Uniformly bounded sets of orthonormal polynomials on the sphere

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With RW-sequences one can show that inner functions do exist (Aleksandrov's second proof, the first is from 81).

It is not known if there exists a uniformly bounded orthonormal basis of holomorphic polynomials in $\mathbb{S}^{2m-1} \subset \mathbb{C}^m$ for $m \geq 3$.

Shiffman's result (14)

Shiffman constructs a uniformly bounded orthonormal system of sections of powers L^N of a positive holomorphic line bundle over a compact Kähler manifold M (i.e. a uniformly bounded orthonormal system of elements of $H^0(M, L^N)$).

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He raises the question whether using kernels peaking at Fekete points one may increase the size of the uniformly bounded orthonormal system of sections.

For $M = \mathbb{CP}^{m-1}$ and L the hyperplane section bundle $\mathcal{O}(1)$ with the Fubini-Study metric one can identify

$$H^0(\mathbb{CP}^{m-1}, L^N) \equiv \boxed{\begin{array}{l}\text{space of homogeneous holomorphic}\\ \text{polynomials of degree } N \text{ on } \mathbb{C}^m\end{array}}$$

i.e.

$$H^0(\mathbb{CP}^1, L^N) \equiv \mathcal{P}_N.$$

The L^p norm of a section is the corresponding norm of the polynomial over the sphere $\mathbb{S}^{2m-1} \subset \mathbb{C}^m$.

Theorem

Let L be a Hermitian holomorphic line bundle over a compact Kähler manifold M with positive curvature. Then for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that for any $N \in \mathbb{Z}^+$, we can find orthonormal holomorphic sections:

$$s_1^N, \dots, s_{n_N}^N \in H^0(M, L^N), \qquad n_N \geq (1-\varepsilon) \dim H^0(M, L^N),$$

such that $\|s_j^N\|_\infty \leq C_\varepsilon$ for $1 \leq j \leq n_N$ and for all $N \in \mathbb{Z}^+$.

Let (M, g) be a compact two-point homogeneous Riemannian manifold of dimension $m \geq 2$. The (discrete) spectrum of the Laplace-Beltrami operator is a sequence of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty,$$

and we consider the corresponding orthonormal basis of eigenvectors ϕ_i (so we have $\Delta \phi_i = -\lambda_i \phi_i$).

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$$E_L = \mathrm{span}_{\lambda_i \leq L} \{\phi_i\}.$$

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Consider the following subspaces of $L^2(M)$:

$$E_L = \mathrm{span}_{\lambda_i \leq L} \{\phi_i\}.$$

We denote $\dim E_L = k_L$. The reproducing kernels of E_L are given by

$$B_L(z, w) = \sum_{i=1}^{k_L} \phi_i(z) \overline{\phi_i(w)}.$$

Observe that $\|B_L(\cdot, w)\|^2_{L^2(M)} = B_L(w, w)$. Hörmander (68) proved that $k_L \sim B_L(w, w) \sim L^m$.

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We denote by $b_L(z, w)$ the normalized reproducing kernels.

The main example is the sphere $M = \mathbb{S}^m$, where the ϕ_i are spherical harmonics and the spaces E_L are the restriction to the sphere of the space of polynomials in $\mathbb{R}^{m+1}$.
Our result is the following:

Theorem

Given $\varepsilon > 0$ and $L \in \mathbb{Z}^+$ there exist $C_\varepsilon > 0$ and a set $\{s_1^L, \dots, s_{n_L}^L\}$ of orthonormal functions in E_L with $n_L \geq (1-\varepsilon) \dim E_L$ such that $\|s_j^L\|_{L^\infty(M)} \leq C_\varepsilon$, for all $L \in \mathbb{Z}^+$ and $1 \leq j \leq n_L$.

Interpolation and Riesz sequences

For degree L we take n_L points in M

$$\mathcal{Z}(L) = \{z_{L,j} \in M : 1 \leq j \leq n_L\}, \quad L \geq 0,$$

and assume that $n_L \to \infty$ as $L \to \infty$. This yields a triangular array of points $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 0}$ in M.

Definition

$\mathcal{Z}$ is interpolating if and only if the normalized reproducing kernel of E_L at the points $\mathcal{Z}(L)$ form a Riesz sequence i.e.

$$C^{-1} \sum_{j=1}^{n_L} |a_{Lj}|^2 \leq \int_{\mathbb{S}^d} \left| \sum_{j=1}^{n_L} a_{Lj} b_L(z, z_{L,j}) \right|^2 d\sigma(z) \leq C \sum_{j=1}^{n_L} |a_{Lj}|^2,$$

for any $\{a_{Lj}\}_{L,j}$ with $C > 0$ independent of L. Observe $n_L \leq k_L$.

$\mathcal{Z}$ is interpolating if and only if the Gramian matrix

$$G = (G_{ij})_{i,j} = (\langle b_L(\cdot, z_{L,i}), b_L(\cdot, z_{L,j})\rangle)_{i,j} = (L^{-m/2} b_L(z_{L,i}, z_{L,j}))_{i,j}$$

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The idea is to take $G^{-1/2} = (G_{ij}^{-1/2})_{ij}$ and

$$\sum_j G_{ij}^{-1/2} b_L(, z_{L,j}) \in E_L$$

for $i = 1, \ldots, n_L$ are orthonormal.

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If one has "good estimates" for the kernel one can see that for $\ell = 1, \ldots, n_L$

$$s_\ell^L = \frac{1}{\sqrt{n_L}} \sum_j \zeta^{i\ell} \sum_j G_{ij}^{-1/2} b_L(, z_{L,j}) \in E_L$$

where $\zeta = e^{2\pi i/n_L}$ is bounded and ON.

Problems to extend the results:

It is also known, see (Lev-Ortega-Cerdà (10)) that there are no Riesz basis of reproducing kernels in the space of sections of $H^0(M, L^N)$. Thus this approach cannot provide uniformly bounded orthonormal basis of sections in $H^0(M, L^N)$.

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For the sphere $\mathbb{S}^m$:

Theorem (Bannai, Damerell 79)

There is no array $\mathcal{Z}$ such that $\{K_L(\cdot, z)/\|K_L(\cdot, z)\|\}_{z\in\mathcal{Z}(L)}$ is an orthonormal basis for the space of spherical harmonics of degree at most L, $m \geq 2$ and $L \geq 3$.

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Open problem

It is not known if there are Riesz basis of reproducing kernels in the spaces of spherical harmonics.

How to get a "maximal" Riesz sequence

Fekete points (extremal systems)

Let $\{\psi_1, \ldots, \psi_{k_L}\}$ be any basis in E_L. A set of points $x_1^*, \ldots, x_{k_L}^* \in M$ such that

$$|\det(\psi_i(x_j^*))_{i,j}| = \max_{x_1,\ldots,x_{k_L} \in M} |\det(\psi_i(x_j))_{i,j}|$$

is a Fekete array of points of degree L for M.

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is a Fekete array of points of degree L for M.

The next result provides us with Riesz sequences of reproducing kernels with cardinality almost optimal.

Theorem (M., Ortega-Cerdà, Pridhnani, Lev)

Given $\varepsilon > 0$ let $L_\epsilon = \lfloor (1-\varepsilon)L \rfloor$ and

$$\mathcal{Z}_\varepsilon(L) = \mathcal{Z}(L_\epsilon) = \{z_{L_\epsilon,1}, \ldots, z_{L_\epsilon,k_{L_\epsilon}}\},$$

where $\mathcal{Z}(L)$ is a set of Fekete points of degree L. Then the array $\mathcal{Z}_\varepsilon = \{\mathcal{Z}_\varepsilon(L)\}_{L\geq 0}$ is interpolating i.e. $\{b_L(\cdot, z)\}_{z\in\mathcal{Z}_\varepsilon(L)}$ form a Riesz sequence.

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For compact complex manifolds one can use directly these kernels to continue with the construction of flat sections. In the real setting the off-diagonal decay of the reproducing kernels is not fast enough. So we need to introduce better kernels.

Changing the kernel

Given $0 < \varepsilon \leq 1$ let $\beta_\varepsilon : [0, +\infty) \mapsto [0,1]$ be a nonincreasing $\mathcal{C}^\infty$ function such that $\beta_\varepsilon(x) = 1$ for $x \in [0, 1-\varepsilon]$ and $\beta_\varepsilon(x) = 0$ if $x > 1$. We consider the following **Bochner-Riesz** type kernels

$$B_L^\varepsilon(z,w) = \sum_{k=1}^{k_L} \beta_\varepsilon\left(\frac{\lambda_i}{L}\right) \phi_k(z)\overline{\phi_k(w)}.$$

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When $\varepsilon = 0$ we “recover” the reproducing kernel for E_L. Observe that $\|B_L^\varepsilon(\cdot, w)\|_2^2 \sim L^m$ for any $w \in M$.
These kernels have better pointwise estimates (Filbir-Mhaskar (10))

$$|B_L^\varepsilon(z, w)| \lesssim \frac{L^m}{(1 + Ld(z, w))^N}, \qquad z, w \in M$$

where one can take any $N > m$.

One can replace the reproducing kernels by the Bochner-Riesz type and still get a Riesz sequence:

Lemma

Given $\varepsilon > 0$ there exist a set of $n_{L,\varepsilon}$ points $\{z_j\}_{j=1,\dots,n_{L,\varepsilon}}$ with $n_{L,\varepsilon} \geq (1-\varepsilon) \dim E_L$ such that the normalized Bochner-Riesz type kernels $\{b_L^{\varepsilon}(\cdot, z_j)\}_{j=1,\dots,n_{L,\varepsilon}}$ form a Riesz sequence (uniformly in L).

Jaffard (90)

Let (X, d) be a metric space such that for all $\epsilon > 0$ there exists C_ϵ such that

$$\sup_{s\in X}\sum_{t\in X} exp(-\epsilon d(s,t)) \leq C_\epsilon.$$

Suppose that

$$\sup_{s\in X}\sum_{t\in X}\frac{1}{|1+d(s,t)|^N} < \infty,$$

and for $\alpha > N$ the matrix $A = (A(s,t))_{s,t\in X}$ is such that

$$|A(s,t)| \leq \frac{C}{|1+d(s,t)|^\alpha}.$$

Then, if A is invertible as an operator in ℓ^2 the matrix A^{-1} (and also $A^{-1/2}$) satisfies the same kind of bound and therefore it is bounded in ℓ^p for $1 \leq p \leq \infty$ by Schur's Lemma.

We define the $n_{L,\varepsilon} \times n_{L,\varepsilon}$ Gramian matrix

$$\Delta = (\Delta_{ij})_{i,j=1,\dots,n_{L,\varepsilon}}, \quad \text{where} \quad \Delta_{ij} = \langle b_L^\varepsilon(\cdot, z_i), b_L^\varepsilon(\cdot, z_j) \rangle,$$

where the points z_j for $j = 1, \dots, n_{L,\varepsilon}$ are given by the previous Lemma.
This matrix defines a bounded operator in ℓ^2 which is also bounded below (uniformly in L).
Because of the structure of the regularized kernel we have the following estimate for the entries of the Gramian:

$$|\Delta_{ij}| = \frac{1}{k_L} \left| \int_M B_L^\varepsilon(z, z_i) \overline{B_L^\varepsilon(z, z_j)} dV(z) \right| \lesssim \frac{1}{(1 + Ld(z_i, z_j))^N}.$$

Then as

Proposition

For $\{z_j\} \subset M$ uniformly separated

$$\sup_i \sum_j \frac{1}{(1 + Ld(z_i, z_j))^N} \lesssim 1.$$

One can apply Jaffard's result getting the estimates

$$\|\Delta^{-1/2}\|_{\ell^\infty \to \ell^\infty} \leq \max_i \sum_j |\Delta_{ij}^{1/2}| \lesssim 1.$$

Denote $\Delta^{-1/2} = (B_{ij})$ and define the orthonormal set of functions from E_L

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$$s_i^L = \frac{1}{\sqrt{n_{L,\varepsilon}}} \sum_j \zeta^{ji} \Psi_j^L,$$

where $\zeta = e^{2\pi i/n_{L,\varepsilon}}$. They are orthonormal because

$$\langle s_i^L, s_k^L \rangle = \frac{1}{n_{L,\varepsilon}} \sum_{j=1}^{n_{L,\varepsilon}} \zeta^{j(i-k)} = \delta_{ik}, \;\; 1 \leq i, k \leq n_{L,\varepsilon}.$$

To verify that the s_i^L are indeed uniformly bounded. Define the linear maps

$$F_L : \mathbb{C}^{n_{L,\varepsilon}} \longrightarrow E_L, \quad v = (v_i) \mapsto \sum_j v_j b_L^{\varepsilon}(\cdot, z_j).$$

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By the previous Proposition

$$\sup_{z \in M} \sum_j |b_L^{\varepsilon}(z, z_j)| \lesssim L^{-m/2} \sup_{z \in M} \sum_j |B_L^{\varepsilon}(z, z_j)| \lesssim L^{m/2}.$$

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So, finally we get

$$\|s_i^L\|_{L^\infty(M)} \leq \frac{1}{\sqrt{n_{L,\varepsilon}}} \|F_L\|_{\ell^\infty \to L^\infty(M)} \|\Delta^{-1/2}\|_{\ell^\infty \to \ell^\infty} \lesssim 1,$$

for all $L \in \mathbb{Z}^+$ and $1 \leq i \leq n_{L,\varepsilon}$.

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