Quantitative description of the dynamics in resonant islands

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Seville 1–3 Setember 2010

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Introduction

We consider a one-parameter family of maps $F_\delta : \mathcal{U} \to \mathbb{R}^2$, $\mathcal{U} \subset \mathbb{R}^2$ domain, such that

1. F_δ real analytic in the (x, y)-coordinates of $\mathcal{U}$,
2. $\det DF_\delta(x, y) = 1$, for all $(x, y) \in \mathbb{R}^2$ and for all $\delta \in \mathbb{R}$, (APMs)
3. F_δ has a fixed point E_0 that will be assumed to be at the origin $\forall \delta \in \mathbb{R}$,
4. spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = \exp(2\pi i \alpha)$, $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$.

$\rightarrow$ We focus on a **semi-global** description of the phase space: we want to describe the dynamics in the **resonant chains** emanating from (but **relatively far** from) the elliptic fixed point E_0.

$\rightarrow$ We are interested in a topological/qualitative description but **our goal is to obtain quantitative information** of the system: size of the chaotic zones, distance to invariant curves, measure of the stability regions within the chaotic zones, transport properties,...

Hénon map

Through the presentation the Hénon map will be used as a paradigm of APM

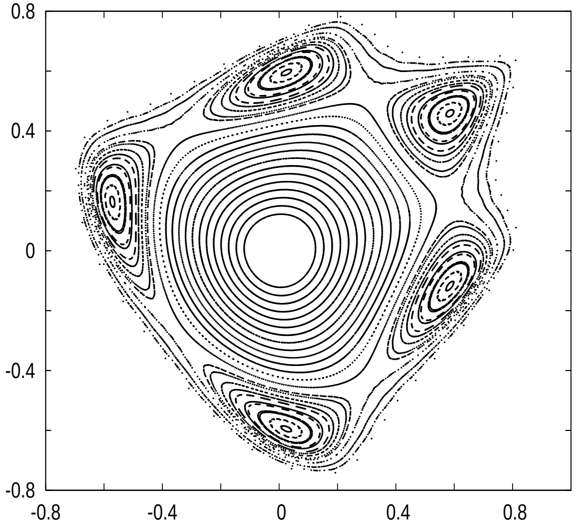
$$H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2), \qquad \alpha \in (0, 1/2)$$

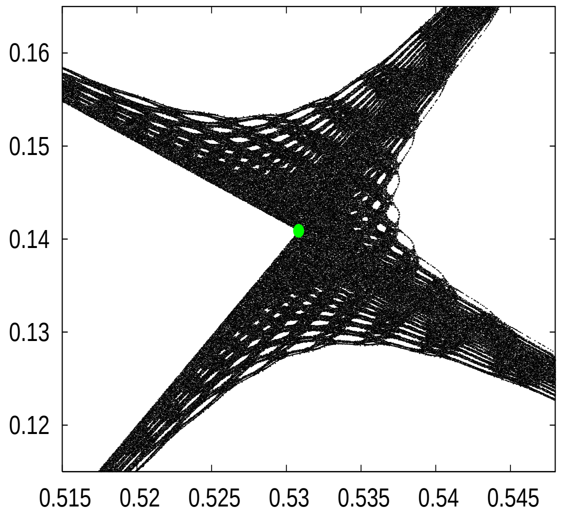
- It has two fixed points:

  the origin is an elliptic fixed point E_0,

  the point $P_h = (2\tan(\pi\alpha), 2\tan^2(\pi\alpha))$ is a hyperbolic fixed point.

- Reversible with respect to $y = x^2/2$ and $y = \tan(\pi\alpha)x$.





$$\alpha = 0.21,$$
$$\sigma_- = \mathcal{O}(10^{-12}),$$
$$\sigma_+ = \mathcal{O}(10^{-3}).$$

PART I

A Hamiltonian model describing the dynamics in resonant islands:

- BNF around the elliptic point
- Interpolation of the BNF
- Localisation around the resonant strip

BNF

F_δ one-parameter δ-family of APMs with $F(E_0) = E_0$ elliptic fixed point.

Spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = e^{2\pi i \alpha}$, $\alpha = q/m + \delta$, δ small enough.

(x, y)-cartesian coord., $(z, \bar{z})$-complex coord. ($z = x + iy$, $\bar{z} = x - iy$).

The Birkhoff NF to order m around E_0 can be expressed as

$$\mathrm{BNF}_m(F)(z) = R_{2\pi\frac{q}{m}} \left(\underbrace{e^{2\pi i \gamma(r)} z}_{\text{unavoidable res.}} + \underbrace{i \bar{z}^{m-1}}_{m\text{-order res.}} \right) + R_{m+1}(z, \bar{z}),$$

where

$$\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \ldots + b_s r^{2s}, \quad r = |z|,$$

being

$s = [(m-1)/2]$,

$b_i \in \mathbb{R}$ are the so-called Birkhoff coefficients,

$R_{m+1}(z, \bar{z})$ denotes the remainder which is of $\mathcal{O}(m+1)$.

Interpolating flow of the BNF

(I,φ)-Poincaré variables ($z=\sqrt{2I}\exp(i\varphi)$).

$$\mathcal{H}_{nr}(I)=\pi\sum_{n=0}^{s}\frac{b_n}{n+1}(2I)^{n+1} \quad \text{and} \quad \mathcal{H}_r(I,\varphi)=\frac{1}{m}(2I)^{\frac{m}{2}}\cos(m\varphi).$$

Let r_* such that $\gamma(r_*)=0$, that is $r_*\approx(-b_0/b_1)^{1/2}$, $b_0=\delta$.

$\rightarrow$ The flow ϕ generated by the Hamiltonian

$$\mathcal{H}(I,\varphi)=\mathcal{H}_{nr}(I)+\mathcal{H}_r(I,\varphi)$$

interpolates K with an error of order $m+1$ with respect to the $(z,\bar{z})$-coordinates, that is,

$$K(I,\varphi)=\phi_{t=1}(I,\varphi)+\mathcal{O}\left(I^{\frac{m+1}{2}}\right).$$

If we assume $b_1\neq 0$ this approximation holds in an annulus centred in the resonance radius r_* of width $r_*^{1+\nu}$, for $\nu>0$.

Description of resonances

Generic case: $\alpha = q/m + \delta$, $m > 5$, δ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then F **has** a resonant island of order m.
- The resonant zone is determined by **two periodic orbits** of period m located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.

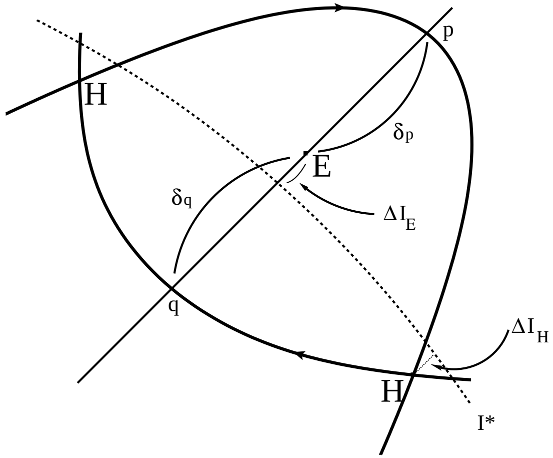
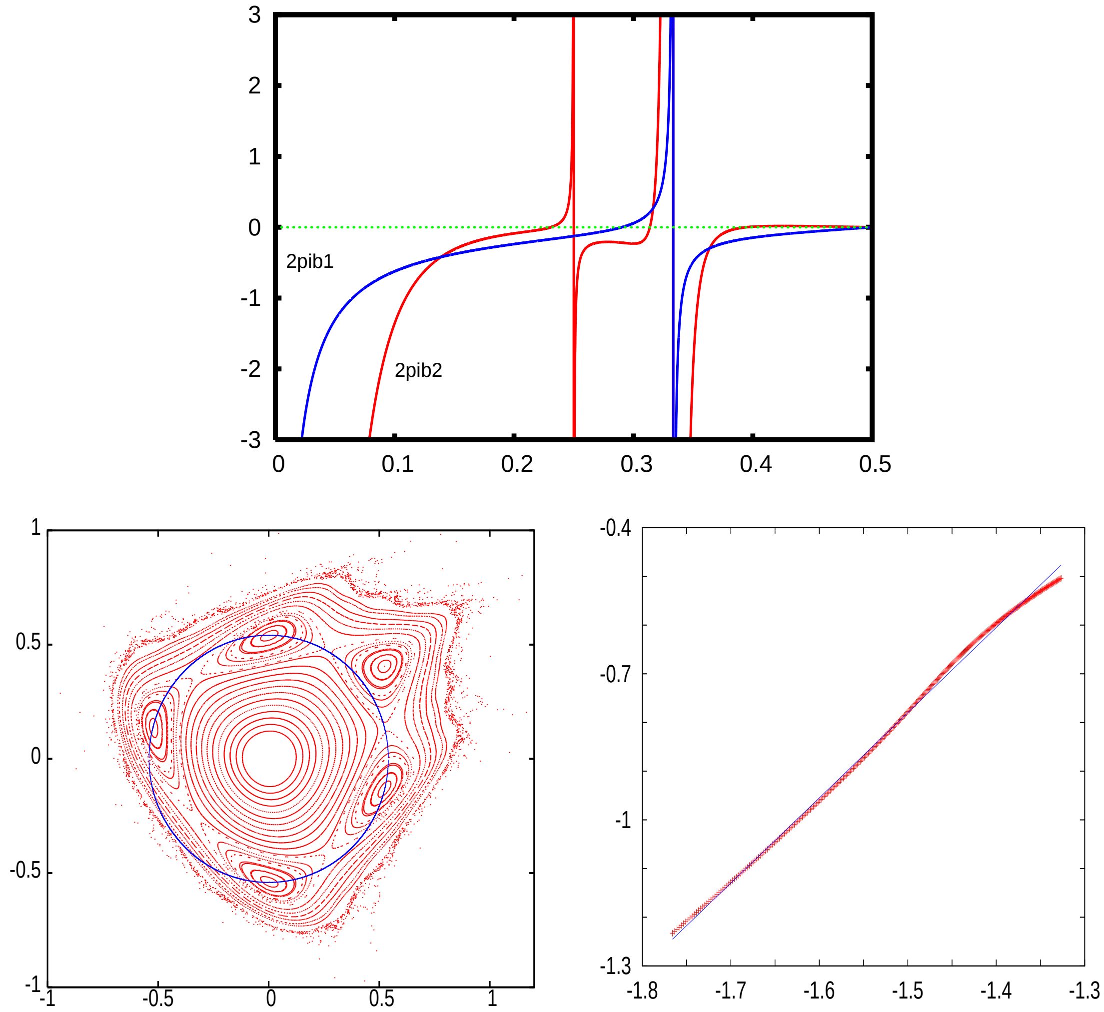


Illustration (Hénon map)



Left: RES 1:5, $\alpha = 0.21$, $b_1 \approx -0.0341669659$ and $r_* \approx 0.5409994$.

Right: RES 1:7, straight line $y = (7/4)x + b$, $b \approx 1.854512$ ($\log_{10}$ plot).

A model around a generic resonance

For a generic APM such that $\alpha = q/m + \delta,\ \delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the m-resonance strip $(m \geq 5)$ can be modelled, after suitable scaling ($J \sim \delta^{-m/4}(I - I_*)$), by the time one map of the flow generated by Hamiltonian

$$\mathcal{H}(J, \psi) = \frac{1}{2} J^2 + \frac{c}{3} J^3 - (1 + dJ)\cos(\psi),$$

where

$$c \approx \frac{\mathrm{sign}(b_1 b_2)|b_2|}{\sqrt{m\pi}\,|b_1|^{\frac{6+m}{4}}}\,|\delta|^{\frac{m}{4}}, \quad d \approx \frac{\sqrt{m}}{2\sqrt{\pi}\,|b_1|^{\frac{m-2}{4}}}\,|\delta|^{\frac{m}{4}-1}.$$

In an annulus domain centred at the radius I_* of width $\mathcal{O}(I_*^{m/4})$ (width of the resonant zone, which is $\mathcal{O}(1)$ in J) the above approximation gives an error $\mathcal{O}(I_*^{\sigma})$, $\sigma = (m+2)/4$.

PART II

Inner and outer splitting of separatrices

Basic Idea

According to our previous results the Hamiltonian $\mathcal{H}(J,\psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1+dJ)\cos(\psi)$ approximates the dynamics in the resonant islands.

$\longrightarrow$ The idea is to used it as a “limit” Hamiltonian to apply Fontich-Simó theorem on exponential small bound of the splitting of separatrices.

That is:

Let σ_- (σ_+) be the inner (outer) separatrices of X_H. We try to compute the location of the singularities of $\sigma_\pm$. Hence, we will determine the exponentially small part of the splitting.

Remark: We will see that F-S Thm cannot be applied directly...

Singularities of $\mathcal{H}(J,\psi)$

Let h the energy level of the separatrices $\sigma_\pm$ then $H(J,\psi) = P(J) - Q(J)\cos(\psi) = h$ implies $\dot{J} = \sqrt{Q^2 - (P-h)^2}$. In our case $P(J) = J^2/2 + cJ^3/3$, $Q(J) = 1 + dJ$ and, hence,

$$\tau = \int_\gamma \frac{dJ}{\sqrt{p_6(J)}}, \ \gamma \text{ path to } \infty \text{ for } J \in \mathbb{C}.$$

We know 4 zeros trivially of $Q^2 - (P-h)^2$: J_h (double), J_M and J_m. Hence,

$$\tau = \int_\gamma \frac{dJ}{(J - J_h)\sqrt{p_4(J)}} \qquad \to \text{Elliptic Integral}$$

The other 2 zeros are located far. We choose the real path of J and we obtain: Im $\tau_+ = \frac{\pi}{2} - d + \ldots$ Im $\tau_- = \frac{\pi}{2} + d + \ldots$ and Re $\tau_\pm = \mathcal{O}(\delta^{m/4})$ (one of them is 0, depending on the sign of c, recall that $d = \mathcal{O}(\delta^{m/4-1})$).
As a consequence, one expects $\sigma_+ >> \sigma_-$ for δ small enough (F-S).

Remarks and difficulties

- Complex extension of the variables: cartesian or Poincaré vbles?

  The map is given in cartesian but the NF in Poincaré vbles. The extensions to complex are not equivalent, but we are interested in **real phenomenon**.

- The Hamiltonian $H(J, \psi)$ is not enough for our purposes.

  We need approximation in a very large domain in the complex (J, ψ) vbles to compare the separatrices.

- We want to reduce the analytic strip not by a constant finite amount η but by a quantity small compared with $d = \mathcal{O}(\delta^{m/4-1})$.

$\Rightarrow$ We have to change the Hamiltonian. This means changing the singularities.

New Hamiltonian

- We consider the Hamiltonian before the localisation. In (J, ψ) vbles.

$$\mathcal{H}_{new}(J, \psi) = \frac{1}{2} J^2 + \frac{\tilde{c}_3}{3} J^3 + \frac{\tilde{c}_4}{4} J^4 + \ldots - (1 + \tilde{d} J)^m \cos \psi .$$

- Is good enough?
  - It is enough to consider $|J| \sim \mathcal{O}(\delta^{1-m/4-\mu})$, $\mu > 0$.
  - In this domain the error of the approximation (both from NF and interpolation) of the above Hamiltonian and the map is below $\delta^{m/4}$ provided $\mu < 1/(m+2)$.
  - We check that the integrals to compute the singularities with the "new" and the "old" Hamiltonians are close enough ($<< \delta^{m/4}$).

Then, we have the following...

Main result: comments on the hypothesis

- **A1.** $b_1(\delta)$ is non-zero for $\delta = 0$.
- **A2.** $W^u = G(W^s)$, G periodic (between homo p and $F(p)$),

  s scaled variable s.t. $G(s) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\, 2\pi s)$

  We assume: c_1, c_{-1} are bounded away from zero, close to the singularity, when the small parameter in the family of maps tends to zero.
- **A3.** There exists a fixed $\alpha > 0$ s.t.

$$\sigma_\pm = \exp\left(-\frac{2\pi \,\mathrm{Re}\, \tau_\pm - \eta_\pm}{\log(\lambda(\epsilon))}\right)\left(\cos\left(\frac{2\pi \,\mathrm{Im}\, \tau_\pm}{\log(\lambda(\epsilon))} - \phi_\pm\right) + o(1)\right),$$

  where $|\eta_\pm| < \log(\lambda(\epsilon))^{1-\alpha}$ for ϵ sufficiently small.
- **A4.** F maybe meromorphic but the singularity remains at a finite distance when δ goes to 0.

Main result

Theorem. Let F be an APM. Assume that it has an m-order resonance strip, $m > 4$, located at an average distance $I = I_* = \mathcal{O}(\delta)$ from the elliptic fixed point and δ is sufficiently small. Under the assumptions **A1**, **A2**, **A3** and **A4**, the following conclusions hold.

a) The outer splitting is larger than the inner one being the difference between the position of the corresponding nearest singularities $\mathcal{O}(\delta^{m/4-1})$.

b) Neither the inner nor the outer splittings oscillate. [a]

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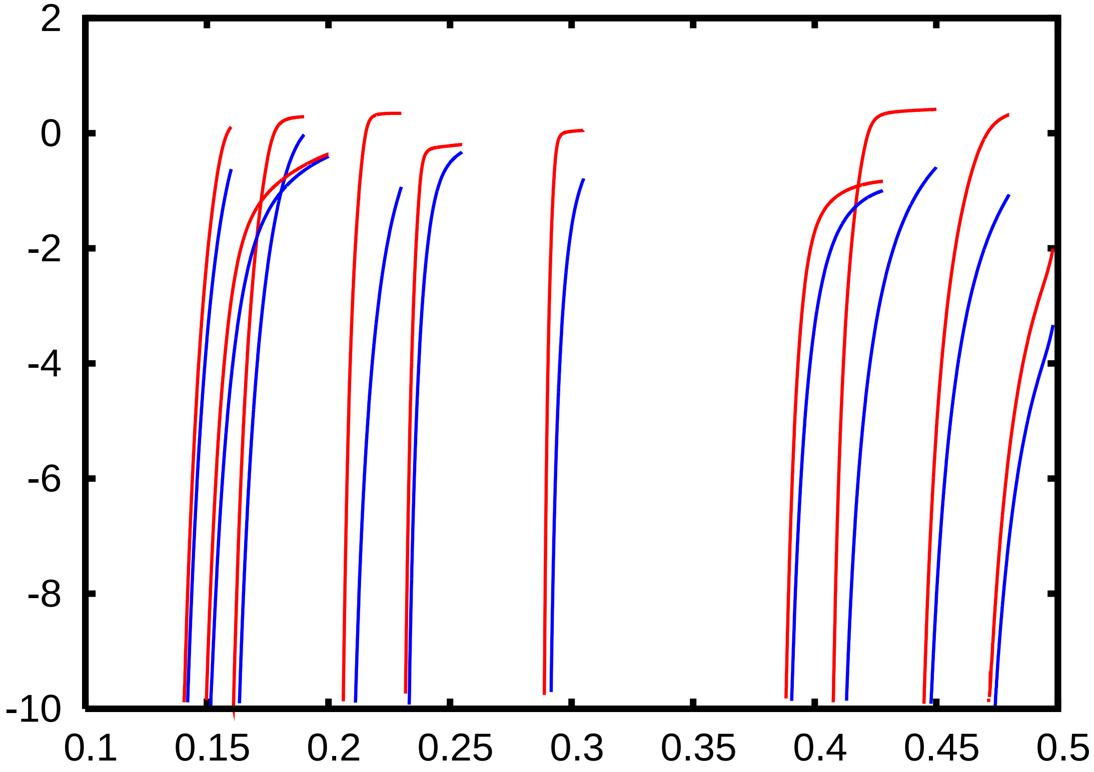
[a] The details of the proof (singularities, suitable Hamiltonian,...) can be found in:
Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps.
Nonlinearity 22, 5:1191–1245, 2009.

Hénon map



From left to right, it is represented the decimal logarithm of the splitting of the resonances 1:9, 1:8, 1:7, 1:5, 2:9, 2:7, 3:8, 2:5, 3:7 and 4:9, respectively. Each pair of red and blue lines corresponds to the outer and inner splitting, respectively, of a different resonance. Note that in all the cases shown the outer splitting (red) is greater than the inner one (blue). In the x-axis it is represented the value of α.

Dynamical consequences I

Generic resonances close to the origin. Assume $b_1\delta < 0$ and that the hypothesis of the theorem concerning the difference of the inner and outer splittings hold. Then,

- The width of the outer chaotic zone is larger than the width of the inner chaotic one if, and only if, sign $b_1 \cdot$ sign $b_2 < 0$.
- Both amplitudes of the stochastic layer are of the order of magnitude of the outer splitting (the largest one).

Basic ideas: Distance to invariant curves from the separatrix: $d_c \sim |b|/k^*$ (SM is approximated by STM, $k^* \approx 0.97/(2\pi)$ Greene value).

When coming back to the original variables: $D_c \sim \sigma\ell/(2\pi k^* \log(\lambda))$,
An “interpolating” Hamiltonian takes into account the re-injection of the dynamics on the distance to curves from the inner and outer parts. [a]

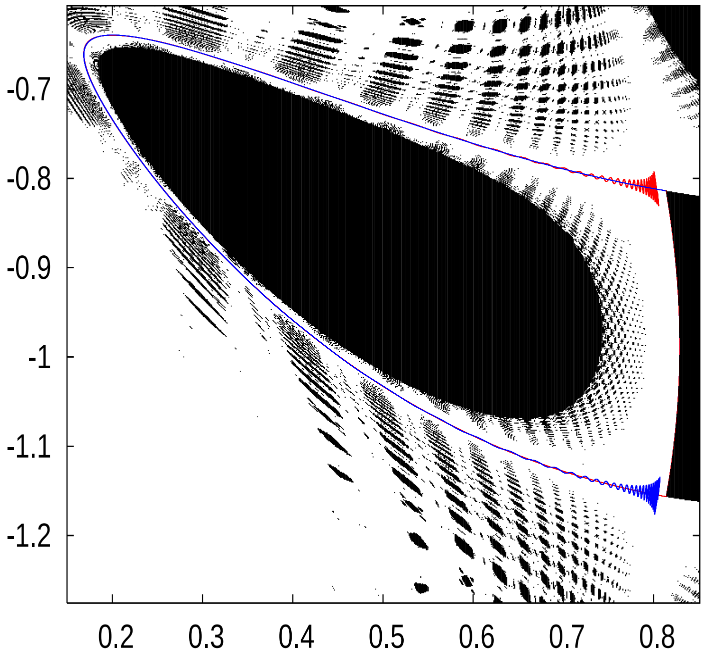
[a] The following results can be found in:

Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.

Submitted to Physica D.

Dynamical consequences II

The same idea applies to resonances far from the origin as well as for strong resonances but, for each case, a suitable interpolating Hamiltonian must be considered.



$c = 1.015$,

$\sigma_+ = \mathcal{O}(10^{-54})$, $\sigma_- = \mathcal{O}(10^{-1})$.

Experimentally, $f \approx -5$. Using interp. Ham. up to order $\delta \approx c - 1$ we obtain $f \approx -5.64$.

But $\delta = 0.015$ is too large. For δ small we obtain better results (even we can predict # tiny islands).

Part III

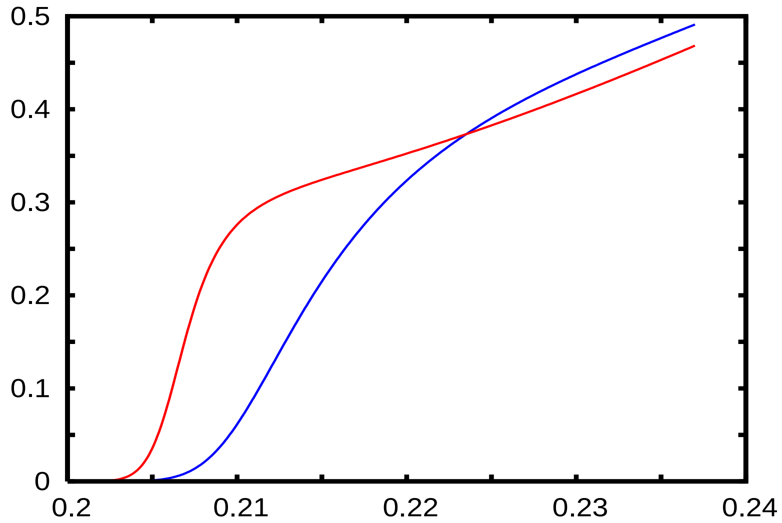
Final comments:

Far from the elliptic point.

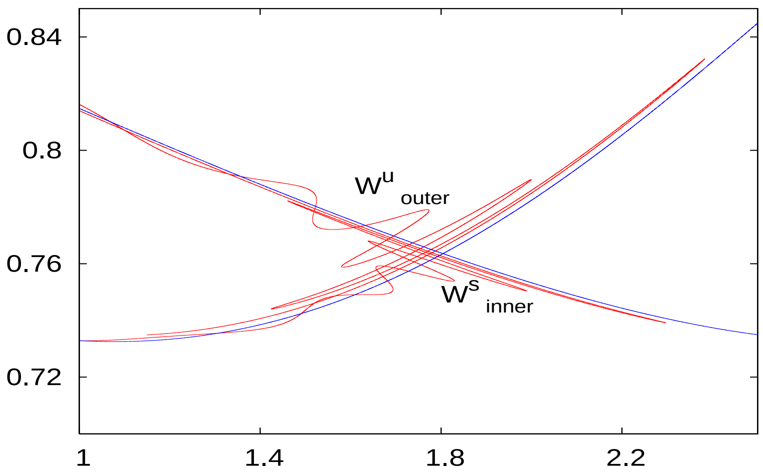
Strong resonances ($m \leq 4$).

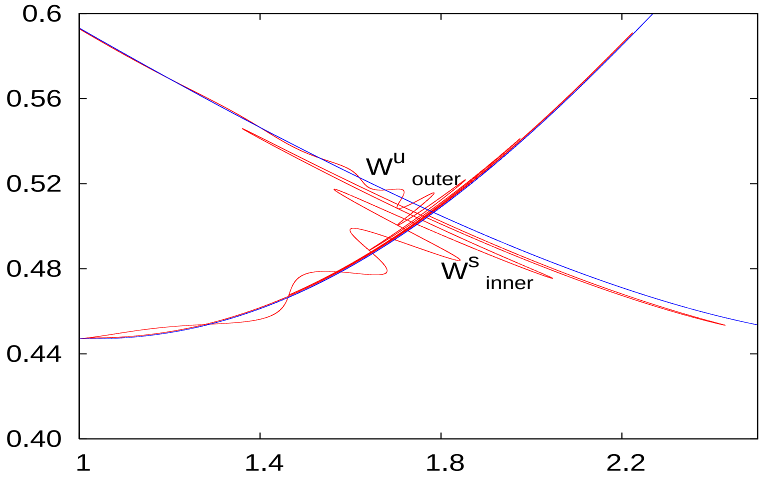
Far from the elliptic point

2:11 Hénon



twist: $(I, \theta) \to (I + 0.14\cos(\theta + \alpha(I)), \theta + \alpha(I)), \alpha(I) = b_1 I + b_2 I^2$.

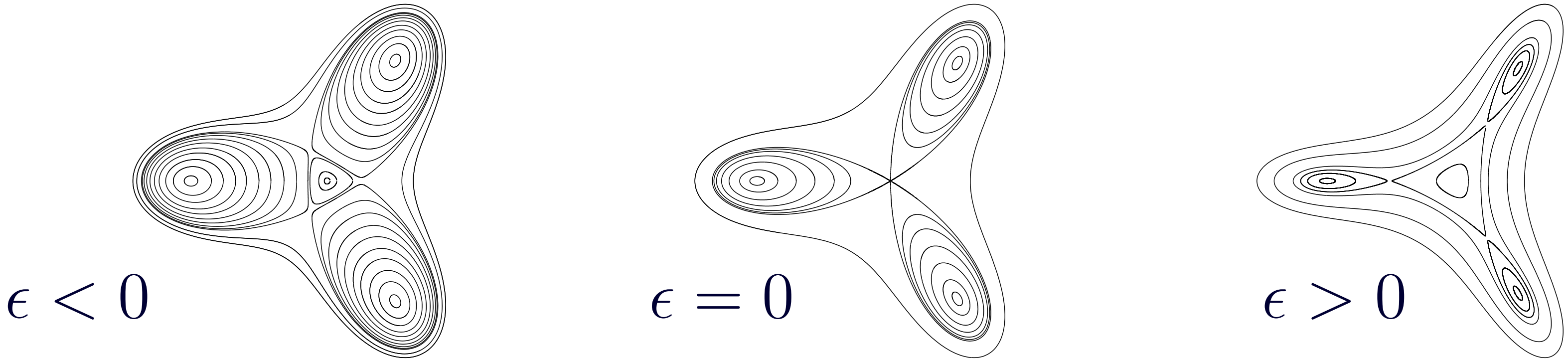




Strong resonances (I)

The description of the resonant structure by means of the interpolating Hamiltonian does not hold if $m \leq 4$.

1:3 resonance: $\mathcal{H}(I, \varphi) = \epsilon I + I^2 + I^{\frac{3}{2}} \cos(3\varphi)$

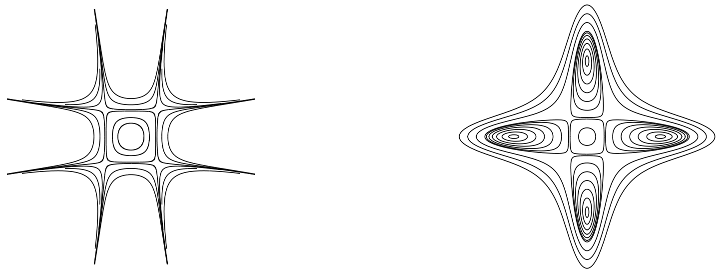


- Hyperbolic points at a distance $\mathcal{O}(\epsilon^2)$. Elliptic points at a **finite** distance.
- Outer splitting non-perturbative since the separatrices remain at a finite distance.
- Inner splitting behaves as described in the generic case $m > 4$.

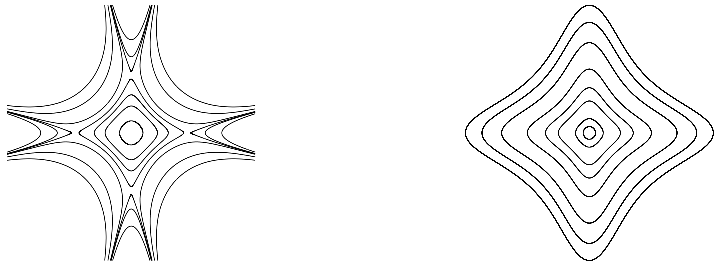
Strong resonances (II)

1:4 resonance: $\mathcal{H}(I,\varphi) = \epsilon I + I^2 + \xi I^2 \cos(4\varphi), \quad \xi < 0.$

$\epsilon < 0$, $\xi < -1$ left,
$-1 < \xi < 0$ right



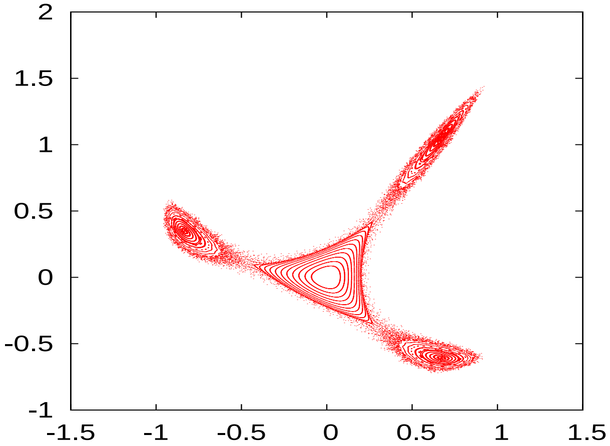
$\epsilon > 0$, $\xi < -1$ left,
$-1 < \xi < 0$ right



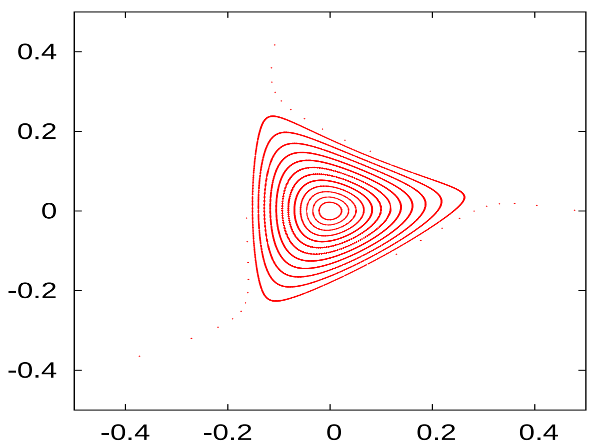
- Elliptic and hyperbolic points located at a distance $\mathcal{O}(\epsilon)$.
- Cases with $\xi < -1$: The splitting **oscillates** and behaves as expected in magnitude in the generic case.
- Case $\epsilon < 0$, $\xi > -1$: The splittings behave as expected in the generic case.

Strong resonances of the Hénon map (I)

1:3 resonance:



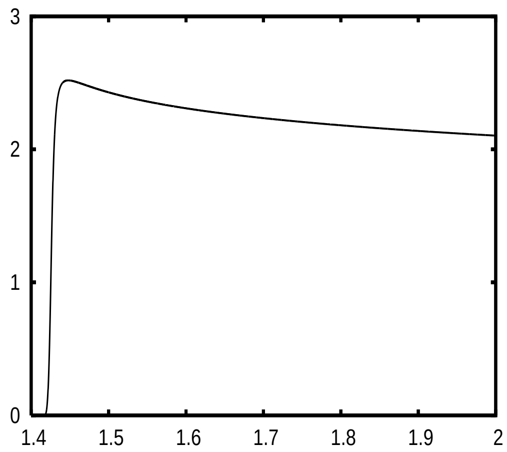
$\alpha = 0.32$



$\alpha = 0.34$

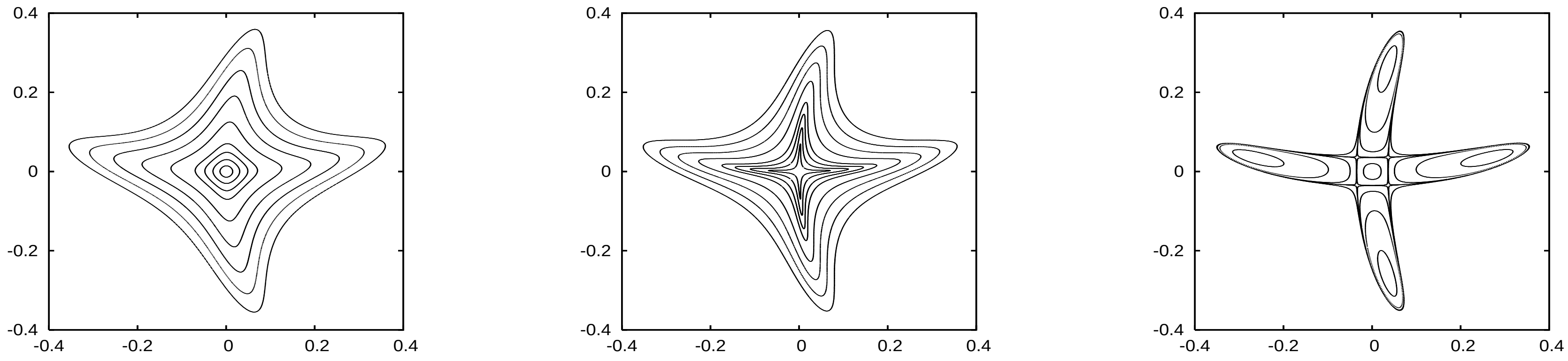
The outer splitting remains finite

($\alpha = 1/3$ corresponds to $c = \sqrt{2}$):



Strong resonances of the Hénon map (II)

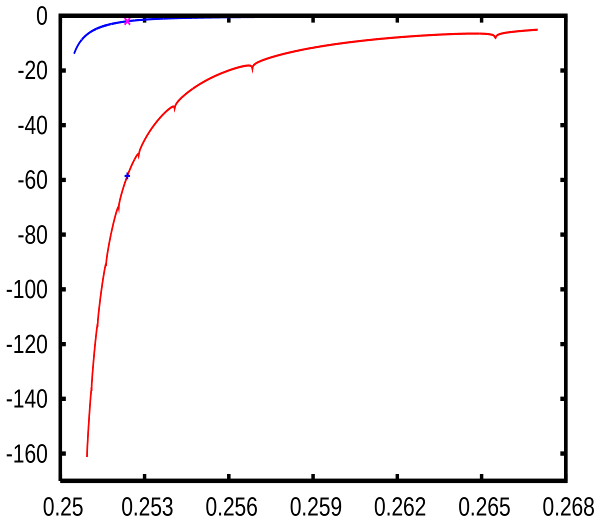
1:4 resonance: **Non-generic!!**

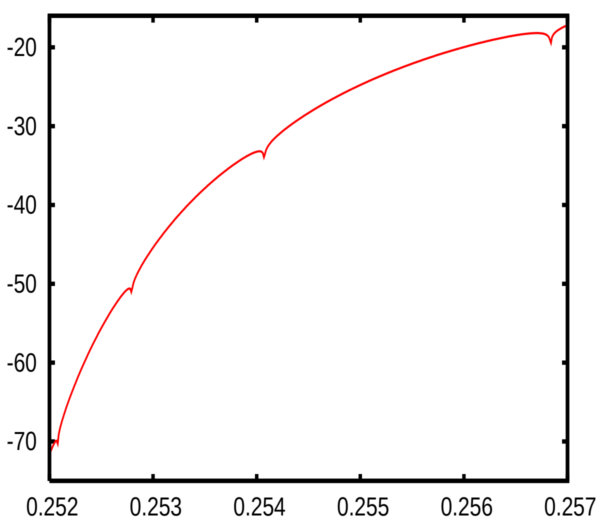


$\alpha = 0.2499$ $\alpha = 0.2500$ $\alpha = 0.2501$

- It corresponds to the case $\xi = -1$ in the Hamiltonian above.
- The elliptic point goes to a distance $\mathcal{O}(\epsilon^{1/2})$ instead $\mathcal{O}(\epsilon)$.
- $H(I, \varphi) = \epsilon I + I^2(1 - \cos(\psi)) + I^3(a + b\cos(\psi) + c\sin(\psi))$.
- Hénon corresponds to $\epsilon < 0$, $a + b > 0$. The inner splitting oscillates and the outer does not. There is a big difference inner-outer splitting magnitude (outer singularity at a distance $\mathcal{O}((\epsilon(a+b))^{1/4})$, inner singularity real part distance 2π).

Strong resonances of the Hénon map (III)

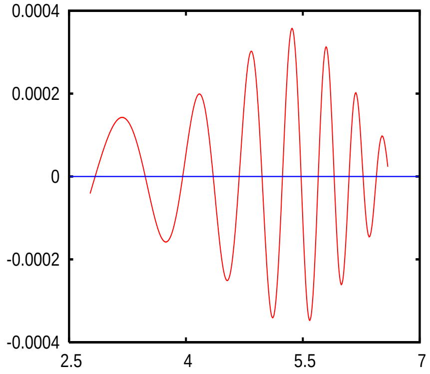




Decimal logarithm of the inner (red) and outer (blue) splittings as a function of α.

- Big difference in the order of the size of the splittings: For $\alpha \approx 0.25238741368$ it is $\sigma_+ \approx 2.5238741368 \times 10^{-1}$ and $\sigma_- \approx -2.986620731 \times 10^{-59}$.

- The inner splitting oscillates ("peaks").



The End

Thank you!