Master: Functional Analysis and PDE

October 2015. List 2

Comment: You have to present at least 7 exercises of the list. These exercises have to be presented before the class of Monday 9 of November. Do not send the exercises by mail.

Remark: On Wednesday 4 of November, you will have a "Solving problem by yourself" class with Prof. Carlos Domingo. On Monday 9, we shall have the last theoretical class of this second part.

1) Compute both the classical and the derivative in the sense of distribution of the following functions:
i) $f(x) = \log|3x-5|, x \in \mathbb{R}$
ii) $f : (-1,1) \to \mathbb{R}$ defined by $f(x) = x^2 \sin(1/x^2)$ if $x \neq 0$ and $f(0) = 0$.
2) Which of the following maps define a distribution?
a) $(\Lambda_1, \varphi) = \sum_{k=0}^{\infty} \varphi(k)$.
b) $(\Lambda_2, \varphi) = \sum_{k=0}^{\infty} \varphi^{k)}(\sqrt{2})$.
c) $(\Lambda_3, \varphi) = \sum_{k=0}^{\infty} \frac{\varphi^{k)}(k)}{k}$.
d) $(\Lambda_4, \varphi) = \int_{\mathbb{R}} \varphi^2(x)dx$.
e) $f_\alpha(x) = \frac{1}{|x|(1+\log^2|x|)^\alpha}$, $\alpha \in \mathbb{R}$.
3) Compute the following limits in $D'(\mathbb{R})$:
a) $\lim_{t\to\infty} t^2 x\cos(tx)$
b) $\lim_{t\to\infty} t^2|x|\cos(tx)$.
4) Compute the following limit in $D'(\mathbb{R})$:

$$\lim_{t\to\infty} \frac{\sin(tx)}{x}.$$

5) Find a distribution in Λ in $D'(\mathbb{R}^3)$ such that $\partial_x\partial_y\partial_z\Lambda = \delta_{(2,4,6)}$.
6) Let $n \geq 3$ and let $E(x) = |x|^{2-n}$. Prove that:
a) E is a distribution.
b) For every $\varphi \in \mathscr{D}$,

$$\lim_{\varepsilon\to 0} \int_{B(0,\varepsilon)} E(x)\varphi(x)dx = 0,$$

7) Prove that if $\Lambda_i \to \Lambda$ in $D'(\Omega)$ and $g_i \to g$ in $C^\infty(\Omega)$, then $(g_i\Lambda_i) \to g\Lambda$ in $D'(\Omega)$.
8) Prove whether the following maps define a distribution:
a) $f(x) = \frac{1}{x}$ with $\Omega = \mathbb{R}$.
b) $\Lambda(\varphi) = \int_0^\infty \frac{\varphi(x)-\varphi(-x)}{x}dx$.
9) Let $f_j \in L^1_{loc}(\mathbb{R})$ and assume that $\lim_j f_j(x) = f(x)$ at almost every point. Suppose that, for every compact set K, there exists $g_K \in L^1$ such that $|f_j(x)| \leq g_K(x)$ at almost every $x \in K$. Prove that f_j converges to f in the sense of distributions and that $f \in L^1_{loc}$.
10) a) Is $\sum_{n=1}^{\infty} \delta_{2^{-n}}$ a well-defined distribution?

b) Show that the distribution $\sum_{n=1}^{\infty} \frac{1}{n^2}\delta_{2^{-n}}$ is well-defined and determine its primitive with support in $[0,\infty)$. Which is its support?

11) Compute the Fourier transform of the distribution $p.v\frac{1}{x}$ and deduce that the Hilbert transform define by $Hf = p.v\frac{1}{x} * f$ is bounded in L^2.

12) If u is a tempered distribution such that $\Delta u = \delta$. Prove that $\hat{u}(\xi) = -|\xi|^{-2}$ in $\mathbb{R}^n$ with $n \geq 3$ and show that u is in fact a function in $L^\infty + L^2$.

13) Let $n \geq 3$ and let $E(x) = |x|^{2-n}$. Prove that:

a) E is a tempered distribution.

b) Compute ΔE whenever exists.

c) Show that $\Delta E = \delta$ in the sense of distributions.

Recall Green's identity: If R is a bounded domain with smooth boundary S and f and g are C^1 functions on $\mathbb{R}$, then:

$$\int_R (f\Delta g - g\Delta f)dx = \int_S (f\delta_\nu g - g\delta_\nu f)d\sigma,$$

where δ_ν is the directional derivative with respect to the outward normal vector to R.

14) Given $g \in L^1(\mathbb{R})$, set $f(x) = \int_{-\infty}^{x} g(t)dt$.

(i) Prove that f is a distribution.

(ii) Compute f' in the sense of distributions.

15) Given a function $\rho \in C_c(\mathbb{R}^n)$ such that $\int \rho = 1$, compute the following limit in the sense of distributions $\lim_{\varepsilon \to 0} \varepsilon^{-n}\rho(x/\varepsilon)$.