phi$:

$$\phi^*\mathrm{d} = \mathrm{d}\phi^*.$$

5. **Poincaré's lemma**

   If $\mathrm{d}\alpha = 0$, then $\alpha$ is locally exact.

   That is, there exist a neighborhood about each point on which $\alpha = \mathrm{d}\beta$.

6. **Interior product**

- $\mathrm{i}_X\alpha$ is real bilinear in $X$,$\alpha$. Also $\mathrm{i}_X\mathrm{i}_X\alpha = 0$, and

$$\mathrm{i}_X\alpha\wedge\beta = \mathrm{i}_X\alpha\wedge\beta + (-1)^k\alpha\wedge\mathrm{i}_X\beta,$$

  for $\alpha$ a $k$-form. So then, fixed $X$, $\mathrm{i}_X$ is an antiderivation.

- For a diffeomorphism $\phi$:

$$\phi^*\mathrm{i}_X\alpha = \mathrm{i}_{\phi^*X}\phi^*\alpha.$$

7. **Lie derivative**

- For $\alpha$ a $k$-form and $X$ a vectorial field, $\mathrm{L}_X\alpha$ is a $k$-form given by

$$\mathrm{L}_X\alpha(X_1,\ldots,X_k) = X(\alpha(X_1,\ldots,X_k)) - \sum_{i=1}^{k}\alpha(X_1,\ldots,[X,X_i]\,,\ldots,X_k).$$

- $\mathrm{L}_X\alpha$ is real bilinear in $X$,$\alpha$, and

$$\mathrm{L}_X\alpha\wedge\beta = \mathrm{L}_X\alpha\wedge\beta + \alpha\wedge\mathrm{L}_X\beta.$$

  Hence, fixed $X$, $\mathrm{L}_X$ is a derivation.

- For a diffeomorphism $\phi$:

$$\phi^* \mathrm{L}_X \alpha \;=\; \mathrm{L}_{\phi^* X} \phi^* \alpha.$$

8. **Cartan's formula**

$$\mathrm{L}_X \alpha \;=\; \mathrm{d}\mathrm{i}_X \alpha + \mathrm{i}_X \mathrm{d}\alpha.$$

9. The following identities hold:

$$\begin{aligned}
\mathrm{L}_{fX} \alpha &= f \mathrm{L}_X \alpha + \mathrm{d}f \wedge \mathrm{i}_X \alpha, \\
\mathrm{L}_{[X,Y]}\, \alpha &= \mathrm{L}_X \mathrm{L}_Y \alpha - \mathrm{L}_Y \mathrm{L}_X \alpha, \\
\mathrm{i}_{fX} \alpha &= f \mathrm{i}_X \alpha = \mathrm{i}_X f \alpha, \\
\mathrm{i}_{[X,Y]}\, \alpha &= \mathrm{L}_X \mathrm{i}_Y \alpha - \mathrm{i}_Y \mathrm{L}_X \alpha, \\
\mathrm{L}_X \mathrm{d}\alpha &= \mathrm{d}\mathrm{L}_X \alpha, \\
\mathrm{L}_X \mathrm{i}_X \alpha &= \mathrm{i}_X \mathrm{L}_X \alpha.
\end{aligned}$$

# Notes on symmetric matrices

As we shall use several properties of symmetric matrices, and specially of positive definite ones, we shall recall some definitions (see [45]).

- **Eigenvalues of a symmetric matrix**

  All the eigenvalues of a symmetric matrix are real, and moreover, it diagonalizes via an orthogonal matrix. That is, if $A$ is a symmetric matrix, there exist a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$, given by the eigenvalues, and a $d \times d$ matrix $U$ satisfying:

  $$UAU^\top = \Lambda, U^\top U = I_d.$$

  The *spectral radius* is $\rho(A) = \max_i |\lambda_i|$.

  If all the eigenvalues of $A$ are positive we say that $A$ is *positive definite*, and we also define $\mu(A) = \min_i \lambda_i$.

- **Inertia, index and signature**

  - The *inertia* of $A$ is the ordered triple

    $$\mathrm{i}(A) \quad = \quad (\mathrm{i}_+(A), \mathrm{i}_-(A), \mathrm{i}_0(A))$$

    of numbers of positive, negative and zero eigenvalues of the matrix $A$, respectively, all counting multiplicity.
  - The *rank* of $A$ is the number of non-zero eigenvalues:

    $$\mathrm{r}(A) \quad = \quad \mathrm{i}_+(A) + \mathrm{i}_-(A).$$

    If the rank coincides with the dimension, that is $\mathrm{i}_0(A) = 0$, we shall say that the matrix is *non degenerated*. Otherwise we shall say that it is *degenerated*.
  - The *signature* of $A$ is the difference

    $$\mathrm{s}(A) \quad = \quad \mathrm{i}_+(A) - \mathrm{i}_-(A).$$

  - If $\mathrm{i}_-(A) = 0$ we shall say that the matrix $A$ is *positive semidefinite*, and if, moreover, $\mathrm{i}_0(A) = 0$, we shall say that it is *positive definite*. If $\mathrm{i}_+(A) = 0$ we shall say that the matrix $A$ is *negative semidefinite*, and if, moreover, $\mathrm{i}_0(A) = 0$, we shall say that it is *negative definite*. Otherwise we shall say that $A$ is undefinite.
  - Finally, the *index* of a non degenerated matrix $A$ is the number of negative eigenvalues.

- **The Loewner partial order**

  In the space of $d \times d$ real symmetric matrices, we consider the next orders:

  - $A \preceq B \Leftrightarrow v^\top A v \leq v^\top B v, \ \forall v \in \mathbb{R}^d \setminus \{0\}$;

- $A \prec B \Leftrightarrow v^\top A v < v^\top B v, \ \forall v \in \mathbb{R}^d \setminus \{0\}$.

Meanwhile $\preceq$ is an order relation (the *Loewner partial order*), $\prec$ does not means $\preceq$ and $\neq$. So, $A \succ 0$ means that $A$ is positive definite, and $A \succeq 0$ means that $A$ is positive semidefinite.

- **Euclidean norm of a symmetric matrix**

If we consider the Euclidean norm on $\mathbb{R}^d$, $\|v\|_2 = \sqrt{\sum_{i=1}^d v_i^2}$, then the Euclidean norm of a symmetric matrix is its spectral radius:

$$\|A\|_2 = \sup_{v \neq 0} \frac{\|Av\|_2}{\|v\|_2} = \rho(A).$$

- **Some formulae**
  - $A \preceq \rho(A) I_d$ ;
  - $B_1 \preceq A \preceq B_2 \Rightarrow \rho(A) \leq \max(\rho(B_1), \rho(B_2))$;
  - $0 \preceq A \Rightarrow \mu(A) I_d \preceq A$;
  - $0 \prec A \Rightarrow 0 \prec A^{-1}$;
  - $0 \prec A \preceq B \Rightarrow 0 \prec B^{-1} \preceq A^{-1}$.

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