Master: Functional Analysis and PDE

September 2015. List 1

Comment: You have to present at least 7 exercises of the list. These exercises have to be presented before the mid-night of Thursday October 1 in my mailbox. Do not send the exercises by mail.

1) If E and F are Banach spaces, prove that $(\mathscr{L}(E,F), ||\cdot||)$ is also a Banach space.

2) a) Prove that the space c_0 of scalar sequences that converge to zero is a Banach space endowed with the topology given by

$$||c|| = \sup_n |c_n|.$$

b) Prove that the space

$$C_o(\mathbb{R}^n) = \{F : \mathbb{R}^n \to \mathbb{K} : \lim_{||x||\to\infty} F(x) = 0\}$$

is a Banach space endowed with the topology given by

$$||F|| = \sup_{x\in\mathbb{R}^n} |F(x)|.$$

3) a) Is the space ℓ^p endowed with the norm in ℓ^∞ a Banach space?.
b) Is the space $(C([0,1]), \|\cdot\|_1)$ with $\|f\|_1 = \int_0^1 |f(x)|dx$ a Banach space? Is it true that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent norms on $C([0,1])$, where $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$?

4) (a) Let $a,b \in \mathbb{R}$ and let $1 \leq p_1 \leq p_2$. Prove that the identity operator $id : L^{p_2}(a,b) \to L^{p_1}(a,b)$ is continuous.
(b) Prove that in the case of sequences spaces $id : \ell^{p_1} \to \ell^{p_2}$ is continuous.
(c) Prove that $id : L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$, for every $p_1 \leq p \leq p_2$.

5) a) Prove that if $(f_n)_n \subset L^p$ for some $1 \leq p < \infty$ and $f = L^p - \lim f_n$, then there exists a subsequence $(n_k)_k$ such that f_{n_k} converges to f at almost every point.
b) Is it true that if (f_n) converges to f at almost every point, then $f = L^p - \lim f_n$?

6) If $\tau_y(f)(x) = f(x+y)$ is the translation operator, then prove that

$$\lim_{y\to 0} \tau_y(f) = f$$

with convergence in $L^p(G)$ and deduce that, if K is an approximation of the identity in $\mathbb{R}^n$, then $K_\lambda * f$ converges to f in $L^p(\mathbb{R}^n)$.

7) a) Prove that the space $C_0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.
b) Prove that the space of continuous functions in $[0,1]$ vanishing at 0 and 1 are dense in $L^2([0,1])$.

8) Given the Fejer's kernel

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)\pi x)}{\sin \pi x}\right)^2 = \frac{1}{N+1}\sum_{k=0}^{N}\left(\sum_{m=-k}^{k} e^{2\pi i m x}\right),$$

prove that:
a) $\int_{-1/2}^{1/2} F_N(x)dx = 1$,
b) Given $0 < \delta < 1/2$, prove that $\lim_{N\to\infty} \int_\delta^{1/2} F_N(x)dx = 0$.
c) Prove that, for every $f \in C(\mathbb{T})$, $f * F_N$ converges uniformly to f.
d) Deduce that the subspace generated by the trigonometric polynomial is dense in $C(\mathbb{T})$.

9) Using exercise 8), prove that the trigonometric system $\{e^{2\pi imx} : m\mathbb{Z}\}$ is a basis in $L^2(\mathbb{T})$.

10) Find the Volterra integral equation that solve the following Cauchy problem and give an expression of the solution:
a)
$$u'' - \frac{t}{2}u' - u + t^2 = 0, \quad u(0) = 0, \quad u'(0) = 1.$$
b)
$$u'' - tu - 2t = 0, \quad u(0) = 0, \quad u'(0) = 0.$$

11) Prove whether the following operators are continuous and, if this is the case, compute its norm and study if it is attained:
(a) $T : (C[0,1], \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Tf = f(a)$, with $a \in [0,1]$.
(b) $T : (C[0,1], \|\cdot\|_\infty) \to (C^1[0,1], \|\cdot\|_{C^1})$ defined by $Tf(x) = \int_0^x f$, where $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$.
(c) $T : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ defined by $Tf(x) = f'$.
(d) $T : (C^1[0,1], \|\cdot\|_{C^1}) \to (C[0,1], \|\cdot\|_\infty)$ defined by $Tf(x) = f'$.
(e) $T : (\ell^2, \|\cdot\|_2) \to (\ell^1, \|\cdot\|_1)$ defined by $Tx = (x_n/n)_n$.
(f) $T : (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{R}$ defined by $Tf = \int_0^1 f(x)dx - \int_{-1}^0 f(x)dx$.

12) Let K be a compact set and let us consider the Banach space $C(K)$ with the usual norm.
a) Prove that if T is a linear functional such that it is positive (that is, $Tf \geq 0$ for every $f \geq 0$), then T is continuous.
b) Prove that if T is a positive linear functional such that $T1 = 1$, then $\|T\|_{C(K)'} = 1$.

13) Given a compact set K and an open set G such that $K \subset G$, construct an Urysohn function $\rho \in C^\infty$.

14) Let $B = \overline{B}(0,1)$ be the closed ball of $C([0,1])$ with the standard norm $\|\cdot\|_\infty$ and let $\|f\|_1 = \int_0^1 |f(x)|dx$.
a) Prove that B is closed in $(C([0,1]), \|\cdot\|_1)$.
b) Prove that $C([0,1]) = \cup_{n=1}^\infty nB$, where $nB = \{nf; f \in B\}$.
c) Is the interior of B in $(C([0,1]), \|\cdot\|_1)$ the empty set?
d) Can we apply Baire's theorem to deduce c)? Explain.

15) a) Prove that ℓ^∞ and ℓ^1 are not isomorphic.
b) Let $(a_n)_n$ be a sequence of positive numbers ($a_n > 0$) such that $\sum_n a_n < \infty$. Prove that there exists (z_n) such that $\overline{\lim}_n z_n = +\infty$ such that $\sum_n z_n a_n < \infty$.
Hint: Consider the operator
$$T : \ell^\infty \to \ell^1$$
defined by $T((y_n)_n) = (a_n y_n)_n$ and use the open mapping theorem and a).